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Research article

Ground states of a Kirchhoff equation with the potential on the lattice graphs

Wenqian Lv*

School of Mathematics, East China University of Science and Technology, Shanghai, 200237, China

* **Correspondence:** Email: 15052018708@163.com.

Abstract: In this paper, we study the nonlinear Kirchhoff equation

$$-\Big(a + b\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\Big)\Delta u + V(x)u = f(u)$$

on lattice graph $\mathbb{Z}^3$, where $a, b > 0$ are constants and $V : \mathbb{Z}^3 \to \mathbb{R}$ is a positive function. Under a Nehari-type condition and 4-superlinearity condition on f, we use the Nehari method to prove the existence of ground-state solutions to the above equation when V is coercive. Moreover, we extend the result to noncompact cases in which V is a periodic function or a bounded potential well.

Keywords: Kirchhoff equation; lattice graph; ground states; Nehari manifold; variational methods
Mathematics Subject Classification: 35J60, 35J20, 35R02

1. Introduction and main results

In this paper, by using the variational methods, we are concerned with the ground-state solutions to the following nonlinear Kirchhoff equation with potentials on lattice graph $\mathbb{Z}^3$:

$$-\Big(a + b\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\Big)\Delta u + V(x)u = f(u), \tag{1.1}$$

where $a, b > 0$ are constants.

Recently, the Kirchhoff equation

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(x)u = f(x, u), \ \ x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

has been extensively and in-depth studied, where a, b are positive constants, $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. Equation (1.2) is a nonlocal problem due to the appearance of the term $\int_{\mathbb{R}^3} |\nabla u|^2$, which means that (1.2) is no longer a pointwise identity. This phenomenon poses some mathematical difficulties that make the study of (1.2) particularly interesting. Problem (1.2) originates from some interesting physical context. Indeed, let $V(x) = 0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^3$ in (1.2); then, we get the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

presented by Kirchhoff [1]. This type of Kirchhoff model takes into account the changes in the length of the string caused by transverse vibrations. An increasing number of researchers have started paying attention to the Kirchhoff equations after the seminal paper by Lions [2], where he proposed a functional analysis approach.

We know that the weak solutions of (1.2) correspond to the critical point of the energy functional given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2 dx\Big)^2 - \int_{\mathbb{R}^3} F(x,u)dx,$$

defined on $E = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 < \infty\}$, where $F(x,u) = \int_0^u f(x,s)ds$, $f(x,u)$ is usually assumed to satisfy the Ambrosetti-Rabinowitz-type condition:

(AR) There exists a positive constant $\theta > 4$ such that

$$0 < \theta F(t) \leq tf(t), \qquad \forall\, t \neq 0, \quad \text{where } F(t) = \int_0^t f(s)ds$$

or $f(x,u)$ is assumed to be subcritical, superlinear at the origin and either 4-superlinear at infinity in the sense that

$$\lim_{|u|\to+\infty}\frac{F(x,u)}{u^4} = +\infty \quad \text{uniformly in } x \in \mathbb{R}^3.$$

Under the above conditions, one can obtain a Palais-Smale ((PS) in short) sequence of I by using the mountain-pass theorem thanks to Ambrosetti and Rabinowitz [3]. Moreover, it can be shown that I satisfies the (PS) condition and (1.2) has at least one nontrivial solution when further conditions are assumed for $f(x,u)$ and $V(x)$ to ensure the compactness of the (PS) sequence.

In [4], given that $V \equiv 1$ and $f(x,u)$ satisfy the conditions of subcriticality, superlinearity at the origin and being 4-superlinear at infinity, Jin and Wu investigated infinitely many radial solutions to (1.2) by using the fountain theorem. By using the symmetric mountain-pass theorem [5], Wu [6] showed that the problem (1.2) has a sequence of high-energy solutions when $\mathbb{R}^3$ is replaced by $\mathbb{R}^N$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf V(x) \geq a_1 > 0$, where a_1 is a constant. And, for each $M > 0$, meas $\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where meas is the Lebesgue measure in $\mathbb{R}^N$. These conditions on $V(x)$, in their note, suffice to ensure the compactness of the embeddings of $E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2 < +\infty\} \hookrightarrow L^q(\mathbb{R}^N)$, where $2 \leq q < 2^* = \frac{2N}{N-2}$. In [7], by applying the Nehari manifold, He and Zou proved the the existence, multiplicity and concentration behavior of positive solutions for the following parameter-perturbed Kirchhoff equation:

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = f(u), & x \in \mathbb{R}^3, \\ u \in H^1\left(\mathbb{R}^3\right), \quad u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a parameter. In [8], Mao and Zhang proved the existence of sign-changing solutions of (1.3) by using the invariant sets of descent flow and minimax methods. For more results about the existence of nontrivial solutions, ground states, the multiplicity of solutions and concentration of solutions and sign-changing solutions, see [9–18] and the references therein.

Recently, many researchers have paid attention to various partial differential equations on discrete spaces. For example, in [19], by using the mountain-pass theorem, Grigor'yan et al. considered the following Yamabe problem:

$$\begin{cases} -\Delta u - \alpha u = |u|^{p-2}u, & in\ \Omega^\circ, \\ u = 0, & on\ \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{V}$ is a bounded domain on a locally finite graph $G = (\mathbb{V}, \mathbb{E})$ and Ω° and $\partial\Omega$ denote the interior and the boundary of Ω, respectively. They proved the existence of a positive solution to this problem, and, in [20], they also showed the existence of positive solutions to the nonlinear equation

$$-\Delta u + hu = f(x, u)$$

on locally finite graphs. In particular, under certain assumptions on h and f, they prove the existence of strictly positive solutions to the above equations. In [21], by applying a Nehari method, Zhang and Zhao studied the convergence of ground-state solutions for the nonlinear Schrödinger equation

$$-\Delta u + (\lambda a(x) + 1)u = |u|^{p-1}u$$

on a locally finite graph $G = (\mathbb{V}, \mathbb{E})$, where the potential $a(x)$ is defined on $\mathbb{V}$. Under the condition that $a(x)$ is coercive, they showed that the above equation admits a ground-state solution u_λ for any $\lambda > 1$ and u_λ converges to a solution for the Dirichlet problem as $\lambda \to \infty$. For further results concerning discrete Sobolev inequalities, p-Laplacian equations and biharmonic equations on graphs, we refer the readers to [22–24] and the references therein.

Motivated by [16, 25, 26], in this paper, we will study the ground-state solutions to the nonlinear Kirchhoff equation (1.1) with potentials on lattice graph $\mathbb{Z}^3$. We generalize some results from the continuous case to the discrete case. Since our problem is discrete, some estimates and results are different from the continuous case.

A function g is called τ-periodic if $g(x + \tau e_i) = g(x)$ for $\tau \in \mathbb{Z}$ and all $x \in \mathbb{Z}^3$, $1 \le i \le 3$, where e_i is the unit vector in the i-th coordinate.

