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Research article

Algebraic Schouten solitons of Lorentzian Lie groups with Yano connections

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Abstract: In this paper, we discuss the beingness conditions for algebraic Schouten solitons associated with Yano connections in the background of three-dimensional Lorentzian Lie groups. By transforming equations of algebraic Schouten solitons into algebraic equations, the existence conditions of solitons are found. In particular, we deduce some formulations for Yano connections and related Ricci operators. Furthermore, we find the detailed categorization for those algebraic Schouten solitons on three-dimensional Lorentzian Lie groups. The major results demonstrate that algebraic Schouten solitons related to Yano connections are present in G_1, G_2, G_3, G_5, G_6 and G_7, while they are not identifiable in G_4.

Keywords: algebraic Schouten solitons; Yano connections; Lorentzian Lie groups
Mathematics Subject Classification: 53C40, 53C42

1. Introduction

In 1982, Hamilton first introduced the Ricci soliton (RS) concept with [1], and Hamilton pointed out the RS serves as a self-similar outcome to the Ricci flow as long as it walks through a one single parameter family with modulo diffeomorphic mappings and grows on a space about Riemannian metrics in [2]. Since then, geometers and physicists turned their attention to discuss RS. For example, in [3], Rovenski found conditions for the existence of an Einstein manifold according to a similar Ricci tensor or generalized RS form in an extremely weak κ-contact manifold. In [4], Arfah presented a condition for RS on semi-Riemannian group manifold and illustrated the applications of group manifold that admit RS. There are some typical works on affine RS [5], algebraic RS [6], as well as generalized RS [7]. Nonetheless, a task about seeking out RS on manifolds is considerably challenging and often necessitates the imposition of limitations. These restrictions can typically be observed in several areas, such as the framework and dimensions of the manifold, the classification of metrics or the classification about vector fields used in the RS equation. An example of this is the utilization of homogeneous spaces, particularly Lie groups (LG) [8]. Following this, several mathematicians delved into the study of algebraic RS on LG, the area that had previously been explored by Lauret. During his research, he investigated the correlation between solvsolitons solitons and RS regarding Riemannian manifolds, ultimately proving that every Riemannian solvsoliton metric constitutes the RS in [9]. With these findings as a foundation, the author was able to derive both steady algebraic RS and diminishing algebraic RS in terms of Lorentzian geometries. It should be noted that Batat together with Onda subsequently investigated RS for three-dimensional Lorentzian Lie groups (LLG) in [10], examining all such Lie groups that qualify as algebraic RS. Furthermore, there also have been certain studies on the LG about Gauss Bonnet theorems in [11, 12].

Motivated by the above research, mathematicians undertook an investigation of algebraic RS that are associated with different affine connections. For example, in [13], Wang presented a novel product structure for three-dimensional LLG, along with a computation for canonical and Kobayashi-Nomizu connections as well as curvature tensor. He went ahead to define algebraic RSs that are related to the above statements. Furthermore, he categorized the algebraic RSs that are related to canonical as well as Kobayashi-Nomizu connections with this specific product structure. Wang also considered the distribution $H = span\{\tilde{q}_1^Y, \tilde{q}_2^Y\}$ and its orthogonal complement $H^{\perp} = span\{\tilde{q}_3^Y\}$, which are relevant to the three-dimensional LLG having a structure $J : J\tilde{q}_1^Y = \tilde{q}_1^Y$, $J\tilde{q}_2^Y = \tilde{q}_2^Y$ and $J\tilde{q}_3^Y = -\tilde{q}_3^Y$. Moreover, other impressive results of RS are found in [14–16]. In [17], Calvaruso performed an in-depth analysis of three-dimensional generalized Rs with regards to Riemannian and Lorentzian frameworks. In order to study the properties associated with such solitons, they introduced a generalized RS in Eq. (1) [18] that can be regarded as the Schouten soliton, based on the Schouten tensor's definition mentioned in [19]. Drawing upon the works of [20], they also defined algebraic Schouten solitons(ASS). Moreover, the study in [21] introduced the concept of Yano connections (YC). Despite the substantial research on ASS, there is limited knowledge about their association with YC on LLG. Inspired by [22], and many studies provide extra incentives for solitons, see [23–26]. In this paper, we attempt to examine ASS associated with YC in the context of three-dimensional LG. The key to solving this problem is to find the existence conditions of ASS associated with YC. Based on this, by transforming equations of ASS into algebraic equations, the existence conditions of solitons are found. In particular, we calculate the curvature of YC and derive expressions for ASS to finish their categorization for three-dimensional LLG. Its main results demonstrate that ASS related to YC are present in G_1, G_2, G_3, G_5, G_6 and G_7, while they are not identifiable in G_4.

The paper is structured as follows. In Sec 2, fundamental concepts for three-dimensional LLG, specifically relating to YC as well ASS, will be introduced. Additionally, we present a succinct depiction of each three-dimensional connected LG, which is both unimodular and non-unimodular. In Sec 3, we obtain all formulas for YC as well their corresponding curvatures tensor in seven LLG. Using this Ricci operator and defining ASS associated to YC, we are able to fully classify three-dimensional LLG that admit the first kind ASS related to YC. In Sec 4, we use this soliton equation in an effort to finish a categorization about three-dimensional LLG that support ASS of the second kind related to YC. In Sec 5, we highlight certain important findings and talk about potential directions regarding research.

2. Preliminaries

In this section, fundamental concepts for three-dimensional LLG, specifically relating to YC as well ASS, will be introduced. Additionally, we present a succinct depiction of each three-dimensional connected LG, which is both unimodular and non-unimodular (for details see [27, 28]).

We designate $\{G_i\}_{i=1,\cdots,7}$ as the collection for three-dimensional LLG, which is connected and simply connected, endowed with left-invariant Lorentzian metric g^Y. Furthermore, the respective Lie algebra(LA) for each group is denoted as $\{g_i^Y\}_{i=1,\cdots,7}$. The LCC will get represented by ∇^L. This is the definition of the YC:

$$\nabla^Y_{U^Y}V^Y = \nabla^L_{U^Y}V^Y - \frac{1}{2}(\nabla^L_{V^Y}J)JU^Y - \frac{1}{4}[(\nabla^L_{U^Y}J)JV^Y - (\nabla^L_{JU^Y}J)V^Y], \tag{2.1}$$

furthermore, $\{G_i\}_{i=1,\cdots,7}$ having a structure $J : J\tilde{q}_1^Y = \tilde{q}_1^Y$, $J\tilde{q}_2^Y = \tilde{q}_2^Y$, $J\tilde{q}_3^Y = -\tilde{q}_3^Y$, followed $J^2 = id$, then $g^Y(J\tilde{q}_j^Y, J\tilde{q}_j^Y) = g^Y(\tilde{q}_j^Y, \tilde{q}_j^Y)$. This is the definitions of the curvature:

$$R^Y(U^Y, V^Y)W^Y = \nabla^Y_{U^Y}\nabla^Y_{V^Y}W^Y - \nabla^Y_{V^Y}\nabla^Y_{U^Y}W^Y - \nabla^Y_{[U^Y,V^Y]}W^Y. \tag{2.2}$$

This definition of the Ricci tensor for (G_i, g^Y), which is related to the YC, can be given as

$$\rho^Y(U^Y, V^Y) = -g^Y(R^Y(U^Y, \tilde{q}_1^Y)V^Y, \tilde{q}_1^Y) - g^Y(R^Y(U^Y, \tilde{q}_2^Y)V^Y, \tilde{q}_2^Y) + g^Y(R^Y(U^Y, \tilde{q}_3^Y)V^Y, \tilde{q}_3^Y), \tag{2.3}$$

the basis $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ is pseudo-orthonormal, $\tilde{q}_3^Y$ is timelike vector fields. This definition of the Ricci operator Ric^Y can be given as

$$\rho^Y(U^Y, V^Y) = g^Y(Ric^Y(U^Y), V^Y). \tag{2.4}$$

One can define the Schouten tensor with the expression given by

$$S^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) - \frac{s^Y}{4}g^Y(\tilde{q}_i^Y, \tilde{q}_j^Y), \tag{2.5}$$

where s^Y represents the scalar curvature. By extending the Schouten tensor's definition, we obtain

$$S^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) - s^Y\lambda_0 g^Y(\tilde{q}_i^Y, \tilde{q}_j^Y), \tag{2.6}$$

where λ_0 is a real-valued constant. By referring to [29], we can obtain

$$s^Y = \rho^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) + \rho^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) - \rho^Y(\tilde{q}_3^Y, \tilde{q}_3^Y), \tag{2.7}$$

for vector fields U^Y, V^Y, W^Y.

Theorem 2.1. *[27, 28] Let (G, g^Y) be three-dimensional LG of connected unimodular that has a left-invariant Lorentzian metric. Thus the LA for G is one of the following if there exists a pseudo-orthonormal basis $\{\tilde{q}_1^Y, \tilde{q}_2^Y, \tilde{q}_3^Y\}$ with $\tilde{q}_3^Y$ timelike:*
(g_1^Y) :

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = \alpha\tilde{q}_1^Y - \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\alpha\tilde{q}_1^Y - \beta\tilde{q}_2^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \beta\tilde{q}_1^Y + \alpha\tilde{q}_2^Y + \alpha\tilde{q}_3^Y, \alpha \neq 0.$$

(g_2^Y) :

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = \gamma\tilde{q}_2^Y - \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\beta\tilde{q}_2^Y - \gamma\tilde{q}_3^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y, \gamma \neq 0.$$

(g_3^Y) :

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = -\gamma\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\beta\tilde{q}_2^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y.$$

(g_4^Y) :

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = -\tilde{q}_2^Y + (2\eta - \beta)\tilde{q}_2^Y, \eta = \pm 1, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\beta\tilde{q}_2^Y + \tilde{q}_3^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y.$$

Theorem 2.2. *[27, 28] Let (G, g^Y) be three-dimensional LG of connected non-unimodular that has a left-invariant Lorentzian metric. Thus the LA for G is one of the following if there exists a pseudo-orthonormal basis $\{\tilde{q}_1^Y, \tilde{q}_2^Y, \tilde{q}_3^Y\}$ with $\tilde{q}_3^Y$ timelike:*

(g_5^Y) :

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = 0, [\tilde{q}_1^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y + \beta\tilde{q}_2^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \gamma\tilde{q}_1^Y + \delta\tilde{q}_2^Y, \alpha + \delta \neq 0, \alpha\gamma + \beta\delta = 0.$$

(g_6^Y) :

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = \alpha\tilde{q}_2^Y + \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = \gamma\tilde{q}_2^Y + \delta\tilde{q}_3^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0.$$

(g_7^Y) :

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = -\alpha\tilde{q}_1^Y - \beta\tilde{q}_2^Y - \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y + \beta\tilde{q}_2^Y + \beta\tilde{q}_3^Y,$$
$$[\tilde{q}_2^Y, \tilde{q}_3^Y] = \gamma\tilde{q}_1^Y + \delta\tilde{q}_2^Y + \delta\tilde{q}_3^Y, \alpha + \delta \neq 0, \alpha\gamma = 0.$$

Definition 2.3. *(G_i, g^Y) is called ASS of the first kind related with YC when it satisfies*

$$Ric^Y = (s^Y\lambda_0 + c)Id + D, \tag{2.8}$$

which c is an actual number, λ_0 is a real-valued constant, as well D is derivation for g^Y, which can be

$$D[U^Y, V^Y] = [DU^Y, V^Y] + [U^Y, DV^Y], \tag{2.9}$$

for $U^Y, V^Y \in g^Y$.

3. The first kind ASS related with YC on three-dimensional LLG

In this section, we aim to obtain the formulas for YC as well their corresponding curvatures in seven LLGs. Using the Ricci operator and defining LLG associated to YC, we are able to fully classify three-dimensional LLG that admit ASS as the first kind associated with YC.