Throughout the paper, we make the following assumptions on the potential $V : \mathbb{Z}^3 \to \mathbb{R}$:

(V_1) There exists a constant $V_0 > 0$ such that $V(x) \ge V_0$ for all $x \in \mathbb{Z}^3$.
(V_2) There exists a vertex $x_0 \in \mathbb{Z}^3$ such that $V(x) \to +\infty$ as $\text{dist}(x, x_0) \to +\infty$.
(V_3) $V(x)$ is τ-periodic in x for all $x \in \mathbb{Z}^3$.
(V_4) $\inf\limits_{x\in\mathbb{Z}^3} V(x) \le \lim\limits_{|x|\to\infty} V(x) = \sup\limits_{x\in\mathbb{Z}^3} V(x) < \infty$.

Moreover, let the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying the following conditions:

(f_1) $\lim\limits_{t\to 0}\frac{f(t)}{t} = 0$.
(f_2) There exists $q \in (3, \infty)$ such that

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|^q} = 0.$$

(f_3) $\frac{F(t)}{t^4} \to \infty$ as $|t| \to \infty$, where $F(t) = \int_0^t f(s)ds$.
(f_4) $\frac{f(t)}{|t|^3}$ is strictly increasing on $\mathbb{R}\backslash\{0\}$.

Condition (V_2) is assumed in [20, 21] to prove the existence of ground-state solutions to nonlinear Schödinger equations on locally finite graphs. Motivated by the papers mentioned above, we shall prove the following theorem.

Theorem 1.1. *Assume that V satisfies* (V_1) *and* (V_2) *and f satisfies* (f_1)–(f_4). *Then, problem* (1.1) *has a ground-state solution.*

In [26], Li et al. consider the nonlinear Schrödinger equation with two potential cases: one is periodic and the other is bounded. Using the Nehari method, they found ground-state solutions without compact embeddings. In [27], Szulkin and Weth presented a unified approach to the Nehari method and proved results similar to Theorem 2.1 and Theorem 3.1 in [26]. In [28], by taking advantage of the generalized Nehari manifold method developed by Szulkin and Weth, Zhang and Zhang proved the existence of semiclassical ground-state solutions of coupled Nonlinear Schrödinger systems with competing potentials. Moreover, they investigated the asymptotic convergence of ground-state solutions under the conditions of scaling and translation. In [25], Hua and Xu extended the results in [27] to the lattice graphs. These inspire us to generalize the above results to the Kirchhoff-type equations on the lattice graphs. More precisely, we have the following theorems.

Theorem 1.2. *Assume that V satisfies* (V_1) *and* (V_3) *and f satisfies* (f_1)–(f_4). *Then, problem* (1.1) *has a ground-state solution.*

Theorem 1.3. *Assume that V satisfies* (V_1) *and* (V_4) *and f satisfies* (f_1)–(f_4). *Then, problem* (1.1) *has a ground-state solution.*

Theorems 1.2 and 1.3 are natural generalizations of the results from Theorem 1.1 to noncompact cases. In both cases, we shall combine the techniques in [26, 27, 29] with the concentration-compactness principle provided by Lions [30, 31] in the discrete space to overcome the loss of compactness.

The paper is organized as follows. In Section 2, we recall the function space settings on the lattice graphs and give some preliminary results. Then, the generalized Nehari manifold is introduced in Section 3. In Section 4, we prove the existence of ground-state solutions to (1.1) with coercive potential. Furthermore, we consider two cases of V without compact embedding, where one is periodic and the other is a bounded potential well. The results will be stated and proved in Section 5.

Notation

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line.
- $o_n(1)$ denotes the quantity that tends to 0 as $n \to +\infty$.

- $\|\cdot\|_q$ and $\|\cdot\|_\infty$ denote the usual norms of the spaces $l^q(\mathbb{Z}^3)$ and $l^\infty(\mathbb{Z}^3)$, respectively, and we may omit the subscript $\mathbb{Z}^3$ if it can be understood from the context.

2. Abstract setting and preliminary results

In this section, we introduce the basic settings on graphs and then give some preliminaries which will be useful for our arguments. For more details on graphs, see [20, 22, 32, 33].

Let $G = (\mathbb{V}, \mathbb{E})$ be a connected, locally finite graph, where $\mathbb{V}$ denotes the vertex set and $\mathbb{E}$ denotes the edge set. We call vertices x and y neighbors, denoted by $x \sim y$, if there exists an edge connecting them, i.e., $(x, y) \in \mathbb{E}$. G is called locally finite if, for any $x \in \mathbb{V}$, the number of vertices connected to x is finite. G is connected if any two vertices in $\mathbb{V}$ can be connected by a finite number of edges in $\mathbb{E}$. If G is connected, then define the graph distance $|x - y|$ between any two distinct vertices x, y as follows: if $x \neq y$, then $|x - y|$ is the minimal path length connecting x and y, and if $x = y$, then $|x - y| = 0$. Let $B_R(x) = \{y \in \mathbb{V} : |x - y| \leq R\}$ be the ball centered at x with radius R in $\mathbb{V}$. We write $B_R = B_R(0)$ and $B_R^c = \mathbb{V} \backslash B_R$ for convenience.

In this paper, we focus on differential equations on the lattice graph $\mathbb{Z}^3$ with the set of vertices consisting of all 3-tuples $(x_1, x_2, , x_3)$, where x_i denotes integers and $(x_1, x_2, , x_3) \sim (y_1, y_2, , y_3)$ if and only if

$$\sum_{i=1}^{3} |x_i - y_i| = 1.$$

That is, x_i is different from y_i for exactly one value of the index i, and $|x_i - y_i| = 1$ for this value of i.

$C(\mathbb{Z}^3)$ is denoted as the space of functions on $\mathbb{Z}^3$. Let μ be the counting measure on V, i.e., for any subset $A \subset \mathbb{Z}^3, \mu(A) := \#\{x : x \in A\}$. For any function $f : \mathbb{Z}^3 \to \mathbb{R}$, integral of f over $\mathbb{Z}^3$ is defined by

$$\int_{\mathbb{Z}^3} f d\mu = \sum_{x \in \mathbb{Z}^3} f(x).$$

For $u \in C(\mathbb{Z}^3)$, we define the difference operator for any $x \sim y$ as

$$\nabla_{xy} u = u(y) - u(x).$$

For any function $u \in C(\mathbb{Z}^3)$ and $x \in \mathbb{Z}^3$, we define the Laplacian of u as

$$\Delta u(x) = \sum_{y \sim x} (u(y) - u(x)).$$

The gradient form, Γ, of two functions u and v on the graph is defined as

$$\Gamma(u, v)(x) = \frac{1}{2} \sum_{y \sim x} (u(y) - u(x))(v(y) - v(x)).$$

In particular, write $\Gamma(u) = \Gamma(u, u)$ and define the length of the discrete gradient as

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2} \sum_{y \sim x} (u(y) - u(x))^2 \right)^{\frac{1}{2}}.$$

The space $l^p(\mathbb{Z}^3)$ is defined as

$$l^p(\mathbb{Z}^3) = \left\{u \in C(\mathbb{Z}^3) : \|u\|_p < \infty\right\},$$

where

$$\|u\|_p = \begin{cases} (\sum_{x\in\mathbb{Z}^3} |u(x)|^p)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty; \\ \sup_{x\in\mathbb{Z}^3} |u(x)|, & \text{if } p = \infty. \end{cases}$$

Let $C_c(\mathbb{Z}^3)$ be the set of all functions on $\mathbb{Z}^3$ with finite support, where $\mathrm{supp}(u) = \{x \in \mathbb{Z}^3 : u(x) \neq 0\}$. In addition, we define the space $W^{1,2}(\mathbb{Z}^3)$ as the completion of $C_c(\mathbb{Z}^3)$ with respect to the norm

$$\|u\|_{1,2} = \left(\int_{\mathbb{Z}^3} \left(|\nabla u|^2 + u^2\right) d\mu\right)^{1/2}.$$

Clearly, $W^{1,2}(\mathbb{Z}^3)$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\mathbb{Z}^3} (\Gamma(u, v) + uv) d\mu, \quad \forall u, v \in W^{1,2}(\mathbb{Z}^3).$$

Let $V(x) \geqslant V_0 > 0$ for all $x \in V$. To study problem (1.1), it is natural to consider the following function space:

$$\mathscr{H} = \left\{u \in W^{1,2}(\mathbb{Z}^3) : \int_{\mathbb{Z}^3} V(x)u^2 d\mu < +\infty\right\},$$

with a norm

$$\|u\| = \left(\int_{\mathbb{Z}^3} \left(a|\nabla u|^2 + V(x)u^2\right) d\mu\right)^{1/2},$$

which is equivalent to the norm of $W^{1,2}(\mathbb{Z}^3)$ under (V_1), (V_3) and (V_4). The space $\mathscr{H}$ is also a Hilbert space; its inner product is

$$\langle u, v\rangle = \int_{\mathbb{Z}^3} (a\Gamma(u, v) + V(x)uv) d\mu, \quad \forall u, v \in \mathscr{H}.$$

We also need another discrete Sobolev space $D^{1,2}(\mathbb{Z}^3)$, which is the completion of $C_c(\mathbb{Z}^3)$ under the norm $\|u\|^2 = \int_{\mathbb{Z}^3} |\nabla u|^2 d\mu$. For some details about $D^{1,2}(\mathbb{Z}^3)$, we refer the reader to [23, 33].

The functional related to problem (1.1) is

$$J(u) = \frac{1}{2}\int_{\mathbb{Z}^3} (a|\nabla u|^2 + V(x)u^2) d\mu + \frac{b}{4}\Big(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\Big)^2 - \int_{\mathbb{Z}^3} F(u) d\mu.$$

$u \in \mathscr{H}$ is said to be the weak solution of (1.1), if for any $\phi \in \mathscr{H}$,

$$0 = \langle J'(u), \phi\rangle = \int_{\mathbb{Z}^3} (a\nabla u\nabla\phi + V(x)u\phi) d\mu + b\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu \int_{\mathbb{Z}^3} \nabla u\nabla\phi d\mu - \int_{\mathbb{Z}^3} f(u)\phi d\mu.$$

Since $C_c\left(\mathbb{Z}^3\right)$ is dense in $\mathscr{H}$, if u is a weak solution of (1.1), then integration by parts gives

$$\int_{\mathbb{Z}^3} (a\nabla u\nabla\phi + V(x)u\phi) d\mu + b\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu \int_{\mathbb{Z}^3} \nabla u\nabla\phi d\mu = \int_{\mathbb{Z}^3} f(u)\phi d\mu, \quad for\ any\ \phi \in C_c\left(\mathbb{Z}^3\right).$$

We say that a nontrivial weak solution $u \in \mathscr{H}$ to (1.1) is a ground-state solution if $J(u) \leq J(v)$ for any nontrivial solution $v \in \mathscr{H}$ to (1.1). To prove our results, we define the Nehari manifold for (1.1) as the set

$$\mathcal{N} = \{u \in \mathscr{H} \backslash \{0\} : \langle J'(u), u \rangle = 0\},$$

namely,

$$\mathcal{N} = \Big\{u \in \mathscr{H} \backslash \{0\} : \int_{\mathbb{Z}^3} \Big(a|\nabla u|^2 + V(x)u^2\Big)\, d\mu + b\Big(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\Big)^2 = \int_{\mathbb{Z}^3} f(u)u d\mu\Big\}.$$

Naturally, all nontrivial critical points of J belong to $\mathcal{N}$. However, because f is only continuous, the Nehari manifold $\mathcal{N}$ is not of class C^1; therefore, we cannot use the Ekeland variational principle on $\mathcal{N}$ or obtain a (PS) sequence for J. In order to overcome this difficulty, we shall apply Szulkin and Weth's method (see [27, 29]) in the discrete setting to show that $\mathcal{N}$ remains as a topological manifold, which is naturally homeomorphic to a unit sphere in $\mathscr{H}$; by differentiability of the unit sphere, we can consider transforming the original problem into finding a critical point of a C^1 functional on it.

Here, we present a compact result which plays a key role in the proof of our theorems; for more details of the proof, see [21].

Lemma 2.1. *If $V(x)$ satisfies (V_1) and (V_2), then $\mathscr{H}$ is compactly embedded into $l^p(\mathbb{Z}^3)$ for any $p \in [2, +\infty]$. Namely, there exists a constant C that depends only on p such that, for any $u \in \mathscr{H}$,*

$$\|u\|_p \leq C\|u\|.$$

Furthermore, for any bounded sequence $\{u_n\} \subset \mathscr{H}$, there exists $u \in \mathscr{H}$ such that, up to a subsequence (still denoted by $\{u_n\}$), we have that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } \mathscr{H}, \\ u_n(x) \to u(x), & \forall\, x \in \mathbb{Z}^3, \\ u_n \to u, & \text{in } l^p(\mathbb{Z}^3). \end{cases}$$

We also present a discrete version of Lions lemma (see [34]); it is useful to show that the weak limit of a (PS) sequence is nontrivial.

Lemma 2.2. (Lions lemma, [34, Lemma 2.5]) Let $1 \leq p < +\infty$. Assume that $\{u_n\}$ is bounded in $l^p(\mathbb{Z}^3)$ and

$$\|u_n\|_\infty \to 0 \text{ as } n \to +\infty.$$

Then, for any $p < q < +\infty$,

$$u_n \to 0 \text{ in } l^q(\mathbb{Z}^3).$$

Proof. For $p < q < +\infty$, by an interpolation inequality, we get that

$$\|u_n\|_q^q \leq \|u_n\|_p^p \|u_n\|_\infty^{q-p}.$$

Since $\{u_n\}$ is bounded in $l^p(\mathbb{Z}^3)$ and $\|u_n\|_\infty^{q-p} \to 0$ as $n \to +\infty$, it is easy to obtain the desired result.

3. Generalized Nehari manifold

This section is devoted to describing the variational framework for our problem (1.1). From now on, we suppose that $V(x)$ satisfies (V_1) and f satisfies $(f_1)-(f_4)$.