3.1. ASS of G_1

In the subsection, we present the LA for G_1 that satisfies the following condition

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = \alpha\tilde{q}_1^Y - \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\alpha\tilde{q}_1^Y - \beta\tilde{q}_2^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \beta\tilde{q}_1^Y + \alpha\tilde{q}_2^Y + \alpha\tilde{q}_3^Y, \alpha \neq 0,$$

the basis vectors $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ form a pseudo-orthonormal basis where $\tilde{q}_3^Y$ is timelike. Four lemmas regarding the formulations of YC as well their corresponding curvatures in G_1 with Lorentzian metric can be derived.

Lemma 3.1 ([10, 30]). *The LCC for G_1 can be given as*

$$\nabla^L_{\tilde{q}_1^Y}\tilde{q}_1^Y = -\alpha\tilde{q}_2^Y - \alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_2^Y = \alpha\tilde{q}_1^Y - \frac{\beta}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_3^Y = -\alpha\tilde{q}_1^Y - \frac{\beta}{2}\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}\tilde{q}_1^Y = \frac{\beta}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_2^Y = \alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_3^Y = \frac{\beta}{2}\tilde{q}_1^Y + \alpha\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}\tilde{q}_1^Y = \frac{\beta}{2}\tilde{q}_2^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\frac{\beta}{2}\tilde{q}_1^Y - \alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_3^Y = -\alpha\tilde{q}_2^Y.$$

Lemma 3.2. *For G_1, the following equalities hold*

$$\nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_1^Y = -2\alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_2^Y = -\beta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_3^Y = 2\alpha\tilde{q}_1^Y + \beta\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_1^Y = \beta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_2^Y = 2\alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_3^Y = -\beta\tilde{q}_1^Y - 2\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_2^Y = -2\alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_3^Y = 2\alpha\tilde{q}_2^Y.$$

Based on (2.1), as well as Lemmas 3.1 and 3.2, one can derive the subsequent lemma.

Lemma 3.3. *The YC for G_1 can be given as*

$$\nabla^Y_{\tilde{q}_1^Y}\tilde{q}_1^Y = -\alpha\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_2^Y = \alpha\tilde{q}_1^Y - \beta\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_3^Y = 0,$$
$$\nabla^Y_{\tilde{q}_2^Y}\tilde{q}_1^Y = \beta\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_2^Y = 0,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_3^Y = \alpha\tilde{q}_3^Y,$$
$$\nabla^Y_{\tilde{q}_3^Y}\tilde{q}_1^Y = \alpha\tilde{q}_1^Y + \beta\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\beta\tilde{q}_1^Y - \alpha\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0.$$

Based on (2.2), as well as Lemma 3.3, one can derive the subsequent lemma.

Lemma 3.4. *The curvature R^Y for (G_1, g^Y) can be given as*

$$R^Y\left(\tilde{q}_1^Y,\tilde{q}_2^Y\right)\tilde{q}_1^Y = \alpha\beta\tilde{q}_1^Y + (\alpha^2+\beta^2)\tilde{q}_2^Y,\ R^Y\left(\tilde{q}_1^Y,\tilde{q}_2^Y\right)\tilde{q}_2^Y = -(\alpha^2+\beta^2)\tilde{q}_1^Y - \alpha\beta\tilde{q}_2^Y + \alpha\beta\tilde{q}_3^Y,$$
$$R^Y\left(\tilde{q}_1^Y,\tilde{q}_2^Y\right)\tilde{q}_3^Y = 0,\ R^Y\left(\tilde{q}_1^Y,\tilde{q}_3^Y\right)\tilde{q}_1^Y = -3\alpha^2\tilde{q}_2^Y,\ R^Y\left(\tilde{q}_1^Y,\tilde{q}_3^Y\right)\tilde{q}_2^Y = -\alpha^2\tilde{q}_1^Y,$$
$$R^Y\left(\tilde{q}_1^Y,\tilde{q}_3^Y\right)\tilde{q}_3^Y = \alpha\beta\tilde{q}_3^Y,\ R^Y\left(\tilde{q}_2^Y,\tilde{q}_3^Y\right)\tilde{q}_1^Y = -\alpha^2\tilde{q}_1^Y,\ R^Y\left(\tilde{q}_2^Y,\tilde{q}_3^Y\right)\tilde{q}_2^Y = \alpha^2\tilde{q}_2^Y,$$
$$R^Y\left(\tilde{q}_2^Y,\tilde{q}_3^Y\right)\tilde{q}_3^Y = -\alpha^2\tilde{q}_3^Y.$$

Using Lemmas 3.3 and 3.4, the following theorem regarding the ASS of the first kind in the first LG with Lorentzian metric can be established.

Theorem 3.5. *(G_1, g^Y, J) is ASS of the first kind related to the YC if it satisfies $\beta = c = 0$, $\alpha \neq 0$. And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}-\alpha^2 & 0 & 0\\ 0 & -\alpha^2 & -\alpha^2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}-\alpha^2+2\alpha^2\lambda_0 & 0 & 0\\ 0 & -\alpha^2+2\alpha^2\lambda_0 & -\alpha^2\\ 0 & 0 & 2\alpha^2\lambda_0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

Proof. According to (2.3), we have

$$\rho^Y(\tilde{q}_1^Y,\tilde{q}_1^Y) = -\alpha^2-\beta^2,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_2^Y) = \alpha\beta,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_3^Y) = -\alpha\beta,$$
$$\rho^Y(\tilde{q}_2^Y,\tilde{q}_1^Y) = \alpha\beta,\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_2^Y) = -(\alpha^2+\beta^2),\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_3^Y) = \alpha^2,$$
$$\rho^Y(\tilde{q}_3^Y,\tilde{q}_1^Y) = 0,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_2^Y) = 0,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_3^Y) = 0.$$

By (2.4), the Ricci operator can be expressed as

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}-\alpha^2-\beta^2 & \alpha\beta & \alpha\beta\\ \alpha\beta & -\alpha^2-\beta^2 & -\alpha^2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y = -2\alpha^2 - 2\beta^2$. If (G_1, g^Y, J) is ASS of the first kind related to the YC, and by $Ric^Y = (s^Y\lambda_0 + c)Id + D$, we can get

$$\begin{cases} D\tilde{q}_1^Y = [-\alpha^2 - \beta^2 + (2\alpha^2 + 2\beta^2)\lambda_0 - c]\tilde{q}_1^Y + \alpha\beta\tilde{q}_2^Y + \alpha\beta\tilde{q}_3^Y, \\ D\tilde{q}_2^Y = \alpha\beta\tilde{q}_1^Y + [-\alpha^2 - \beta^2 + (2\alpha^2 + 2\beta^2)\lambda_0 - c]\tilde{q}_2^Y - \alpha^2\tilde{q}_3^Y, \\ D\tilde{q}_3^Y = [(2\alpha^2 + 2\beta^2)\lambda_0 - c]\tilde{q}_3^Y. \end{cases}$$

Therefore, by (2.9) and ASS of the first kind related to the YC can be established if it satisfies

$$\begin{cases} 2\alpha^3\lambda_0 - 2\alpha\beta^2 + 2\alpha\beta^2\lambda_0 - \alpha c = 0, \\ \alpha^2\beta = 0, \\ \beta^3 - \alpha^2\beta = 0, \\ 2\beta^3\lambda_0 - 2\alpha^2\beta + 2\alpha^2\beta\lambda_0 - \beta c = 0. \end{cases} \tag{3.1}$$

Considering that $\alpha \neq 0$, by solving the first and second equations in (3.1) leads to the conclusion that $\beta = 0$ and $c = 0$. Thus we get Theorem 3.5.

3.2. ASS of G_2

In the subsection, we present the LA for G_2 that satisfies the following condition

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = \gamma\tilde{q}_2^Y - \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\beta\tilde{q}_2^Y - \gamma\tilde{q}_3^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y, \gamma \neq 0,$$

the basis vectors $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ form a pseudo-orthonormal basis where $\tilde{q}_3^Y$ is timelike. Four lemmas regarding the formulations of YC as well their corresponding curvatures in G_2 with Lorentzian metric can be derived.

Lemma 3.6 ([10, 30]). *The LCC for G_2 can be given as*

$$\begin{aligned} &\nabla^L_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_2^Y = (\frac{\alpha}{2} - \beta)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_3^Y = (\frac{\alpha}{2} - \beta)\tilde{q}_2^Y, \\ &\nabla^L_{\tilde{q}_2^Y}\tilde{q}_1^Y = -\gamma\tilde{q}_2^Y + \frac{\alpha}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_2^Y = \gamma\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_3^Y = \frac{\alpha}{2}\tilde{q}_1^Y, \\ &\nabla^L_{\tilde{q}_3^Y}\tilde{q}_1^Y = \frac{\alpha}{2}\tilde{q}_2^Y + \gamma\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\frac{\alpha}{2}\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_3^Y = \gamma\tilde{q}_1^Y. \end{aligned}$$

Lemma 3.7. *For G_2, the following equalities hold*

$$\begin{aligned} &\nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_2^Y = (\alpha - 2\beta)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_3^Y = -(\alpha - 2\beta)\tilde{q}_2^Y, \\ &\nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_1^Y = \alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_3^Y = -\alpha\tilde{q}_1^Y, \\ &\nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_1^Y = 2\gamma\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_3^Y = -2\gamma\tilde{q}_1^Y. \end{aligned}$$

Based on (2.1), as well as Lemmas 3.6 and 3.7, one can derive the subsequent lemma.

Lemma 3.8. *The YC for G_2 can be given as*

$$\begin{aligned} &\nabla^Y_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_2^Y = -\beta\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_3^Y = -2\beta\tilde{q}_2^Y - \gamma\tilde{q}_3^Y, \\ &\nabla^Y_{\tilde{q}_2^Y}\tilde{q}_1^Y = -\gamma\tilde{q}_2^Y + \beta\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_2^Y = \gamma\tilde{q}_1^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_3^Y = 0, \\ &\nabla^Y_{\tilde{q}_3^Y}\tilde{q}_1^Y = \beta\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\alpha\tilde{q}_1^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0. \end{aligned}$$

Based on (2.2), as well as Lemma 3.8, one can derive the subsequent lemma.

Lemma 3.9. *The curvature R^Y for (G_2, g^Y) can be given as*

$$
\begin{aligned}
&R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_1^Y = (\gamma^2 - \beta^2)\tilde{q}_2^Y - \beta\gamma\tilde{q}_3^Y,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_2^Y = -(\gamma^2 + \alpha\beta)\tilde{q}_1^Y,\\
&R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_3^Y = 2\beta\gamma\tilde{q}_1^Y,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = 0,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = (\beta\gamma - \alpha\gamma)\tilde{q}_1^Y,\\
&R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = -2\alpha\beta\tilde{q}_1^Y,\ R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = (\beta\gamma - \alpha\gamma)\tilde{q}_1^Y,\\
&R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = -\beta\gamma\tilde{q}_2^Y + \alpha\beta\tilde{q}_3^Y,\ R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = 2\alpha\beta\tilde{q}_2^Y + \alpha\gamma\tilde{q}_3^Y.
\end{aligned}
$$

Using Lemmas 3.8 and 3.9, the following theorem regarding the ASS of the first kind in the second LG with Lorentzian metric can be established.