In what follows, we shall prove some elementary properties for $\mathcal{N}$. To do this, let us start with some elementary observations. By (f_1) and (f_2), for any $\varepsilon > 0$ that is sufficiently small, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^q \quad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

From (f_1) and (f_4), it is easy to verify that

$$F(t) > 0 \quad \text{and} \quad \frac{1}{4}f(t)t > F(t) > 0 \quad \text{for all } t \ne 0. \tag{3.2}$$

We now establish several properties of J on $\mathcal{N}$ that are beneficial to the study of our problem.

Lemma 3.1. *Under the assumptions of (V_1) and $(f_1)-(f_4)$, the following conclusions hold:*

(i) For each $u \in \mathscr{H}\backslash\{0\}$, there exists a unique $s_u > 0$ such that $m(u) := s_u u \in \mathcal{N}$ and $J(m(u)) = \max\limits_{s>0} J(su)$.
(ii) There is $\alpha_0 > 0$ such that $\|u\| \ge \alpha_0$ for each $u \in \mathcal{N}$.
(iii) J is bounded from below on $\mathcal{N}$ by a positive constant.
(iv) J is coercive on $\mathcal{N}$, i.e., $J(u) \to \infty$ as $\|u\| \to \infty, u \in \mathcal{N}$.
(v) Suppose that $\mathcal{V} \subset \mathscr{H}\backslash\{0\}$ is a compact subset; then, there exists $R > 0$ such that $J \le 0$ on $\mathbb{R}^+\mathcal{V}\backslash B_R(0)$.

Proof. (i) For any $u \in \mathscr{H}\backslash\{0\}$ and $s > 0$,

$$\begin{aligned} J(su) &= \frac{s^2}{2}\int_{\mathbb{Z}^3}(a|\nabla u|^2 + V(x)u^2)d\mu + \frac{bs^4}{4}\Big(\int_{\mathbb{Z}^3}|\nabla u|^2 d\mu\Big)^2 - \int_{\mathbb{Z}^3}F(su)d\mu \\ &= \frac{s^2}{2}\|u\|^2 + \frac{bs^4}{4}\Big(\int_{\mathbb{Z}^3}|\nabla u|^2 d\mu\Big)^2 - \int_{\mathbb{Z}^3}F(su)d\mu. \end{aligned}$$

By (3.1) and the Sobolev embedding $W^{1,2}(\mathbb{Z}^3) \hookrightarrow l^p(\mathbb{Z}^3)$, $p \ge 2$, we have

$$\begin{aligned} J(su) &\ge \frac{s^2}{2}\|u\|^2 - \varepsilon s^2\int_{\mathbb{Z}^3}|u|^2 d\mu - C_\varepsilon s^{q+1}\int_{\mathbb{Z}^3}|u|^{q+1}d\mu \\ &\ge \frac{s^2}{2}\|u\|^2 - C_1 s^2\varepsilon\|u\|^2 - C_2 C_\varepsilon s^{q+1}\|u\|^{q+1}. \end{aligned}$$

Fix $\varepsilon > 0$ to be small; since $u \in \mathscr{H}\backslash\{0\}$ and $q > 3$, we easily conclude that $J(su) > 0$ for $s > 0$ small enough.

On the other hand, we have that $|su| \to \infty$ as $s \to \infty$ if $u \ne 0$. Then, by (f_3), we obtain

$$\begin{aligned} J(su) &\le \frac{s^2}{2}\|u\|^2 + \frac{bs^4}{4}\Big(\int_{\mathbb{Z}^3}|\nabla u|^2 d\mu\Big)^2 - s^4\int_{\mathbb{Z}^3}\frac{F(su)}{|su|^4}u^4 d\mu \\ &\to -\infty \quad \text{as} \quad s \to \infty. \end{aligned}$$

Thus, $\max_{s>0} J(su)$ is achieved at some $s_u > 0$ with $s_u u \in \mathcal{N}$.

Next, we show the uniqueness of s_u by a contradiction. Suppose that there exist $s'_u > s_u > 0$ such that $s'_u u, s_u u \in \mathcal{N}$. Then, one has

$$\frac{1}{(s'_u)^2} + b\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 = \int_{\mathbb{Z}^3} \frac{f(s'_u u)}{(s'_u u)^3} u^4 d\mu,$$

$$\frac{1}{(s_u)^2} + b\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 = \int_{\mathbb{Z}^3} \frac{f(s_u u)}{(s_u u)^3} u^4 d\mu.$$

We see that

$$\frac{1}{(s'_u)^2} - \frac{1}{(s_u)^2} = \int_{\mathbb{Z}^3} \left(\frac{f(s'_u u)}{(s'_u u)^3} - \frac{f(s_u u)}{(s_u u)^3}\right) u^4,$$

which is absurd in view of (f_4) and $s'_u > s_u > 0$. We have completed the proof of (i).

(ii) Let $u \in \mathcal{N}$; by (3.1) and the Sobolev embedding, we have

$$\begin{aligned}\langle J'(u), u\rangle = 0 &= \|u\|^2 + b\Big(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\Big)^2 - \int_{\mathbb{Z}^3} f(u) u d\mu \\ &\geq \|u\|^2 - \varepsilon \int_{\mathbb{Z}^3} |u|^2 - C_\varepsilon \int_{\mathbb{Z}^3} |u|^{q+1} \\ &\geq \|u\|^2 - C_1 \varepsilon \|u\|^2 - C_2 C_\varepsilon \|u\|^{q+1}.\end{aligned}$$

Choose $C_1\varepsilon = \frac{1}{2}$; then, there exists a constant $\alpha_0 > 0$ such that $\|u\| \geq \alpha_0 > 0$ for each $u \in \mathcal{N}$.

(iii) For any $u \in \mathcal{N}$, from (ii) and (3.2), we deduce that

$$\begin{aligned}J(u) &= J(u) - \frac{1}{4}\langle J'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \int_{\mathbb{Z}^3} \Big(\frac{1}{4} f(u) u - F(u)\Big) d\mu \\ &\geq \frac{1}{4}\|u\|^2 \geq \frac{1}{4}\alpha_0^2 > 0.\end{aligned}$$

(iv) For any $u \in \mathcal{N}$, it follows from (iii) that

$$J(u) \geq \frac{1}{4}\|u\|^2.$$

This gives that J is coercive on $\mathcal{N}$.

(v) Without loss of generality, we may assume that $\|u\| = 1$ for every $u \in \mathcal{V}$. Suppose, by contradiction, that there exist $u_n \in \mathcal{V}$ and $v_n = t_n u_n$ such that $J(v_n) \geq 0$ for all $n \in \mathbb{N}$ and $t_n \to \infty$ as $n \to \infty$. Passing to a subsequence, there exists $u \in \mathscr{H}$ with $\|u\| = 1$ such that $u_n \to u$ in $\mathscr{H}$. Notice that $|v_n(x)| \to \infty$ if $u(x) \neq 0$. Combining (f_3) and Fatou's lemma, we obtain that

$$\int_{\mathbb{R}^3} \frac{F(v_n)}{v_n^4} u_n^4 \to +\infty,$$

which implies that

$$0 \leq \frac{J(v_n)}{\|v_n\|^4} = \frac{1}{2\|v_n\|^2} + \frac{b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2}{4\|v_n\|^4} - \int_{\mathbb{R}^3} \frac{F(v_n)}{v_n^4} u_n^4 \to -\infty,$$

which is a contradiction.

Now, we define the map

$$\widehat{m} : S \to \mathcal{N},$$
$$w \mapsto \widehat{m}(w) = s_w w,$$

where s_w is as in Lemma 3.1(i). As in [29, Lemma 2.8], we have from Lemma 3.1(i),(ii),(iv),(v) that the map $\widehat{m}$ is continuous; moreover, $\widehat{m}$ is a homeomorphism between S and $\mathcal{N}$, where the inverse of $\widehat{m}$ is given by

$$\widehat{m}^{-1}(u) = \frac{u}{\|u\|}. \tag{3.3}$$

Define the functional

$$\Psi : S \to \mathbb{R}, \quad \Psi(w) := J(\widehat{m}(w)). \tag{3.4}$$

Since we are not assuming that f is differentiable and satisfies the (AR) condition, $\mathcal{N}$ may not be of class C^1 in our case. Nevertheless, we observe that Ψ is of class C^1 and there is a one-to-one correspondence between critical points of Ψ and nontrivial critical points of J. Furthermore, as in [29, Proposition 2.9 and Corollary 2.10], we have the following lemma.

Lemma 3.2. *Under the assumptions of Lemma 3.1, we have the following:*

(i) $\Psi(w) \in C^1(S, \mathbb{R})$ and

$$\Psi'(w)z = \|\widehat{m}(w)\|\langle J'(\widehat{m}(w)), z\rangle \quad \text{for } z \in T_w S = \{v \in \mathscr{H} : \langle v, w\rangle = 0\}.$$

(ii) $\{w_n\}$ is a Palais-Smale sequence for Ψ if and only if $\{\widehat{m}(w_n)\}$ is a Palais-Smale sequence for J.

(iii) We have

$$c = \inf_{\mathcal{N}} J = \inf_{S} \Psi.$$

Moreover, $w \in S$ is a critical point of Ψ if and only if $\widehat{m}(w) \in \mathcal{N}$ is a nontrivial critical point of J and the corresponding critical values coincide.

Now, we set the infimum of J on $\mathcal{N}$ by

$$c = \inf_{\mathcal{N}} J = \inf_{S} \Psi.$$

Remark 3.3. *We point out that the ground-state energy of J has a minimax characterization given by*

$$c = \inf_{\mathcal{N}} J = \inf_{w \in \mathscr{H}\setminus\{0\}} \max_{s>0} J(sw) = \inf_{w \in S\setminus\{0\}} \max_{s>0} J(sw).$$

4. The compact case

In this section, we focus on studying the ground states of (1.1) under the coercive condition (V_2) on $V(x)$. Now, for the minimizing sequence for J on $\mathcal{N}$, we have the following lemma.

Lemma 4.1. *Let $\{w_n\} \subset S$ be a minimizing sequence for Ψ. Then, $\{\widehat{m}(w_n)\}$ is bounded in $\mathscr{H}$. Moreover, there exists $u \in \mathcal{N}$ such that $\widehat{m}(w_n) \rightharpoonup u$ and $J(u) = \inf_{\mathcal{N}} J$.*

Proof. Take a minimizing sequence $\{w_n\} \subset S$ for Ψ. By Ekeland's variational principle in [35], we may assume that $\Psi(w_n) \to c$ and $\Psi'(w_n) \to 0$ as $n \to \infty$. Consequently, without loss of generality, we may assume that $\Psi'(w_n) \to 0$ as $n \to \infty$. Put $u_n = \widehat{m}(w_n) \in \mathcal{N}$ for all $n \in \mathbb{N}$; from Lemma 3.2(ii), we have that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. Moreover, it is easy to show that $\{u_n\}$ is bounded in $\mathscr{H}$ from Lemma 3.1(iv), and that there exists $u \in \mathscr{H}$ such that, up to a subsequence (still denoted by $\{u_n\}$), we have that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } \mathscr{H}, \\ u_n(x) \to u(x), & \forall\ x \in \mathbb{Z}^3, \\ u_n \to u, & \text{in } l^p(\mathbb{Z}^3). \end{cases}$$

We prove that $u \neq 0$. Since $u_n \in \mathcal{N}$, we have that $\langle J'(u_n), u_n \rangle = 0$, that is,

$$\|u_n\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u_n|^2 d\mu\right)^2 = \int_{\mathbb{Z}^3} f(u_n)u_n d\mu. \tag{4.1}$$

By (3.1) and Lemma 3.1(ii), one has

$$\alpha_0^2 \leq \|u_n\|^2 \leq \int_{\mathbb{Z}^3} f(u_n)u_n d\mu \leq \varepsilon \int_{\mathbb{Z}^3} |u_n|^2 d\mu + C_\varepsilon \int_{\mathbb{Z}^3} |u_n|^{q+1} d\mu.$$

By the boundedness of $\{u_n\}$, there is $C_3 > 0$ such that

$$\alpha_0^2 \leq C_3\varepsilon + C_\varepsilon \int_{\mathbb{Z}^3} |u_n|^{q+1} d\mu.$$

Choosing $\varepsilon = \frac{\alpha_0^2}{2C_3}$, we get

$$\int_{\mathbb{Z}^3} |u_n|^{q+1} d\mu \geq \frac{\alpha_0^2}{2C_4},$$

where C_4 is a positive constant. Because of the compact embedding from Lemma 2.1, we obtain

$$\int_{\mathbb{Z}^3} |u|^{q+1} d\mu \geq \frac{\alpha_0^2}{2C_4};$$

thus, $u \neq 0$.

Now, we prove that u is a critical point of J. By (3.1), Lemma 2.1 and a variant of the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_{\mathbb{Z}^3} f(u_n)u_n d\mu = \int_{\mathbb{Z}^3} f(u)u d\mu, \tag{4.2}$$

$$\lim_{n\to\infty} \int_{\mathbb{Z}^3} F(u_n) d\mu = \int_{\mathbb{Z}^3} F(u) d\mu.$$

Moreover, by the weak semi-continuity of norms of $\mathscr{H}$ and $D^{1,2}(\mathbb{Z}^3)$, one has

$$\liminf_{n\to\infty} \left\{\|u_n\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u_n|^2 d\mu\right)^2\right\} \geq \|u\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2.$$