Theorem 3.10. *(G_2, g^Y, J) is ASS of the first kind related to YC if it satisfies $\alpha = \beta = 0$, $\gamma \neq 0$, $c = \gamma^2(2\lambda_0 - 1)$. And specifically*

$$
Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}-\gamma^2 & 0 & 0\\ 0 & -\gamma^2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},
$$

$$
D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \gamma^2\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.
$$

Proof. According to (2.3), we have

$$
\begin{aligned}
&\rho^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) = \beta^2 - \gamma^2,\ \rho^Y(\tilde{q}_1^Y, \tilde{q}_2^Y) = 0,\ \rho^Y(\tilde{q}_1^Y, \tilde{q}_3^Y) = 0,\\
&\rho^Y(\tilde{q}_2^Y, \tilde{q}_1^Y) = 0,\ \rho^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) = -\gamma^2 - 2\alpha\beta,\ \rho^Y(\tilde{q}_2^Y, \tilde{q}_3^Y) = 2\beta\gamma - \alpha\gamma,\\
&\rho^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0,\ \rho^Y(\tilde{q}_3^Y, \tilde{q}_2^Y) = -\alpha\gamma,\ \rho^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0.
\end{aligned}
$$

By (2.4), the Ricci operator can be expressed as

$$
Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}\beta^2 - \gamma^2 & 0 & 0\\ 0 & -\gamma^2 - 2\alpha\beta & \alpha\gamma - 2\beta\gamma\\ 0 & -\alpha\gamma & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.
$$

As a result, the scalar curvature can be obtained as $s^Y = \beta^2 - 2\gamma^2 - 2\alpha\beta$. If (G_2, g^Y, J) is ASS of the first kind related to the YC, and by $Ric^Y = (s^Y\lambda_0 + c)Id + D$, we can get

$$
\begin{cases}
D\tilde{q}_1^Y = (\beta^2 - \gamma^2 - \beta^2\lambda_0 + 2\gamma^2\lambda_0 + 2\alpha\beta\lambda_0 - c)\tilde{q}_1^Y,\\
D\tilde{q}_2^Y = (-\gamma^2 - 2\alpha\beta - \beta^2\lambda_0 + 2\gamma^2\lambda_0 + 2\alpha\beta\lambda_0 - c)\tilde{q}_2^Y + (\alpha\gamma - 2\beta\gamma)\tilde{q}_3^Y,\\
D\tilde{q}_3^Y = -\alpha\gamma\tilde{q}_2^Y + (-\beta^2\lambda_0 + 2\gamma^2\lambda_0 + 2\alpha\beta\lambda_0 - c)\tilde{q}_3^Y.
\end{cases}
$$

Therefore, by (2.9) and ASS of the first kind related to the YC can be established if it satisfies

$$
\begin{cases}
\beta^3 - \beta^3\lambda_0 + 2\alpha\gamma^2 - 6\beta\gamma^2 - 2\alpha\beta^2 + 2\beta\gamma^2\lambda_0 + 2\alpha\beta^2\lambda_0 - \beta c = 0,\\
\beta^3 - \beta^3\lambda_0 + 2\alpha\gamma^2 + 2\alpha\beta^2 + 2\beta\gamma^2\lambda_0 + 2\alpha\beta^2\lambda_0 - \beta c = 0,\\
\gamma^3 - 2\gamma^3\lambda_0 - 3\beta^2\gamma + \beta^2\gamma\lambda_0 + 2\alpha\beta\gamma - 2\alpha\beta\gamma\lambda_0 + \gamma c = 0,\\
\alpha\beta^2 - 2\alpha^2\beta - \alpha\beta^2\lambda_0 + 2\alpha\gamma^2\lambda_0 + 2\alpha^2\beta\lambda_0 - \alpha c = 0.
\end{cases}
\tag{3.2}
$$

By solving the first and second equations of (3.2) imply that

$$2\alpha\beta^2 + 3\beta\gamma^2 = 0.$$

As $\gamma \neq 0$, it follows that β must be zero. On this basis, the second equation of (3.2) reduces to

$$2\alpha\gamma^2 = 0,$$

we have $\alpha = 0$. In this case, the third equation of (3.2) can be simplified to

$$\gamma^3 - 2\gamma^3\lambda_0 + \gamma c = 0,$$

then we obtain $c = \gamma^2(2\lambda_0 - 1)$. Thus we get Theorem 3.10.

3.3. ASS of G_3

In the subsection, we present the LA for G_3 that satisfies the following condition

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = -\gamma\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\beta\tilde{q}_2^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y,$$

the basis vectors $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ form a pseudo-orthonormal basis where $\tilde{q}_3^Y$ is timelike. Four lemmas regarding the formulations of YC as well their corresponding curvatures in G_3 with Lorentzian metric can be derived.

Lemma 3.11 ([10, 30]). *The LCC for G_3 can be given as*

$$\begin{aligned}
&\nabla^L_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_2^Y = \frac{\alpha-\beta-\gamma}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_3^Y = \frac{\alpha-\beta-\gamma}{2}\tilde{q}_2^Y,\\
&\nabla^L_{\tilde{q}_2^Y}\tilde{q}_1^Y = \frac{\alpha-\beta+\gamma}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_3^Y = \frac{\alpha-\beta+\gamma}{2}\tilde{q}_1^Y,\\
&\nabla^L_{\tilde{q}_3^Y}\tilde{q}_1^Y = \frac{\alpha+\beta-\gamma}{2}\tilde{q}_2^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\frac{\alpha+\beta-\gamma}{2}\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0.
\end{aligned}$$

Lemma 3.12. *For G_3, the following equalities hold*

$$\begin{aligned}
&\nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_2^Y = (\alpha-\beta-\gamma)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_3^Y = -(\alpha-\beta-\gamma)\tilde{q}_2^Y,\\
&\nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_1^Y = (\alpha-\beta+\gamma)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_3^Y = -(\alpha-\beta+\gamma)\tilde{q}_1^Y,\\
&\nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_3^Y = 0.
\end{aligned}$$

Based on (2.1), as well as Lemmas 3.11 and 3.12, one can derive the subsequent lemma.

Lemma 3.13. *The YC for G_3 can be given as*

$$\begin{aligned}
&\nabla^Y_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_2^Y = -\gamma\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_3^Y = 0,\\
&\nabla^Y_{\tilde{q}_2^Y}\tilde{q}_1^Y = \gamma\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_2^Y = 0,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_3^Y = -\gamma\tilde{q}_1^Y,\\
&\nabla^Y_{\tilde{q}_3^Y}\tilde{q}_1^Y = \beta\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\alpha\tilde{q}_1^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0.
\end{aligned}$$

Based on (2.2), as well as Lemma 3.13, one can derive the subsequent lemma.

Lemma 3.14. *The curvature R^Y for (G_3, g^Y) can be given as*

$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_1^Y = \beta\gamma\tilde{q}_2^Y,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_2^Y = -(\gamma^2 + \alpha\gamma)\tilde{q}_1^Y,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_3^Y = 0,$$
$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = 0,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = 0,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = -\beta\gamma\tilde{q}_1^Y,$$
$$R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = 0,\ R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = 0,\ R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = \beta\gamma\tilde{q}_2^Y.$$

Using Lemmas 3.13 and 3.14, the following theorem regarding the ASS of the first kind in the third LG with Lorentzian metric can be established.

Theorem 3.15. *(G_3, g^Y, J) is ASS of the first kind related to YC if it satisfies*
(1)$\alpha = \beta = \gamma = 0$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}-c & 0 & 0\\ 0 & -c & 0\\ 0 & 0 & -c\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

(2)$\alpha = \beta = 0,\ \gamma \neq 0,\ c = \gamma^2\lambda_0 - \gamma^2$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & -\gamma^2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}\gamma^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \gamma^2\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

(3)$\gamma = c = 0$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

(4)$\beta = 0,\ \alpha \neq 0,\ \gamma \neq 0,\ \gamma^3 - \gamma^3\lambda_0 - \alpha^2\gamma + \alpha^2\gamma\lambda_0 + \gamma c - \alpha c = 0$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & -\gamma^2 - \alpha\gamma & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}\gamma^2\lambda_0+\alpha\gamma\lambda_0-c & 0 & 0\\ 0 & -\gamma^2+\gamma^2\lambda_0-\alpha\gamma+\alpha\gamma\lambda_0-c & 0\\ 0 & 0 & \gamma^2\lambda_0+\alpha\gamma\lambda_0-c\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

(5)$\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$, $\gamma^3 - \gamma^3\lambda_0 - 2\beta\gamma^2\lambda_0 + \beta^2\gamma - \beta^2\gamma\lambda_0 + \gamma c - \beta c = 0$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\beta\gamma & 0 & 0\\ 0 & -\gamma^2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}\gamma^2\lambda_0-\beta\gamma+\beta\gamma\lambda_0-c & 0 & 0\\ 0 & -\gamma^2+\gamma^2\lambda_0+\beta\gamma\lambda_0-c & 0\\ 0 & 0 & \gamma^2\lambda_0+\beta\gamma\lambda_0-c\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

Proof. According to (2.3), we have

$$\begin{aligned}
&\rho^Y(\tilde{q}_1^Y,\tilde{q}_1^Y) = -\beta\gamma,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_2^Y) = 0,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_3^Y) = 0,\\
&\rho^Y(\tilde{q}_2^Y,\tilde{q}_1^Y) = 0,\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_2^Y) = -\gamma^2-\alpha\gamma,\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_3^Y) = 0,\\
&\rho^Y(\tilde{q}_3^Y,\tilde{q}_1^Y) = 0,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_2^Y) = 0,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_3^Y) = 0.
\end{aligned}$$

By (2.4), the Ricci operator can be expressed as

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\beta\gamma & 0 & 0\\ 0 & -\gamma^2-\alpha\gamma & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y = -(\gamma^2 + \alpha\gamma + \beta\gamma)$. If (G_3, g^Y, J) is ASS of the first kind related to the YC, and by $Ric^Y = (s^Y\lambda_0 + c)Id + D$, we can get

$$\begin{cases}
D\tilde{q}_1^Y = (\gamma^2\lambda_0 - \beta\gamma + \alpha\gamma\lambda_0 + \beta\gamma\lambda_0 - c)\tilde{q}_1^Y,\\
D\tilde{q}_2^Y = (-\gamma^2 + \gamma^2\lambda_0 - \alpha\gamma + \alpha\gamma\lambda_0 + \beta\gamma\lambda_0 - c)\tilde{q}_2^Y,\\
D\tilde{q}_3^Y = (\gamma^2\lambda_0 + \alpha\gamma\lambda_0 + \beta\gamma\lambda_0 - c)\tilde{q}_3^Y.
\end{cases}$$

Therefore, by (2.9) and ASS of the first kind related to the YC can be established if it satisfies

$$\begin{cases}
\gamma^3 - \gamma^3\lambda_0 + \beta\gamma^2 + \alpha\gamma^2 - \alpha\gamma^2\lambda_0 - \beta\gamma^2\lambda_0 + \gamma c = 0,\\
\beta^2\gamma - \beta\gamma^2 - \beta\gamma^2\lambda_0 - \beta^2\gamma\lambda_0 - \alpha\beta\gamma - \alpha\beta\gamma\lambda_0 - \beta c = 0,\\
\alpha\gamma^2 + \alpha^2\gamma - \alpha\gamma^2\lambda_0 - \alpha^2\gamma\lambda_0 - \alpha\beta\gamma - \alpha\beta\gamma\lambda_0 + \alpha c = 0.
\end{cases} \tag{3.3}$$

Assuming that $\gamma = 0$, we get

$$\begin{cases}
\beta c = 0,\\
\alpha c = 0.
\end{cases}$$

If $\beta = 0$, we obtain two cases (1) and (2) of Theorem 3.15 holds. If $\beta \neq 0$, for the case (3) of Theorem 3.15 holds. Next assuming that $\gamma \neq 0$, If $\beta = 0$, and (3.3) can be simplified to

$$\begin{cases}
\gamma^3 - \gamma^3\lambda_0 + \alpha\gamma^2 - \alpha\gamma^2\lambda_0 + \gamma c = 0,\\
\alpha\gamma^2 + \alpha^2\gamma - \alpha\gamma^2\lambda_0 - \alpha^2\gamma\lambda_0 + \alpha c = 0.
\end{cases}$$

We get two cases (3) and (4) of Theorem 3.15 holds. If $\beta \neq 0$ and $\alpha = 0$, then a direct calculation reveals that (3.3) reduces to

$$\begin{cases} \gamma^3 - \gamma^3\lambda_0 + \beta\gamma^2 - \beta\gamma^2\lambda_0 + \gamma c = 0, \\ \beta^2\gamma - \beta\gamma^2 - \beta\gamma^2\lambda_0 - \beta^2\gamma\lambda_0 - \beta c = 0. \end{cases}$$

We have case (5) of Theorem 3.15 holds. Thus we get Theorem 3.15.

3.4. ASS of G_4

In the subsection, we present the LA for G_4 that satisfies the following condition

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = -\tilde{q}_2^Y + (2\eta - \beta)\tilde{q}_2^Y, \eta = \pm 1, [\tilde{q}_1^Y, \tilde{q}_3^Y] = -\beta\tilde{q}_2^Y + \tilde{q}_3^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y,$$

the basis vectors $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ form a pseudo-orthonormal basis where $\tilde{q}_3^Y$is timelike. Four lemmas regarding the formulations of YC as well their corresponding curvatures in G_4 with Lorentzian metric can be derived.

Lemma 3.16 ([10, 30]). *The LCC for G_4 can be given as*

$$\begin{aligned}
&\nabla^L_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_2^Y = (\frac{\alpha}{2} + \eta - \beta)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_3^Y = (\frac{\alpha}{2} + \eta - \beta)\tilde{q}_2^Y,\\
&\nabla^L_{\tilde{q}_2^Y}\tilde{q}_1^Y = \tilde{q}_2^Y + (\frac{\alpha}{2} - \eta)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_2^Y = -\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_3^Y = (\frac{\alpha}{2} - \eta)\tilde{q}_1^Y,\\
&\nabla^L_{\tilde{q}_3^Y}\tilde{q}_1^Y = (\frac{\alpha}{2} + \eta)\tilde{q}_2^Y - \tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_2^Y = -(\frac{\alpha}{2} + \eta)\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_3^Y = -\tilde{q}_1^Y.
\end{aligned}$$

Lemma 3.17. *For G_4, the following equalities hold*

$$\begin{aligned}
&\nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_2^Y = (\alpha + 2\eta - 2\beta)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_3^Y = -(\alpha + 2\eta - 2\beta)\tilde{q}_2^Y,\\
&\nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_1^Y = (\alpha - 2\eta)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_3^Y = -(\alpha - 2\eta)\tilde{q}_1^Y,\\
&\nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_1^Y = -2\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_3^Y = 2\tilde{q}_1^Y.
\end{aligned}$$

Based on (2.1), as well as Lemmas 3.16 and 3.17, one can derive the subsequent lemma.

Lemma 3.18. *The YC for G_4 can be given as*

$$\begin{aligned}
&\nabla^Y_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_2^Y = (2\eta - \beta)\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_3^Y = \tilde{q}_3^Y,\\
&\nabla^Y_{\tilde{q}_2^Y}\tilde{q}_1^Y = \tilde{q}_2^Y + (\beta - 2\eta)\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_2^Y = -\tilde{q}_1^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_3^Y = 0,\\
&\nabla^Y_{\tilde{q}_3^Y}\tilde{q}_1^Y = \beta\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\alpha\tilde{q}_1^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0.
\end{aligned}$$

Based on (2.2), as well as Lemma 3.18, one can derive the subsequent lemma.

Lemma 3.19. *The curvature R^Y for (G_4, g^Y) can be given as*

$$\begin{aligned}
&R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_1^Y = (\beta^2 - 2\beta\eta + 1)\tilde{q}_2^Y,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_2^Y = (2\alpha\eta - \alpha\beta - 1)\tilde{q}_1^Y,\\
&R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_3^Y = 0,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = 0,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = (\alpha - \beta)\tilde{q}_1^Y,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = 0,\\
&R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = (\alpha - \beta)\tilde{q}_1^Y,\ R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = (\beta - \alpha)\tilde{q}_2^Y,\ R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = -\alpha\tilde{q}_3^Y.
\end{aligned}$$

Using Lemmas 3.18 and 3.19, the following theorem regarding the ASS of the first kind in the fourth LG with Lorentzian metric can be established.

Theorem 3.20. *The LG G_4 cannot be ASS of a first kind related to the YC.*

Proof. According to (2.3), we have

$$\rho^Y(\tilde{q}_1^Y,\tilde{q}_1^Y)=2\beta\eta-\beta^2-1,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_2^Y)=0,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_3^Y)=0,$$
$$\rho^Y(\tilde{q}_2^Y,\tilde{q}_1^Y)=0,\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_2^Y)=2\alpha\eta-\alpha\beta-1,\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_3^Y)=\alpha,$$
$$\rho^Y(\tilde{q}_3^Y,\tilde{q}_1^Y)=0,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_2^Y)=0,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_3^Y)=0.$$

By (2.4), the Ricci operator can be expressed as

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\beta^2+2\beta\eta-1 & 0 & 0\\ 0 & 2\alpha\eta-\alpha\beta-1 & -\alpha\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y=-\beta^2+2\alpha\eta+2\beta\eta-\alpha\beta-2$.If (G_4,g^Y,J) is ASS of the first kind related to the YC, and by $Ric^Y=(s^Y\lambda_0+c)Id+D$, we can get

$$\begin{cases} D\tilde{q}_1^Y=(-\beta^2+\beta^2\lambda_0+2\beta\eta-2\alpha\eta\lambda_0-2\beta\eta\lambda_0+\alpha\beta\lambda_0+2\lambda_0-1-c)\tilde{q}_1^Y,\\ D\tilde{q}_2^Y=(\beta^2\lambda_0+2\alpha\eta-\alpha\beta-2\alpha\eta\lambda_0-2\beta\eta\lambda_0+\alpha\beta\lambda_0+2\lambda_0-1-c)\tilde{q}_2^Y-\alpha\tilde{q}_3^Y,\\ D\tilde{q}_3^Y=(\beta^2\lambda_0-2\alpha\eta\lambda_0-2\beta\eta\lambda_0+\alpha\beta\lambda_0+2\lambda_0-c)\tilde{q}_3^Y.\end{cases}$$

Therefore, by (2.9) and ASS of the first kind related to the YC can be established if it satisfies

$$\begin{cases} 2\alpha+(2\eta-\beta)(\beta^2-\beta^2\lambda_0-2\alpha\eta-2\beta\eta+\alpha\beta+2\alpha\eta\lambda_0+2\beta\eta\lambda_0-\alpha\beta\lambda_0-2\lambda_0+2+c)=0,\\ \beta^3-\beta^3\lambda_0-\alpha\beta^2-2\beta^2\eta+2\beta^2\eta\lambda_0-\alpha\beta^2\lambda_0+2\alpha\beta\eta+2\alpha\beta\eta\lambda_0-2\beta\lambda_0+\beta c=0,\\ \beta^2\lambda_0-\beta^2+2\beta\eta-\alpha\beta-2\alpha\eta\lambda_0-2\beta\eta\lambda_0+\alpha\beta\lambda_0+2\lambda_0-1-c=0,\\ 2\alpha^2\eta-\alpha^2\beta+\alpha\beta^2-2\alpha^2\eta\lambda_0+\alpha^2\beta\lambda_0+\alpha\beta^2\lambda_0-2\alpha\beta\eta-2\alpha\beta\eta\lambda_0+2\alpha\lambda_0-\alpha c=0.\end{cases} \tag{3.4}$$

By the first equation of (3.4), we assume that

$$\alpha=0,\beta=2\eta.$$

On this basis, by the second equation of (3.4), we have $c=2\lambda_0$. By the third equation of (3.4), we get $c=2\lambda_0-1$, and there is a contradiction. One can prove Theorem 3.20.

3.5. ASS of G_5

In the subsection, we present the LA for G_5 that satisfies the following condition

$$[\tilde{q}_1^Y,\tilde{q}_2^Y]=0,[\tilde{q}_1^Y,\tilde{q}_3^Y]=\alpha\tilde{q}_1^Y+\beta\tilde{q}_2^Y,[\tilde{q}_2^Y,\tilde{q}_3^Y]=\gamma\tilde{q}_1^Y+\delta\tilde{q}_2^Y,\alpha+\delta\neq 0,\alpha\gamma+\beta\delta=0,$$

the basis vectors $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ form a pseudo-orthonormal basis where $\tilde{q}_3^Y$ is timelike. Four lemmas regarding the formulations of YC as well their corresponding curvatures in G_5 with Lorentzian metric can be derived.

Lemma 3.21 ([10,30]). *The LCC for G_5 can be given as*

$$\nabla^L_{\tilde{q}_1^Y}\tilde{q}_1^Y = \alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_2^Y = \frac{\beta+\gamma}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_3^Y = \alpha\tilde{q}_1^Y + \frac{\beta+\gamma}{2}\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}\tilde{q}_1^Y = \frac{\beta+\gamma}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_2^Y = \delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_3^Y = \frac{\beta+\gamma}{2}\tilde{q}_1^Y + \delta\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}\tilde{q}_1^Y = -\frac{\beta-\gamma}{2}\tilde{q}_2^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_2^Y = \frac{\beta-\gamma}{2}\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0.$$

Lemma 3.22. *For G_5, the following equalities hold*

$$\nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_1^Y = 2\alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_2^Y = (\beta+\gamma)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_3^Y = -2\alpha\tilde{q}_1^Y - (\beta+\gamma)\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_1^Y = (\beta+\gamma)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_2^Y = 2\delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_3^Y = -(\beta+\gamma)\tilde{q}_1^Y - 2\delta\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_3^Y = 0.$$

Based on (2.1), as well as Lemmas 3.21 and 3.22, one can derive the subsequent lemma.

Lemma 3.23. *The YC for G_5 can be given as*

$$\nabla^Y_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_2^Y = 0,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_3^Y = 0,$$
$$\nabla^Y_{\tilde{q}_2^Y}\tilde{q}_1^Y = 0,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_2^Y = 0,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_3^Y = 0,$$
$$\nabla^Y_{\tilde{q}_3^Y}\tilde{q}_1^Y = -\alpha\tilde{q}_1^Y + (\beta+\gamma)\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\gamma\tilde{q}_1^Y - \delta\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0.$$

Based on (2.2), as well as Lemma 3.23, one can derive the subsequent lemma.

Lemma 3.24. *The curvature R^Y for (G_5, g^Y) can be given as*

$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_j^Y = R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_j^Y = R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_j^Y = 0,$$

where $1 \le j \le 3$.

Using Lemmas 3.23 and 3.24, the following theorem regarding the ASS of the first kind in the fifth LG with Lorentzian metric can be established..

Theorem 3.25. (G_5, g^Y, J) *is ASS of the first kind related to YC if it satisfies $c = 0$. And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

Proof. According to (2.3), we have

$$\rho^Y(\tilde{q}_1^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_2^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_3^Y, \tilde{q}_j^Y) = 0,$$

where $1 \leq j \leq 3$.
By (2.4), the Ricci operator can be expressed as

$$Ric^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y = 0$. If (G_5, g^Y, J) is ASS of the first kind related to the YC, and by $Ric^Y = (s^Y \lambda_0 + c)Id + D$, we can get

$$\begin{cases} D\tilde{q}_1^Y = -c\tilde{q}_1^Y, \\ D\tilde{q}_2^Y = -c\tilde{q}_2^Y, \\ D\tilde{q}_3^Y = -c\tilde{q}_3^Y. \end{cases}$$

Therefore, by (2.9) and ASS of the first kind related to the YC can be established if it satisfies

$$\begin{cases} \alpha c = 0, \\ \beta c = 0, \\ \gamma c = 0, \\ \delta c = 0. \end{cases} \tag{3.5}$$

Since $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$, by solving (3.5), we have $c = 0$. Thus we get Theorem 3.25.

3.6. ASS of G_6

In the subsection, we present the LA for G_6 that satisfies the following condition

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = \alpha\tilde{q}_2^Y + \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = \gamma\tilde{q}_2^Y + \delta\tilde{q}_3^Y, [\tilde{q}_2^Y, \tilde{q}_3^Y] = 0, \alpha + \delta \neq 0, \alpha\gamma - \beta\delta = 0.$$

The basis vectors $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ form a pseudo-orthonormal basis where $\tilde{q}_3^Y$ is timelike. Four lemmas regarding the formulations of YC as well their corresponding curvatures in G_6 with Lorentzian metric can be derived.