Then, from (4.1) and (4.2), we obtain

$$\|u\|^2 + b\left(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\right)^2 \leq \int_{\mathbb{Z}^3} f(u)u d\mu,$$

which implies that $\langle J'(u), u\rangle \leq 0$. Define $g(\theta) = \langle J'(\theta u), \theta u\rangle$ for $\theta > 0$. Since $g(1) = \langle J'(u), u\rangle < 0$, from (f_1) and (f_2), we also have that $g(\theta) > 0$ for $\theta > 0$ small. Hence, there exists $\theta_0 \in (0, 1)$ such that $g(\theta_0) = 0$, that is, $\langle J'(\theta_0 u), \theta_0 u\rangle = 0$. Moreover, combining (f_3) and (f_4), we can see that $J(\theta_0 u) = \max_{\theta>0} J(\theta u)$. It is easy to obtain from (f_4) that $\frac{1}{4} f(t)t - F(t) > 0$ is strictly increasing in $t > 0$ and identically equal to zero for $t < 0$. Hence, it follows from the above arguments and Fatou's lemma that

$$\begin{aligned}
c &\leq J(\theta_0 u) = J(\theta_0 u) - \frac{1}{4}\langle J'(\theta_0 u), \theta_0 u\rangle \\
&= \frac{\theta_0^2}{4}\int_{\mathbb{Z}^3}\left(a|\nabla u|^2 + V(x)u^2\right) d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4} f(\theta_0 u)\theta_0 u - F(\theta_0 u)\right) d\mu \\
&< \frac{1}{4}\int_{\mathbb{Z}^3}\left(a|\nabla u|^2 + V(x)u^2\right) d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4} f(u)u - F(u)\right) d\mu \\
&\leq \liminf_{n\to\infty}\left[\frac{1}{4}\int_{\mathbb{Z}^3}\left(a|\nabla u_n|^2 + V(x)u_n^2\right) d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4} f(u_n)u_n - F(u_n)\right) d\mu\right] \\
&= \liminf_{n\to\infty}\left[J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle\right] \\
&= c,
\end{aligned}$$

which is a contradiction. Therefore, $\langle J'(u), u\rangle = 0$, which implies that $u \in \mathcal{N}$ and $J(u) \geq c$. Moreover, by Fatou's lemma and $u \not\equiv 0$, it follows that

$$\begin{aligned}
c &\leq J(u) - \frac{1}{4}\langle J'(u), u\rangle \\
&= \frac{1}{4}\int_{\mathbb{Z}^3}\left(a|\nabla u|^2 + V(x)u^2\right) d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4} f(u)u - F(u)\right) d\mu \\
&\leq \liminf_{n\to\infty}\left[\frac{1}{4}\int_{\mathbb{Z}^3}\left(a|\nabla u_n|^2 + V(x)u_n^2\right) d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4} f(u_n)u_n - F(u_n)\right) d\mu\right] \\
&= \liminf_{n\to\infty}\left[J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle\right] \\
&= c.
\end{aligned}$$

Thus, $J(u) = c$ and $\|u_n\| \to \|u\|$ as $n \to \infty$. Since $\mathscr{H}$ is a Hilbert space, we can obtain that $u_n \to u$ in $\mathscr{H}$. The proof is completed.

Now, we shall prove Theorem 1.1.

Proof of Theorem 1.1. Let $c = \inf_{\mathcal{N}} J$ as described above. By Lemma 3.1(iii), we obtain that $c > 0$. Moreover, if $u_0 \in \mathcal{N}$ satisfies that $J(u_0) = c$, then $\widehat{m}^{-1}(u_0) \in S$ is a minimizer of Ψ and thus a critical point of Ψ, where $\widehat{m}^{-1}$ is given in (3.3) and Ψ is given in (3.4). Therefore, combining this with Lemma 3.2(iii), u_0 is a critical point of J. Now, it suffices to show that there exists a minimizer u of $J|_{\mathcal{N}}$. Using Ekeland's variational principle, we find a sequence $\{w_n\} \subset S$ such that $\Psi(w_n) \to c$ and $\Psi'(w_n) \to 0$ as $n \to \infty$. Put $u_n = \widehat{m}(w_n) \in \mathcal{N}$ for all $n \in \mathbb{N}$. Hence, we deduce from Lemma 3.2(ii) that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. Consequently, $\{u_n\}$ is a minimizing sequence for J on $\mathcal{N}$. Therefore, by Lemma 4.1, there exists a minimizer u of $J|_{\mathcal{N}}$, as desired.

5. Noncompact cases

In this section, we generalize our results in Section 4 to noncompact cases. We consider two cases of the potentials, where one is periodic, i.e., the x-dependence is periodic, and the other is that V has a bounded potential well. The discrete version of the Lions lemma will be useful in subsequent proofs.

5.1. The periodic potential case

Throughout this subsection, we consider problem (1.1) with the potential $V(x)$ satisfying the periodic condition.

We now discuss the minimizing sequence for J on $\mathcal{N}$ in a similar but slightly different way than Lemma 4.1.

Lemma 5.1. *Let $\{w_n\} \subset S$ be a minimizing sequence for Ψ. Then, $\{\widehat{m}(w_n)\}$ is bounded in $\mathscr{H}$. Moreover, after a suitable $\mathbb{Z}^3$-translation, up to a subsequence, there exists $u \in \mathcal{N}$ such that $\widehat{m}(w_n) \rightharpoonup u$ and $J(u) = \inf_{\mathcal{N}} J$.*

Proof. Let $\{w_n\} \subset \mathcal{N}$ be a minimizing sequence such that $\Psi(w_n) \to c$. By Ekeland's variational principle, we may assume that $\Psi'(w_n) \to 0$ as $n \to \infty$. Put $u_n = \widehat{m}(w_n) \in \mathcal{N}$ for all $n \in \mathbb{N}$. Then, from Lemma 3.2(ii), we have that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. Consequently, $\{u_n\}$ is a minimizing sequence for J on $\mathcal{N}$. By Lemma 3.1(iv), it is easy to show that $\{u_n\}$ is bounded in $\mathscr{H}$; therefore, $u_n \rightharpoonup u$ for some $u \in \mathscr{H}$, up to a subsequence if necessary. If

$$\|u_n\|_\infty \to 0 \text{ as } n \to \infty, \tag{5.1}$$

from Lemma 2.2, we have that $u_n \to 0$ in $l^{q+1}(\mathbb{Z}^3)$. Moreover, by (3.1), it is easy to obtain that $\int_{\mathbb{Z}^3} f(u_n)u_n d\mu = o_n(1)$ as $n \to \infty$. Hence,

$$\begin{aligned} 0 = \langle J'(u_n), u_n\rangle &= \|u_n\|^2 + b\Big(\int_{\mathbb{Z}^3} |\nabla u_n|^2 d\mu\Big)^2 - \int_{\mathbb{Z}^3} f(x, u_n)u_n d\mu \\ &\geq \|u_n\|^2 + o_n(1), \end{aligned}$$

which implies that $\|u_n\| \to 0$ as $n \to \infty$, which is a contradiction with $\|u_n\| \geq \alpha_0 > 0$ in Lemma 3.1(ii). Therefore, (5.1) does not hold, and there exists $\delta > 0$ such that

$$\liminf_{n\to\infty} \|u_n\|_\infty \geq \delta > 0. \tag{5.2}$$

Hence, there exists a sequence $\{y_n\} \subset \mathbb{Z}^3$ such that

$$|u_n(y_n)| \geq \frac{\delta}{2} \tag{5.3}$$

for $n \in \mathbb{N}$ sufficiently large. For every $y_n \in \mathbb{Z}^3$, let $k_n = (k_n^1, k_n^2, k_n^3) \in \mathbb{Z}^3$ be a vector such that $\{y_n - k_n\tau\} \subset \Omega$, where $\Omega = [0, \tau)^3$ is a finite subset in $\mathbb{Z}^3$. By translations, define $v_n(y) := u_n(y + k_n\tau)$; then, for each v_n,

$$\|v_n\|_{l^\infty(\Omega)} \geq |v_n(y_n - k_n\tau)| = |u_n(y_n)| \geq \frac{\delta}{2} > 0. \tag{5.4}$$

Since $V(x)$ is τ-periodic, J and $\mathcal{N}$ are invariant under the translation; we obtain that $\{v_n\}$ is also a minimizing sequence for J and bounded in $\mathscr{H}$. By passing to a subsequence, $v_n \rightharpoonup v \neq 0$.