Lemma 3.26 ([10, 30]). *The LCC for G_6 can be given as*

$$\nabla^L_{\tilde{q}_1^Y}\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_2^Y = \frac{\beta+\gamma}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_3^Y = \frac{\beta+\gamma}{2}\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}\tilde{q}_1^Y = -\alpha\tilde{q}_2^Y - \frac{\beta-\gamma}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_2^Y = \alpha\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_3^Y = -\frac{\beta-\gamma}{2}\tilde{q}_1^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}\tilde{q}_1^Y = \frac{\beta-\gamma}{2}\tilde{q}_2^Y - \delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\frac{\beta-\gamma}{2}\tilde{q}_1^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_3^Y = -\delta\tilde{q}_1^Y.$$

Lemma 3.27. *For G_6, the following equalities hold*

$$\nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_1^Y = 0,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_2^Y = (\beta+\gamma)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_3^Y = -(\beta+\gamma)\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_1^Y = -(\beta-\gamma)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_3^Y = (\beta-\gamma)\tilde{q}_1^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_1^Y = -2\delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_2^Y = 0,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_3^Y = 2\delta\tilde{q}_1^Y.$$

Based on (2.1), as well as Lemmas 3.26 and 3.27, one can derive the subsequent lemma.

Lemma 3.28. *The YC for G_6 can be given as*

$$\begin{aligned}
&\nabla^Y_{\tilde{q}^Y_1}\tilde{q}^Y_1 = 0,\ \nabla^Y_{\tilde{q}^Y_1}\tilde{q}^Y_2 = \beta\tilde{q}^Y_3,\ \nabla^Y_{\tilde{q}^Y_1}\tilde{q}^Y_3 = \delta\tilde{q}^Y_3,\\
&\nabla^Y_{\tilde{q}^Y_2}\tilde{q}^Y_1 = -\alpha\tilde{q}^Y_2 - \beta\tilde{q}^Y_3,\ \nabla^Y_{\tilde{q}^Y_2}\tilde{q}^Y_2 = \alpha\tilde{q}^Y_1,\ \nabla^Y_{\tilde{q}^Y_2}\tilde{q}^Y_3 = 0,\\
&\nabla^Y_{\tilde{q}^Y_3}\tilde{q}^Y_1 = -\gamma\tilde{q}^Y_2,\ \nabla^Y_{\tilde{q}^Y_3}\tilde{q}^Y_2 = 0,\ \nabla^Y_{\tilde{q}^Y_3}\tilde{q}^Y_3 = 0.
\end{aligned}$$

Based on (2.2), as well as Lemma 3.28, one can derive the subsequent lemma.

Lemma 3.29. *The curvature R^Y for (G_6, g^Y) can be given as*

$$\begin{aligned}
&R^Y\left(\tilde{q}^Y_1,\tilde{q}^Y_2\right)\tilde{q}^Y_1 = (\beta\gamma+\alpha^2)\tilde{q}^Y_2 - \beta\delta\tilde{q}^Y_3,\ R^Y\left(\tilde{q}^Y_1,\tilde{q}^Y_2\right)\tilde{q}^Y_2 = -\alpha^2\tilde{q}^Y_1,\ R^Y\left(\tilde{q}^Y_1,\tilde{q}^Y_2\right)\tilde{q}^Y_3 = 0,\\
&R^Y\left(\tilde{q}^Y_1,\tilde{q}^Y_3\right)\tilde{q}^Y_1 = (\alpha\gamma+\delta\gamma)\tilde{q}^Y_2,\ R^Y\left(\tilde{q}^Y_1,\tilde{q}^Y_3\right)\tilde{q}^Y_2 = -\alpha\gamma\tilde{q}^Y_1,\ R^Y\left(\tilde{q}^Y_1,\tilde{q}^Y_3\right)\tilde{q}^Y_3 = 0,\\
&R^Y\left(\tilde{q}^Y_2,\tilde{q}^Y_3\right)\tilde{q}^Y_1 = -\alpha\gamma\tilde{q}^Y_1,\ R^Y\left(\tilde{q}^Y_2,\tilde{q}^Y_3\right)\tilde{q}^Y_2 = \alpha\gamma\tilde{q}^Y_2,\ R^Y\left(\tilde{q}^Y_2,\tilde{q}^Y_3\right)\tilde{q}^Y_3 = 0.
\end{aligned}$$

Using Lemmas 3.28 and 3.29, the following theorem regarding the ASS of the first kind in the sixth LG with Lorentzian metric can be established.

Theorem 3.30. *(G_6, g^Y, J) is ASS of the first kind related to YC if it satisfies*
(1)$\alpha = \beta = c = 0$, $\delta \neq 0$. And specifically

$$Ric^Y\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix}.$$

(2)$\alpha \neq 0$, $\beta = \gamma = 0$, $\alpha + \delta \neq 0$, $c = 2\alpha^2\lambda_0 - \alpha^2$. And specifically

$$Ric^Y\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix} = \begin{pmatrix}-\alpha^2&0&0\\0&-\alpha^2&0\\0&0&0\end{pmatrix}\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&\alpha^2\end{pmatrix}\begin{pmatrix}\tilde{q}^Y_1\\ \tilde{q}^Y_2\\ \tilde{q}^Y_3\end{pmatrix}.$$

Proof. According to (2.3), we have

$$\begin{aligned}
&\rho^Y(\tilde{q}^Y_1,\tilde{q}^Y_1) = -(\beta\gamma+\alpha^2),\ \rho^Y(\tilde{q}^Y_1,\tilde{q}^Y_2) = 0,\ \rho^Y(\tilde{q}^Y_1,\tilde{q}^Y_3) = 0,\\
&\rho^Y(\tilde{q}^Y_2,\tilde{q}^Y_1) = 0,\ \rho^Y(\tilde{q}^Y_2,\tilde{q}^Y_2) = -\alpha^2,\ \rho^Y(\tilde{q}^Y_2,\tilde{q}^Y_3) = 0,\\
&\rho^Y(\tilde{q}^Y_3,\tilde{q}^Y_1) = 0,\ \rho^Y(\tilde{q}^Y_3,\tilde{q}^Y_2) = 0,\ \rho^Y(\tilde{q}^Y_3,\tilde{q}^Y_3) = 0.
\end{aligned}$$

By (2.4), the Ricci operator can be expressed as

$$Ric^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} -(\alpha^2+\beta\gamma) & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y = -2\alpha^2 - \beta\gamma$. If (G_6, g^Y, J) is ASS of the first kind related to the YC, and by $Ric^Y = (s^Y\lambda_0 + c)Id + D$, we can get

$$\begin{cases} D\tilde{q}_1^Y = (2\alpha^2\lambda_0 - \alpha^2 - \beta\gamma + \beta\gamma\lambda_0 - c)\tilde{q}_1^Y, \\ D\tilde{q}_2^Y = (2\alpha^2\lambda_0 - \alpha^2 + \beta\gamma\lambda_0 - c)\tilde{q}_2^Y, \\ D\tilde{q}_3^Y = (2\alpha^2\lambda_0 + \beta\gamma\lambda_0 - c)\tilde{q}_3^Y. \end{cases}$$

Therefore, by (2.9) and ASS of the first kind related to the YC can be established if it satisfies

$$\begin{cases} 2\alpha^3\lambda_0 - \alpha^3 - \alpha\beta\gamma + \alpha\beta\gamma\lambda_0 - \alpha c = 0, \\ \beta^2\gamma\lambda_0 - \beta^2\gamma - 2\alpha^2\beta + 2\alpha^2\beta\lambda_0 - \beta c = 0, \\ -\beta\gamma^2 + \beta\gamma^2\lambda_0 + 2\alpha^2\gamma\lambda_0 - \gamma c = 0, \\ 2\alpha^2\delta\lambda_0 - \alpha^2\delta - \beta\delta\gamma + \beta\gamma\delta\lambda_0 - \delta c = 0. \end{cases} \tag{3.6}$$

Because $\alpha + \delta \neq 0$ as well $\alpha\gamma - \beta\delta = 0$, we suppose first that $\alpha = \beta = 0$, $\delta \neq 0$. On this basis, the fourth equation of (3.6) can be simplified to

$$\delta c = 0,$$

we get $c = 0$, for the case (1) of Theorem 3.30 holds. Suppose second that $\gamma = 0$, $\alpha \neq 0$, $\alpha + \delta \neq 0$, on this basis, the first and fourth equations of (3.6) reduces to

$$-\alpha^2 + 2\alpha^2\lambda_0 - c = 0,$$

and the second equation of (3.6) can be simplified to

$$\beta(2\alpha^2\lambda_0 - 2\alpha^2 - c) = 0,$$

we have $\alpha^2\beta = 0$, thus $\beta = 0$, for the case (2) of Theorem 3.30 holds. It turns out Theorem 3.30.

3.7. ASS of G_7

In the subsection, we present the LA for G_7 that satisfies the following condition

$$[\tilde{q}_1^Y, \tilde{q}_2^Y] = -\alpha\tilde{q}_1^Y - \beta\tilde{q}_2^Y - \beta\tilde{q}_3^Y, [\tilde{q}_1^Y, \tilde{q}_3^Y] = \alpha\tilde{q}_1^Y + \beta\tilde{q}_2^Y + \beta\tilde{q}_3^Y,$$

$$[\tilde{q}_2^Y, \tilde{q}_3^Y] = \gamma\tilde{q}_1^Y + \delta\tilde{q}_2^Y + \delta\tilde{q}_3^Y, \alpha + \delta \neq 0, \alpha\gamma = 0,$$

the basis vectors $\tilde{q}_1^Y$, $\tilde{q}_2^Y$ and $\tilde{q}_3^Y$ form a pseudo-orthonormal basis where $\tilde{q}_3^Y$ is timelike. Four lemmas regarding the formulations of YC as well their corresponding curvatures in G_7 with Lorentzian metric can be derived.

Lemma 3.31 ([10,30]). *The LCC for G_7 can be given as*

$$\nabla^L_{\tilde{q}_1^Y}\tilde{q}_1^Y = \alpha\tilde{q}_2^Y + \alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_2^Y = -\alpha\tilde{q}_1^Y + \frac{\gamma}{2}\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}\tilde{q}_3^Y = \alpha\tilde{q}_1^Y + \frac{\gamma}{2}\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}\tilde{q}_1^Y = \beta\tilde{q}_2^Y + (\beta + \frac{\gamma}{2})\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_2^Y = -\beta\tilde{q}_1^Y + \delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}\tilde{q}_3^Y = (\beta + \frac{\gamma}{2})\tilde{q}_1^Y + \delta\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}\tilde{q}_1^Y = -(\beta - \frac{\gamma}{2})\tilde{q}_2^Y - \beta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_2^Y = (\beta - \frac{\gamma}{2})\tilde{q}_1^Y - \delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}\tilde{q}_3^Y = -\beta\tilde{q}_1^Y - \delta\tilde{q}_2^Y.$$

Lemma 3.32. *For G_7, the following equalities hold*

$$\nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_1^Y = 2\alpha\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_2^Y = \gamma\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_1^Y}(J)\tilde{q}_3^Y = -2\alpha\tilde{q}_1^Y - \gamma\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_1^Y = (2\beta + \gamma)\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_2^Y = 2\delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_2^Y}(J)\tilde{q}_3^Y = -(2\beta + \gamma)\tilde{q}_1^Y - 2\delta\tilde{q}_2^Y,$$
$$\nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_1^Y = -2\beta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_2^Y = -2\delta\tilde{q}_3^Y,\ \nabla^L_{\tilde{q}_3^Y}(J)\tilde{q}_3^Y = 2\beta\tilde{q}_1^Y + 2\delta\tilde{q}_2^Y.$$

Based on (2.1), as well as Lemmas 3.31 and 3.32, one can derive the subsequent lemma.

Lemma 3.33. *The YC for G_7 can be given as*

$$\nabla^Y_{\tilde{q}_1^Y}\tilde{q}_1^Y = \alpha\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_2^Y = -\alpha\tilde{q}_1^Y - \beta\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_1^Y}\tilde{q}_3^Y = \beta\tilde{q}_3^Y,$$
$$\nabla^Y_{\tilde{q}_2^Y}\tilde{q}_1^Y = \beta\tilde{q}_2^Y + \beta\tilde{q}_3^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_2^Y = -\beta\tilde{q}_1^Y,\ \nabla^Y_{\tilde{q}_2^Y}\tilde{q}_3^Y = \delta\tilde{q}_3^Y,$$
$$\nabla^Y_{\tilde{q}_3^Y}\tilde{q}_1^Y = -\alpha\tilde{q}_1^Y - \beta\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_2^Y = -\gamma\tilde{q}_1^Y - \delta\tilde{q}_2^Y,\ \nabla^Y_{\tilde{q}_3^Y}\tilde{q}_3^Y = 0.$$

Based on (2.2), as well as Lemma 3.33, one can derive the subsequent lemma.

Lemma 3.34. *The curvature R^Y for (G_7, g^Y) can be given as*

$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_1^Y = -\alpha\beta\tilde{q}_1^Y + \alpha^2\tilde{q}_2^Y + \beta\tilde{q}_3^Y,$$
$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_2^Y = -(\alpha^2 + \beta^2 + \beta\gamma)\tilde{q}_1^Y - \beta\delta\tilde{q}_2^Y + \beta\delta\tilde{q}_3^Y,$$
$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_2^Y\right)\tilde{q}_3^Y = (\beta\delta + \alpha\beta)\tilde{q}_3^Y,\ R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = (2\alpha\beta + \alpha\gamma)\tilde{q}_1^Y + (\alpha\delta - 2\alpha^2)\tilde{q}_2^Y,$$
$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = (\beta^2 + \beta\gamma + \alpha\delta)\tilde{q}_1^Y + (-\alpha\beta - \alpha\gamma + \beta\delta)\tilde{q}_2^Y + (\beta\delta + \alpha\beta)\tilde{q}_3^Y,$$
$$R^Y\left(\tilde{q}_1^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = -(\alpha\beta + \beta\delta)\tilde{q}_3^Y,$$
$$R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_1^Y = (\beta^2 + \beta\gamma + \alpha\delta)\tilde{q}_1^Y + (\beta\delta - \alpha\beta - \alpha\gamma)\tilde{q}_2^Y - (\alpha\beta + \beta\delta)\tilde{q}_3^Y,$$
$$R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_2^Y = (2\beta\delta - \alpha\beta + \alpha\gamma + \gamma\delta)\tilde{q}_1^Y + (\delta - \beta\gamma - \beta^2)\tilde{q}_2^Y,$$
$$R^Y\left(\tilde{q}_2^Y, \tilde{q}_3^Y\right)\tilde{q}_3^Y = -(\beta\gamma + \delta^2)\tilde{q}_3^Y.$$

The following theorem regarding the ASS of the first kind in the seventh LG with Lorentzian metric can be established.

Theorem 3.35. *(G_7, g^Y, J) is ASS of the first kind related to YC if it satisfies*
(1)$\alpha = \beta = \gamma = 0,\ \delta \neq 0,\ \delta = -1, c = 1$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\delta^2\\ 0 & \delta & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-c & 0 & 0\\ 0 & -c & -\delta^2\\ 0 & \delta & -c\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

(2)$\alpha=\beta=c=0$, $\delta\neq 0$, $\gamma\neq 0$, $\delta=-1$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\delta^2\\ 0 & \delta & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\delta^2\\ 0 & \delta & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

(3)$\alpha\neq 0$, $\beta=\gamma=0$, $\alpha+\delta\neq 0$, $\alpha=\lambda\delta$, $\delta=\dfrac{1}{\lambda^2-\lambda-1}$, $c=\dfrac{1-\lambda^2+2\lambda^2\lambda_0}{(\lambda^2-\lambda-1)^2}$, $\lambda\neq 0$. *And specifically*

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\alpha^2 & 0 & 0\\ 0 & -\alpha^2 & -\delta^2\\ 0 & \alpha\delta+\delta & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\delta^2 & 0 & 0\\ 0 & -\delta^2 & -\delta^2\\ 0 & \alpha\delta+\delta & \alpha^2-\delta^2\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

Proof. According to (2.3), we have

$$\begin{aligned}&\rho^Y(\tilde{q}_1^Y,\tilde{q}_1^Y)=-\alpha^2,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_2^Y)=-\alpha\beta,\ \rho^Y(\tilde{q}_1^Y,\tilde{q}_3^Y)=\alpha\beta+\beta\delta,\\ &\rho^Y(\tilde{q}_2^Y,\tilde{q}_1^Y)=\beta\delta,\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_2^Y)=-\alpha^2-\beta^2-\beta\gamma,\ \rho^Y(\tilde{q}_2^Y,\tilde{q}_3^Y)=\beta\gamma+\delta^2,\\ &\rho^Y(\tilde{q}_3^Y,\tilde{q}_1^Y)=\alpha\beta+\beta\delta,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_2^Y)=\alpha\delta+\delta,\ \rho^Y(\tilde{q}_3^Y,\tilde{q}_3^Y)=0.\end{aligned}$$

By (2.4), the Ricci operator can be expressed as

$$Ric^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\alpha^2 & -\alpha\beta & -(\alpha\beta+\beta\delta)\\ \beta\delta & -\alpha^2-\beta^2-\beta\gamma & -(\delta^2+\beta\gamma)\\ \alpha\beta+\beta\delta & \alpha\delta+\delta & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y=-2\alpha^2-\beta^2-\beta\gamma$. If (G_7,g^Y,J) is ASS of the first kind related to the YC, and by $Ric^Y=(s^Y\lambda_0+c)Id+D$, we can get

$$\begin{cases}D\tilde{q}_1^Y=(-\alpha^2+2\alpha^2\lambda_0+\beta^2\lambda_0+\beta\gamma\lambda_0-c)\tilde{q}_1^Y-\alpha\beta\tilde{q}_2^Y-(\alpha\beta+\beta\delta)\tilde{q}_3^Y,\\ D\tilde{q}_2^Y=\beta\delta\tilde{q}_1^Y+(-\alpha^2-\beta^2+2\alpha^2\lambda_0+\beta^2\lambda_0-\beta\gamma+\beta\gamma\lambda_0-c)\tilde{q}_2^Y-(\delta^2+\beta\gamma)\tilde{q}_3^Y,\\ D\tilde{q}_3^Y=(\alpha\beta+\beta\delta)\tilde{q}_1^Y+(\alpha\delta+\delta)\tilde{q}_2^Y+(2\alpha^2\lambda_0+\beta^2\lambda_0+\beta\gamma\lambda_0-c)\tilde{q}_3^Y.\end{cases}$$

Therefore, by (2.9) and ASS of the first kind related to the YC can be established if it satisfies

$$\begin{cases} \alpha^3 - 2\alpha^3\lambda_0 + 2\alpha\beta^2 - \alpha\delta^2 + 2\beta^2\delta - \alpha\beta^2\lambda_0 + \alpha\beta\gamma + \beta\delta\gamma - \alpha\beta\gamma\lambda_0 + \alpha c = 0, \\ \beta^3\lambda_0 + 2\alpha^2\beta\lambda_0 + \beta^2\gamma\lambda_0 + \beta^2\gamma - 2\alpha\beta\delta - \beta\delta - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 + \alpha^2\beta - \beta^2\gamma - \beta\delta^2 - 2\alpha^2\beta\lambda_0 - \beta^2\gamma\lambda_0 + \beta c = 0, \\ 2\alpha^3\lambda_0 - \alpha^2\delta - 2\beta^2\delta - \alpha\beta^2 + \alpha\beta^2\lambda_0 - \alpha\delta - \alpha\beta\gamma + \alpha\beta\gamma\lambda_0 - \alpha c = 0, \\ \beta^3 + \beta^3\lambda_0 + \alpha^2\beta + \beta^2\gamma + 2\alpha^2\beta\lambda_0 + \beta^2\gamma\lambda_0 - 3\alpha\beta\delta - 2\beta\delta - \beta c = 0, \\ \beta^3\lambda_0 + \beta^2\gamma + \beta\delta^2 + 2\alpha^2\beta\lambda_0 + \beta^2\gamma\lambda_0 - \alpha\beta\delta - \beta\delta - \beta c = 0, \\ \alpha^2\beta - \beta^2\gamma - \beta\gamma^2 - 2\beta\delta^2 + 2\alpha^2\gamma\lambda_0 + \beta^2\gamma\lambda_0 + \beta\gamma^2\lambda_0 + \alpha\beta\delta - \gamma c = 0, \\ \alpha\beta^2 + 2\beta^2\delta - \alpha\delta^2 - \delta^2 + 2\alpha^2\delta\lambda_0 + \beta^2\delta\lambda_0 + \alpha\beta\gamma + \beta\delta\gamma\lambda_0 - \delta c = 0, \\ \delta^3 + \alpha\beta^2 + \beta^2\delta - \alpha^2\delta + 2\alpha^2\delta\lambda_0 + \beta^2\delta\lambda_0 + \alpha\beta\gamma + \beta\gamma\delta + \beta\gamma\delta\lambda_0 - \delta c = 0. \end{cases} \tag{3.7}$$

Because $\alpha + \delta \neq 0$ as well $\alpha\gamma = 0$. Let's first suppose that $\alpha = 0$. On this basis, (3.7) can be simplified to

$$\begin{cases} 2\beta^2\delta + \beta\delta\gamma = 0, \\ \beta^3\lambda_0 + \beta^2\gamma\lambda_0 + \beta^2\gamma - \beta\delta - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 - \beta^2\gamma - \beta\delta^2 - \beta^2\gamma\lambda_0 + \beta c = 0, \\ \beta^2\delta = 0, \\ \beta^3 + \beta^3\lambda_0 + \beta^2\gamma + \beta^2\gamma\lambda_0 - 2\beta\delta - \beta c = 0, \\ \beta^3\lambda_0 + \beta^2\gamma + \beta\delta^2 + \beta^2\gamma\lambda_0 - \beta\delta - \beta c = 0, \\ \beta^2\gamma + \beta\gamma^2 + 2\beta\delta^2 - \beta^2\gamma\lambda_0 - \beta\gamma^2\lambda_0 + \gamma c = 0, \\ 2\beta^2\delta - \delta^2 + \beta^2\delta\lambda_0 + \beta\delta\gamma\lambda_0 - \delta c = 0, \\ \delta^3 + \beta^2\delta + \beta^2\delta\lambda_0 + \beta\gamma\delta + \beta\gamma\delta\lambda_0 - \delta c = 0. \end{cases}$$