Now, we prove that v is a critical point of J. Since $\{v_n\}$ is bounded, then, passing to a subsequence, $v_n \to v$ in $l^p_{loc}(\mathbb{Z}^3)$, $p \geq 2$ and $v_n \to v$ pointwise in $\mathbb{Z}^3$. We may assume that there exists a nonnegative constant A such that $\int_{\mathbb{Z}^3} |\nabla v_n|^2 d\mu \to A^2$ as $n \to \infty$. Notice that

$$\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{Z}^3} |\nabla v_n|^2 d\mu = A^2.$$

Moreover, we show that

$$\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu = A^2.$$

Suppose, by contradiction, that $\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu < A^2$. For any $\varphi \in C_c(\mathbb{Z}^3)$, we have that $J'(v_n)\varphi = o_n(1)$, that is,

$$\int_{\mathbb{Z}^3} (a\nabla v_n \nabla\varphi + V(x)v_n\varphi)d\mu + b\int_{\mathbb{Z}^3} |\nabla v_n|^2 d\mu \int_{\mathbb{Z}^3} \nabla v_n \nabla\varphi d\mu - \int_{\mathbb{Z}^3} f(v_n)\varphi d\mu = o_n(1). \tag{5.5}$$

Passing to a limit as $n \to \infty$, then we have

$$0 = \int_{\mathbb{Z}^3} (a\nabla v\nabla\varphi + V(x)v\varphi)d\mu + bA^2 \int_{\mathbb{Z}^3} \nabla v\nabla\varphi d\mu - \int_{\mathbb{Z}^3} f(v)\varphi d\mu. \tag{5.6}$$

Thus,

$$\begin{aligned} 0 &= \int_{\mathbb{Z}^3} \left(a|\nabla v|^2 + V(x)v^2\right) d\mu + bA^2 \int_{\mathbb{Z}^3} |\nabla v|^2 d\mu - \int_{\mathbb{Z}^3} f(v)v d\mu \\ &> \int_{\mathbb{Z}^3} \left(a|\nabla v|^2 + V(x)v^2\right) d\mu + b\left(\int_{\mathbb{Z}^3} |\nabla v|^2 d\mu\right)^2 - \int_{\mathbb{Z}^3} f(v)v d\mu, \end{aligned}$$

which implies that $\langle J'(v), v\rangle < 0$. (f_1) and (f_2) imply that $\langle J'(\theta v), \theta v\rangle > 0$ for $\theta > 0$ sufficiently small. Therefore, following a similar argument as in the proof of Lemma 4.1, there exists $\theta_0 \in (0, 1)$ such that $\langle J'(\theta_0 v), \theta_0 v\rangle = 0$ and $J(\theta_0 v) = \max_{\theta>0} J(\theta v)$. Consequently, it follows from the above arguments and

Fatou's lemma that

$$\begin{aligned}
c &\leq J(\theta_0 v) = J(\theta_0 v) - \frac{1}{4}\langle J'(\theta_0 v), \theta_0 v\rangle \\
&= \frac{\theta_0^2}{4}\int_{\mathbb{Z}^3}\left(a|\nabla v|^2 + V(x)v^2\right)d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4}f(\theta_0 v)\theta_0 v - F(\theta_0 v)\right)d\mu \\
&< \frac{1}{4}\int_{\mathbb{Z}^3}\left(a|\nabla v|^2 + V(x)v^2\right)d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4}f(v)v - F(v)\right)d\mu \\
&\leq \liminf_{n\to\infty}\left[\frac{1}{4}\int_{\mathbb{Z}^3}\left(a|\nabla v_n|^2 + V(x)v_n^2\right)d\mu + \int_{\mathbb{Z}^3}\left(\frac{1}{4}f(v_n)v_n - F(v_n)\right)d\mu\right] \\
&= \liminf_{n\to\infty}\left[J(v_n) - \frac{1}{4}\langle J'(v_n), v_n\rangle\right] \\
&= c,
\end{aligned}$$

which is a contradiction. Therefore,

$$\int_{\mathbb{Z}^3}|\nabla v_n|^2 d\mu \to \int_{\mathbb{Z}^3}|\nabla v|^2 d\mu = A^2. \tag{5.7}$$

From (5.5) and (5.6), we have that $J'(v) = 0$. Thus, $v \in \mathcal{N}$ and $J(v) \geq c$.

It remains to prove that $J(v) \leq c$. In fact, from Fatou's lemma, the boundedness of $\{v_n\}$ and the weakly lower semi-continuity of $\|\cdot\|$, we obtain that

$$\begin{aligned}
c &= \lim_{n\to\infty}\left\{J(v_n) - \frac{1}{4}\langle J'(v_n), v_n\rangle\right\} \\
&= \liminf_{n\to\infty}\left\{\frac{1}{4}\|v_n\|^2 + \int_{\mathbb{Z}^3}\left(\frac{1}{4}f(v_n)v_n - F(v_n)\right)d\mu\right\} \\
&\geq \frac{1}{4}\|v\|^2 + \int_{\mathbb{Z}^3}\left(\frac{1}{4}f(v)v - F(v)\right) \\
&= J(v) - \frac{1}{4}\langle J'(v), v\rangle \\
&= J(v)
\end{aligned}$$

which implies that $J(v) \leq c$. Thus, we have that $J(v) = c$. This ends the proof.

Finally, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1; here, we summarize it. Let $c = \inf_{\mathcal{N}} J$. By Lemma 3.1, we obtain that $c > 0$. Furthermore, if $u_0 \in \mathcal{N}$ satisfies that $J(u_0) = c$, then $\widehat{m}^{-1}(u_0) \in S$ is a minimizer of Ψ and thus a critical point of Ψ. Then, combining this with Lemma 3.2(iii), we get a critical point u_0 of J. Now, it suffices to show that there exists a minimizer u of $J|_{\mathcal{N}}$. Using Ekeland's variational principle [35], we find a sequence $\{w_n\} \subset S$ such that $\Psi(w_n) \to c$ and $\Psi'(w_n) \to 0$ as $n \to \infty$. Put $u_n = \widehat{m}(w_n) \in \mathcal{N}$ for all $n \in \mathbb{N}$. Hence, we deduce from Lemma 3.2(ii) that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. Consequently, $\{u_n\}$ is a minimizing sequence for J on $\mathcal{N}$. Moreover, by Lemma 5.1, there exists a minimizer u of $J|_{\mathcal{N}}$, as desired.