If $\gamma \neq 0$ and $\delta \neq 0$, we get case (2) of Theorem 3.35 holds. If $\gamma = 0$ as well $\delta \neq 0$, on this basis, we calculate that

$$\begin{cases} \beta^2\delta = 0, \\ \beta^3\lambda_0 - \beta\delta - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 - \beta\delta^2 + \beta c = 0, \\ \beta^3 + \beta^3\lambda_0 - 2\beta\delta - \beta c = 0, \\ \beta^3\lambda_0 + \beta\delta^2 - \beta\delta - \beta c = 0, \\ \beta\delta^2 = 0, \\ 2\beta^2\delta - \delta^2 + \beta^2\delta\lambda_0 - \delta c = 0, \\ \delta^3 + \beta^2\delta + \beta^2\delta\lambda_0 - \delta c = 0. \end{cases}$$

we obtain case (1) of Theorem 3.35 holds. Assume second that $\alpha \neq 0$, $\alpha + \delta \neq 0$ and $\gamma = 0$. In this case, (3.7) reduces to

$$\begin{cases} \alpha^3 - 2\alpha^3\lambda_0 + 2\alpha\beta^2 - \alpha\delta^2 + 2\beta^2\delta - \alpha\beta^2\lambda_0 + \alpha c = 0, \\ \beta^3\lambda_0 + 2\alpha^2\beta\lambda_0 - 2\alpha\beta\delta - \beta\delta - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 + \alpha^2\beta - \beta\delta^2 - 2\alpha^2\beta\lambda_0 + \beta c = 0, \\ 2\alpha^3\lambda_0 - \alpha^2\delta - 2\beta^2\delta - \alpha\beta^2 + \alpha\beta^2\lambda_0 - \alpha\delta - \alpha c = 0, \\ \beta^3 + \beta^3\lambda_0 + \alpha^2\beta + 2\alpha^2\beta\lambda_0 - 3\alpha\beta\delta - 2\beta\delta - \beta c = 0, \\ \beta^3\lambda_0 + \beta\delta^2 + 2\alpha^2\beta\lambda_0 - \alpha\beta\delta - \beta\delta - \beta c = 0, \\ \alpha^2\beta - 2\beta\delta^2 + \alpha\beta\delta = 0, \\ \alpha\beta^2 + 2\beta^2\delta - \alpha\delta^2 - \delta^2 + 2\alpha^2\delta\lambda_0 + \beta^2\delta\lambda_0 - \delta c = 0, \\ \delta^3 + \alpha\beta^2 + \beta^2\delta - \alpha^2\delta + 2\alpha^2\delta\lambda_0 + \beta^2\delta\lambda_0 - \delta c = 0. \end{cases}$$

Next suppose that $\beta = 0$, we have

$$\begin{cases} \alpha^3 - 2\alpha^3\lambda_0 - \alpha\delta^2 + \alpha c = 0, \\ 2\alpha^3\lambda_0 - \alpha^2\delta - \alpha\delta - \alpha c = 0, \\ \alpha\delta^2 + \delta^2 - 2\alpha^2\delta\lambda_0 + \delta c = 0, \\ \delta^3 - \alpha^2\delta + 2\alpha^2\delta\lambda_0 - \delta c = 0. \end{cases}$$

Then we get

$$\alpha^3 - \delta^3 - 2\alpha\delta^2 - \delta^2 - \alpha\delta = 0.$$

Let $\alpha = \lambda\delta$, $\lambda \neq 0$, it becomes

$$(\lambda^3 - 2\lambda - 1)\delta^3 - (\lambda + 1)\delta^2 = 0,$$

for the cases (3) of Theorem 3.35 holds. Thus it turns out Theorem 3.35.

4. The second kind ASS related to YC on three-dimensional LLG

In the section, we use the soliton equation in an effort to finish a categorization about three-dimensional LLG that support ASS of the second kind associated with YC.

Let

$$\tilde{\rho}^Y(U^Y, V^Y) = \frac{\rho^Y(U^Y, V^Y) + \rho^Y(V^Y, U^Y)}{2}, \tag{4.1}$$

and

$$\tilde{\rho}^Y(U^Y, V^Y) = g^Y(\widetilde{Ric}^Y(U^Y), V^Y). \tag{4.2}$$

Similar to the formulae (2.6), we have

$$\tilde{S}^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \tilde{\rho}^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) - s^Y\lambda_0 g^Y(\tilde{q}_i^Y, \tilde{q}_j^Y), \tag{4.3}$$

where λ_0 is a real number. Refer to [29], we can get

$$s^Y = \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) + \tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) - \tilde{\rho}^Y(\tilde{q}_3^Y, \tilde{q}_3^Y). \tag{4.4}$$

for vector fields U^Y, V^Y.

Definition 4.1. *(G_i, g^Y) is called ASS of the second kind related with YC when it satisfies*

$$\widetilde{Ric}^Y = (s^Y\lambda_0 + c)Id + D, \tag{4.5}$$

which c is an actual number, λ_0 is a real-valued constant, as well D is derivation for g^Y, which can be

$$D[U^Y, V^Y] = [DU^Y, V^Y] + [U^Y, DV^Y], \tag{4.6}$$

for U^Y, $V^Y \in g^Y$.

Theorem 4.2. *(G_1, g^Y, J) is ASS of the second kind related to YC if it satisfies $\alpha \neq 0$, $\beta = 0$, $\frac{\alpha^2}{2} - 2\alpha^2\lambda_0 + c = 0$. And specifically*

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} -\alpha^2 & 0 & 0 \\ 0 & -\alpha^2 & -\frac{\alpha^2}{2} \\ 0 & \frac{\alpha^2}{2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\frac{\alpha^2}{2} & 0 & 0\\ 0 & \frac{\alpha^2}{2} & -\frac{\alpha^2}{2}\\ 0 & \frac{\alpha^2}{2} & \frac{\alpha^2}{2}\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

Proof. For (G_1, ∇^Y), according to (4.1), we have

$$\begin{aligned}&\tilde{\rho}^Y(\tilde{q}_1^Y,\tilde{q}_1^Y)=-(\alpha^2+\beta^2),\ \tilde{\rho}^Y(\tilde{q}_1^Y,\tilde{q}_2^Y)=\alpha\beta,\ \tilde{\rho}^Y(\tilde{q}_1^Y,\tilde{q}_3^Y)=-\frac{\alpha\beta}{2},\\ &\tilde{\rho}^Y(\tilde{q}_2^Y,\tilde{q}_2^Y)=-(\alpha^2+\beta^2),\ \tilde{\rho}^Y(\tilde{q}_2^Y,\tilde{q}_3^Y)=\frac{\alpha^2}{2},\ \tilde{\rho}^Y(\tilde{q}_3^Y,\tilde{q}_3^Y)=0.\end{aligned}\tag{4.7}$$

By (4.2), the Ricci operator can be expressed as

$$\widetilde{Ric}^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\alpha^2-\beta^2 & \alpha\beta & \frac{\alpha\beta}{2}\\ \alpha\beta & -\alpha^2-\beta^2 & -\frac{\alpha^2}{2}\\ -\frac{\alpha\beta}{2} & \frac{\alpha^2}{2} & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y=-2\alpha^2-2\beta^2$. If (G_1, g^Y, J) is ASS of the second kind related to the YC, and by $\widetilde{Ric}^Y=(s^Y\lambda_0+c)Id+D$, we can get

$$\begin{cases}D\tilde{q}_1^Y=[-\alpha^2-\beta^2+(2\alpha^2+2\beta^2)\lambda_0-c]\tilde{q}_1^Y+\alpha\beta\tilde{q}_2^Y+\frac{\alpha\beta}{2}\tilde{q}_3^Y,\\ D\tilde{q}_2^Y=\alpha\beta\tilde{q}_1^Y+[-\alpha^2-\beta^2+(2\alpha^2+2\beta^2)\lambda_0-c]\tilde{q}_2^Y-\frac{\alpha^2}{2}\tilde{q}_3^Y,\\ D\tilde{q}_3^Y=-\frac{\alpha\beta}{2}\tilde{q}_1^Y+\frac{\alpha^2}{2}\tilde{q}_2^Y+[(2\alpha^2+2\beta^2)\lambda_0-c]\tilde{q}_3^Y.\end{cases}$$

Therefore, by (4.6) and ASS of the second kind related to the YC can be established if it satisfies

$$\begin{cases}2\alpha^3\lambda_0-\frac{\alpha^3}{2}-2\alpha\beta^2+2\alpha\beta^2\lambda_0-\alpha c=0,\\ \alpha^2\beta=0,\\ \beta^3=0,\\ 2\beta^3\lambda_0-2\alpha^2\beta+2\alpha^2\beta\lambda_0-\beta c=0,\\ 2\beta^3\lambda_0-\alpha^2\beta+2\alpha^2\beta\lambda_0-\beta c=0.\end{cases}\tag{4.8}$$

Since $\alpha\neq 0$, by solving the second and third equations of (4.8) imply that $\beta=0$. In this case, the first equation of (4.8) can be simplified to

$$2\alpha^3\lambda_0-\frac{\alpha^3}{2}-\alpha c=0,$$

we have $\frac{\alpha^2}{2}-2\alpha^2\lambda_0+c=0$. Thus we get Theorem 4.2.

Theorem 4.3. *(G_2, g^Y, J) is ASS of the second kind related to YC if it satisfies $\alpha=\beta=0$, $\gamma\neq 0$, $c=\gamma^2(2\lambda_0-1)$. And specifically*

$$\widetilde{Ric}^Y\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}-\gamma^2 & 0 & 0\\ 0 & -\gamma^2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix},$$

$$D\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \gamma^2\end{pmatrix}\begin{pmatrix}\tilde{q}_1^Y\\ \tilde{q}_2^Y\\ \tilde{q}_3^Y\end{pmatrix}.$$

Proof. For (G_2, ∇^Y), according to (4.1), we can get

$$\tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) = \beta^2 - \gamma^2,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_2^Y) = 0,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_3^Y) = 0,$$
$$\tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) = -\gamma^2 - 2\alpha\beta,\ \tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_3^Y) = \beta\gamma - \alpha\gamma,\ \tilde{\rho}^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0.$$

By (4.2), the Ricci operator can be expressed as

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 \\ 0 & -\gamma^2 - 2\alpha\beta & \alpha\gamma - \beta\gamma \\ 0 & \beta\gamma - \alpha\gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y = \beta^2 - 2\gamma^2 - 2\alpha\beta$. If (G_2, g^Y, J) is ASS of the second kind related to the YC, and by $\widetilde{Ric}^Y = (s^Y\lambda_0 + c)Id + D$, we can get

$$\begin{cases} D\tilde{q}_1^Y = (\beta^2 - \gamma^2 - \beta^2\lambda_0 + 2\gamma^2\lambda_0 + 2\alpha\beta\lambda_0 - c)\tilde{q}_1^Y, \\ D\tilde{q}_2^Y = (-\gamma^2 - 2\alpha\beta - \beta^2\lambda_0 + 2\gamma^2\lambda_0 + 2\alpha\beta\lambda_0 - c)\tilde{q}_2^Y + (\alpha\gamma - \beta\gamma)\tilde{q}_3^Y, \\ D\tilde{q}_3^Y = (\beta\gamma - \alpha\gamma)\tilde{q}_2^Y + (-\beta^2\lambda_0 + 2\gamma^2\lambda_0 + 2\alpha\beta\lambda_0 - c)\tilde{q}_3^Y. \end{cases}$$

Therefore, by (4.6) and ASS of the second kind related to the YC can be established if it satisfies

$$\begin{cases} \beta^3 - \beta^3\lambda_0 + 2\alpha\gamma^2 - 4\beta\gamma^2 - 2\alpha\beta^2 + 2\beta\gamma^2\lambda_0 + 2\alpha\beta^2\lambda_0 - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 - 2\beta\gamma^2 + 2\alpha\gamma^2 + 2\alpha\beta^2 + 2\beta\gamma^2\lambda_0 + 2\alpha\beta^2\lambda_0 - \beta c = 0, \\ \gamma^3 - 2\gamma^3\lambda_0 - 3\beta^2\gamma + \beta^2\gamma\lambda_0 + 2\alpha\beta\gamma - 2\alpha\beta\gamma\lambda_0 + \gamma c = 0, \\ \alpha\beta^2 - 2\alpha^2\beta - \alpha\beta^2\lambda_0 + 2\alpha\gamma^2\lambda_0 + 2\alpha^2\beta\lambda_0 - \alpha c = 0. \end{cases} \tag{4.9}$$

By solving the first and second equations of (4.9) imply that

$$2\alpha\beta^2 + \beta\gamma^2 = 0.$$

Since $\gamma \neq 0$, we have $\beta = 0$. In this case, the first equation of (4.9) reduces to

$$2\alpha\gamma^2 = 0,$$

we get $\alpha = 0$. In this case, the third equation of (4.9) can be simplified to

$$\gamma^3 - 2\gamma^3\lambda_0 + \gamma c = 0,$$

then we have $c = \gamma^2(2\lambda_0 - 1)$. Thus we get Theorem 4.3.