Remark 5.2. *The conclusion of Theorem 1.2 remains valid if $V(x) \equiv 1$.*

5.2. The potential well case

In this subsection, we show that there exists a ground-state solution to (1.1) for the case that the function $V(x)$ has a bounded potential well.

Proof of Theorem 1.3. We state that $V_\infty = \sup_{x\in\mathbb{Z}^3} V(x) = \lim_{|x|\to\infty} V(x)$. Consider the limit equation

$$-\Big(a + b\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\Big)\Delta u + V_\infty u = f(u),\ x \in \mathbb{Z}^3. \tag{5.8}$$

The energy functional is as follows:

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{Z}^3} (a|\nabla u|^2 + V_\infty u^2)d\mu + \frac{b}{4}\Big(\int_{\mathbb{Z}^3} |\nabla u|^2 d\mu\Big)^2 - \int_{\mathbb{Z}^3} F(u)d\mu.$$

Define

$$c_\infty := \inf_{\mathcal{N}_\infty} J_\infty(u),$$

where

$$\mathcal{N}_\infty := \{u \in \mathscr{H}\backslash\{0\} : \langle J'_\infty(u), u\rangle = 0\}.$$

From Remark 3.3, we know that c_∞ has the following minimax characterization:

$$c_\infty = \inf_{w\in\mathcal{S}\backslash\{0\}} \max_{s>0} J_\infty(sw).$$

It is easy to see that $c_\infty \geq c > 0$. If $V(x) = V_\infty$, this is a special case of periodic potential. Then, c_∞ is achieved for a nontrivial function $u_\infty \in \mathcal{N}$, i.e., $J_\infty(u_\infty) = c_\infty$. Without loss of generality, we shall assume that V is strictly less than V_∞ at some point. Then, $\langle J'(u_\infty), u_\infty\rangle < 0$, and there is $s > 0$ such that $su_\infty \in \mathcal{N}$. Therefore, we have

$$c \leq J(su_\infty) < J_\infty(su_\infty) \leq J_\infty(u_\infty) = c_\infty.$$

Let $\{w_n\} \subset \mathcal{S}$ be a minimizing sequence for Ψ, where Ψ is given in (3.4). Again, by Ekeland's variational principle, we may assume that $\Psi'(w_n) \to 0$ as $n \to \infty$. Let $u_n = \widehat{m}(w_n) \in \mathcal{N}$ for all $n \in \mathbb{N}$; then, from Lemma 3.2(ii), we have that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. By Lemma 3.1(iv), $\{u_n\}$ is bounded. Similar to the arguments used in the proof of Lemma 5.1, we obtain a new subsequence $\{u_n\}$ and a corresponding new sequence of points $\{y_n\} \subset \mathbb{Z}^3$ such that $|u_n(y_n)| \geq \delta > 0$ for all $n \in \mathbb{N}$. Therefore, $\widetilde{u}_n \rightharpoonup \widetilde{u} \neq 0$ for the translated functions $\widetilde{u}_n := u_n(\cdot - y_n)$.

It suffices to show that $\{y_n\}$ is bounded. Suppose that $|y_n| \to \infty$ for a subsequence; we claim that $\widetilde{u}$ is a critical point of J_∞. Indeed, for any $\varphi \in C_c(\mathbb{Z}^3)$, let $\varphi = \varphi_n(\cdot - y_n)$; observe that

$$|J'(u_n)\varphi_n| \leq \|J'(u_n)\|\|\varphi_n\| = \|J'(u_n)\|\|\varphi\| \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\begin{aligned} J'(u_n)\varphi_n &= \int_{\mathbb{Z}^3} (a\nabla u_n\nabla\varphi_n + V(x)u_n\varphi_n)\, d\mu + b\int_{\mathbb{Z}^3} |\nabla u_n|^2 d\mu \int_{\mathbb{Z}^3} \nabla u_n\nabla\varphi_n d\mu - \int_{\mathbb{Z}^3} f(u_n)\varphi_n d\mu \\ &= \int_{\mathbb{Z}^3} (a\nabla\widetilde{u}_n\nabla\varphi + V(x-y_n)\widetilde{u}_n\varphi)\, d\mu + b\int_{\mathbb{Z}^3} |\nabla\widetilde{u}_n|^2 d\mu \int_{\mathbb{Z}^3} \nabla\widetilde{u}_n\nabla\varphi d\mu - \int_{\mathbb{Z}^3} f(\widetilde{u}_n)\varphi d\mu \end{aligned}$$

$$
\begin{aligned}
&\to \int_{\mathbb{Z}^3} (a\nabla\widetilde{u}\nabla\varphi + V_\infty\widetilde{u}\varphi)\, d\mu + b\int_{\mathbb{Z}^3} |\nabla\widetilde{u}|^2 d\mu \int_{\mathbb{Z}^3} \nabla\widetilde{u}\nabla\varphi d\mu - \int_{\mathbb{Z}^3} f(\widetilde{u})\varphi d\mu \\
&= J'_\infty(\widetilde{u})\varphi.
\end{aligned}
$$

Consequently, it follows again from Fatou's lemma that

$$
\begin{aligned}
c + o(1) &= J(u_n) - \frac{1}{4}J'(u_n)\, u_n \\
&= \int_{\mathbb{Z}^3} \left(\frac{1}{4} f(u_n)\, u_n - F(u_n)\right) d\mu \\
&= \int_{\mathbb{Z}^3} \left(\frac{1}{4} f(\widetilde{u}_n)\,\widetilde{u}_n - F(\widetilde{u}_n)\right) d\mu \\
&\geq \int_{\mathbb{Z}^3} \left(\frac{1}{4} f(\widetilde{u})\widetilde{u} - F(\widetilde{u})\right) d\mu + o_n(1) \\
&= J_\infty(\widetilde{u}) - \frac{1}{4}J'_\infty(\widetilde{u})\widetilde{u} + o_n(1) \\
&= J_\infty(\widetilde{u}) + o_n(1) \\
&\geq c_\infty + o_n(1), \quad n \to \infty,
\end{aligned}
$$

which contradicts $c < c_\infty$. Thus, $\{y_n\}$ is bounded. Without loss of generality, we may assume that $y_n = 0 \in \mathbb{Z}^3$; therefore, $\widetilde{u}_n = u_n$ for all $n \in \mathbb{N}$. Then, using the same arguments as in Lemma 4.1 and the proof of Theorem 1.1, we can show that $\widetilde{u}$ is a ground-state solution of problem (1.1).

Use of AI tools declaration

The author declares that no artificial intelligence tools were used in the creation of this article.

Conflict of interest

The author declares that there is no conflict of interest.

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