Theorem 4.4. (G_3, g^Y, J) *is ASS of the second kind related to YC if it satisfies ASS of the first kind related to YC.*

Proof. For (G_3, ∇^Y), according to (4.1), we have

$$\tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) = -\beta\gamma,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_2^Y) = 0,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_3^Y) = 0,$$
$$\tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) = -\gamma^2 - \alpha\gamma,\ \tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_3^Y) = 0,\ \tilde{\rho}^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0.$$

By (4.2), the Ricci operator can be expressed as

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} -\beta\gamma & 0 & 0 \\ 0 & -\gamma^2 - \alpha\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

Since $\rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_j^Y, \tilde{q}_i^Y)$, then $\tilde{\rho}^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y)$. So (G_3, g^Y, J) is ASS of the second kind related to YC if it satisfies ASS of the first kind related to YC.

Theorem 4.5. *The LG G_4 cannot be ASS of a second kind related to the YC.*

Proof. For (G_4, ∇^Y), according to (4.1), we can get

$$\tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) = 2\beta\eta - \beta^2 - 1,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_2^Y) = 0,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_3^Y) = 0,$$
$$\tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) = 2\alpha\eta - \alpha\beta - 1,\ \tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_3^Y) = \frac{\alpha}{2},\ \tilde{\rho}^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0.$$

By (4.2), the Ricci operator can be expressed as

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} -\beta^2 + 2\beta\eta - 1 & 0 & 0 \\ 0 & 2\alpha\eta - \alpha\beta - 1 & -\frac{\alpha}{2} \\ 0 & \frac{\alpha}{2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y = -\beta^2 + 2\alpha\eta + 2\beta\eta - \alpha\beta - 2$. If (G_4, g^Y, J) is ASS of the second kind related to the YC, and by $\widetilde{Ric}^Y = (s^Y\lambda_0 + c)Id + D$, we can get

$$\begin{cases} D\tilde{q}_1^Y = (-\beta^2 + \beta^2\lambda_0 + 2\beta\eta - 2\alpha\eta\lambda_0 - 2\beta\eta\lambda_0 + \alpha\beta\lambda_0 + 2\lambda_0 - 1 - c)\tilde{q}_1^Y, \\ D\tilde{q}_2^Y = (\beta^2\lambda_0 + 2\alpha\eta - \alpha\beta - 2\alpha\eta\lambda_0 - 2\beta\eta\lambda_0 + \alpha\beta\lambda_0 + 2\lambda_0 - 1 - c)\tilde{q}_2^Y - \frac{\alpha}{2}\tilde{q}_3^Y, \\ D\tilde{q}_3^Y = \frac{\alpha}{2}\tilde{q}_2^Y + (\beta^2\lambda_0 - 2\alpha\eta\lambda_0 - 2\beta\eta\lambda_0 + \alpha\beta\lambda_0 + 2\lambda_0 - c)\tilde{q}_3^Y. \end{cases}$$

Therefore, by (4.6) and ASS of the second kind related to the YC can be established if it satisfies

$$\begin{cases} \alpha + (2\eta - \beta)(\beta^2 - \beta^2\lambda_0 - 2\alpha\eta - 2\beta\eta + \alpha\beta + 2\alpha\eta\lambda_0 + 2\beta\eta\lambda_0 - \alpha\beta\lambda_0 - 2\lambda_0 + 2 + c) = 0, \\ \beta^3 - \beta^3\lambda_0 - \alpha\beta^2 - 2\beta^2\eta + 2\beta^2\eta\lambda_0 - \alpha\beta^2\lambda_0 + 2\alpha\beta\eta + 2\alpha\beta\eta\lambda_0 - 2\beta\lambda_0 - \alpha + \beta c = 0, \\ \beta^2\lambda_0 - \beta^2 + \alpha\eta + 2\beta\eta - \alpha\beta - 2\alpha\eta\lambda_0 - 2\beta\eta\lambda_0 + \alpha\beta\lambda_0 + 2\lambda_0 - 1 - c = 0, \\ 2\alpha^2\eta - \alpha^2\beta + \alpha\beta^2 - 2\alpha^2\eta\lambda_0 + \alpha^2\beta\lambda_0 + \alpha\beta^2\lambda_0 - 2\alpha\beta\eta - 2\alpha\beta\eta\lambda_0 + 2\alpha\lambda_0 - \alpha c = 0. \end{cases} \tag{4.10}$$

By the first equation of (4.10), we assume that

$$\alpha = 0, \beta = 2\eta.$$

On this basis, by the second equation of (4.10), we get $c = 2\lambda_0$. By the third equation of (4.10), we have $c = 2\lambda_0 - 1$, and there is a contradiction. One can prove Theorem 4.5.

Theorem 4.6. *(G_5, g^Y, J) is ASS of the second kind related to YC if it satisfies ASS of the first kind related to YC.*

Proof. For (G_5, ∇^Y), according to (4.1), we can get

$$\tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) = \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_2^Y) = \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_3^Y) = 0,$$
$$\tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) = \tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_3^Y) = \tilde{\rho}^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0.$$

By (4.2), the Ricci operator can be expressed as

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

Since $\rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_j^Y, \tilde{q}_i^Y) = 0$, then $\tilde{\rho}^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = 0$. So (G_5, g^Y, J) is ASS of the second kind related to YC if it satisfies ASS of the first kind related to YC.

Theorem 4.7. *(G_6, g^Y, J) is ASS of the second kind related to YC if it satisfies ASS of the first kind related to YC.*

Proof. For (G_6, ∇^Y), according to (4.1), we have

$$\tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) = -(\beta\gamma + \alpha^2),\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_2^Y) = 0,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_3^Y) = 0,$$
$$\tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) = -\alpha^2,\ \tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_3^Y) = 0,\ \tilde{\rho}^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0.$$

By (4.2), the Ricci operator can be expressed as

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} -\alpha^2 - \beta\gamma & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

Since $\rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_j^Y, \tilde{q}_i^Y)$, then $\tilde{\rho}^Y(\tilde{q}_i^Y, \tilde{q}_j^Y) = \rho^Y(\tilde{q}_i^Y, \tilde{q}_j^Y)$. So (G_6, g^Y, J) is ASS of the second kind related to YC if it satisfies ASS of the first kind related to YC.

Theorem 4.8. *(G_7, g^Y, J) is ASS of the second kind related to YC if it satisfies $\alpha = \beta = \gamma = c = 0$, $\delta \neq 0$. And specifically*

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix},$$

$$D \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\delta^2+\delta}{2} \\ 0 & \frac{\delta^2+\delta}{2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

Proof. For (G_7, ∇^Y), according to (4.1), we can get

$$\tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_1^Y) = -\alpha^2,\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_2^Y) = \frac{\beta\delta - \alpha\beta}{2},\ \tilde{\rho}^Y(\tilde{q}_1^Y, \tilde{q}_3^Y) = \alpha\beta + \beta\delta,$$
$$\tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_2^Y) = -\alpha^2 - \beta^2 - \beta\gamma,\ \tilde{\rho}^Y(\tilde{q}_2^Y, \tilde{q}_3^Y) = \frac{\delta^2 + \delta + \alpha\delta + \beta\gamma}{2},\ \tilde{\rho}^Y(\tilde{q}_3^Y, \tilde{q}_3^Y) = 0.$$

According to (4.2), the Ricci operator can be expressed as

$$\widetilde{Ric}^Y \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix} = \begin{pmatrix} -\alpha^2 & \frac{\beta\delta-\alpha\beta}{2} & -(\alpha\beta + \beta\delta) \\ \frac{\beta\delta-\alpha\beta}{2} & -\alpha^2 - \beta^2 - \beta\gamma & -\frac{\delta^2+\delta+\alpha\delta+\beta\gamma}{2} \\ \alpha\beta + \beta\delta & \frac{\delta^2+\delta+\alpha\delta+\beta\gamma}{2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1^Y \\ \tilde{q}_2^Y \\ \tilde{q}_3^Y \end{pmatrix}.$$

As a result, the scalar curvature can be obtained as $s^Y = -2\alpha^2 - \beta^2 - \beta\gamma$. If (G_7, g^Y, J) is ASS of the second kind related to the YC, and by $\widetilde{Ric}^Y = (s^Y\lambda_0 + c)Id + D$, we can get

$$\begin{cases} D\tilde{q}_1^Y = (-\alpha^2 + 2\alpha^2\lambda_0 + \beta^2\lambda_0 + \beta\gamma\lambda_0 - c)\tilde{q}_1^Y + \frac{\beta\delta-\alpha\beta}{2}\tilde{q}_2^Y - (\alpha\beta + \beta\delta)\tilde{q}_3^Y, \\ D\tilde{q}_2^Y = \frac{\beta\delta-\alpha\beta}{2}\tilde{q}_1^Y + (-\alpha^2 - \beta^2 + 2\alpha^2\lambda_0 + \beta^2\lambda_0 - \beta\gamma + \beta\gamma\lambda_0 - c)\tilde{q}_2^Y - \frac{\delta^2+\delta+\alpha\delta+\beta\gamma}{2}\tilde{q}_3^Y, \\ D\tilde{q}_3^Y = (\alpha\beta + \beta\delta)\tilde{q}_1^Y + \frac{\delta^2+\delta+\alpha\delta+\beta\gamma}{2}\tilde{q}_2^Y + (2\alpha^2\lambda_0 + \beta^2\lambda_0 + \beta\gamma\lambda_0 - c)\tilde{q}_3^Y. \end{cases}$$

Therefore, by (4.6) and ASS of the second kind related to the YC can be established if it satisfies

$$\begin{cases} \alpha^3 - 2\alpha^3\lambda_0 + \frac{3}{2}\alpha\beta^2 - \frac{1}{2}\alpha\delta^2 - \frac{1}{2}\alpha^2\delta + \frac{3}{2}\beta^2\delta - \alpha\beta^2\lambda_0 - \frac{1}{2}\alpha\delta + \frac{3}{2}\alpha\beta\gamma + \beta\delta\gamma - \alpha\beta\gamma\lambda_0 + \alpha c = 0, \\ \beta^3\lambda_0 + 2\alpha^2\beta\lambda_0 + \beta^2\gamma\lambda_0 - \frac{1}{2}\alpha^2\beta - \beta\delta^2 - \frac{3}{2}\alpha\beta\delta - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 + \alpha^2\beta - 2\alpha^2\beta\lambda_0 - \beta^2\gamma\lambda_0 - \alpha\beta\delta - \beta\delta + \beta c = 0, \\ 2\alpha^3\lambda_0 - \frac{1}{2}\alpha^2\delta - \frac{1}{2}\alpha\delta^2 - \frac{3}{2}\beta^2\delta - \frac{1}{2}\alpha\beta^2 + \alpha\beta^2\lambda_0 - \frac{1}{2}\alpha\delta - \alpha\beta\gamma + \frac{1}{2}\beta\delta\gamma + \alpha\beta\gamma\lambda_0 - \alpha c = 0, \\ \beta^3 + \beta^3\lambda_0 + \frac{1}{2}\alpha^2\beta - \frac{1}{2}\delta^2\beta + 2\alpha^2\beta\lambda_0 + \beta^2\gamma\lambda_0 - 2\alpha\beta\delta - \beta c = 0, \\ \beta^3\lambda_0 + \frac{1}{2}\beta\delta^2 + 2\alpha^2\beta\lambda_0 + \beta^2\gamma\lambda_0 + \frac{1}{2}\alpha\beta\delta - \beta c = 0, \\ \frac{1}{2}\alpha^2\beta - \beta^2\gamma - \beta\gamma^2 - \frac{3}{2}\beta\delta^2 + 2\alpha^2\gamma\lambda_0 + \beta^2\gamma\lambda_0 + \beta\gamma^2\lambda_0 + \alpha\beta\delta - \gamma c = 0, \\ \frac{1}{2}\delta^3 - \frac{1}{2}\alpha\beta^2 - \frac{3}{2}\beta^2\delta + \frac{1}{2}\alpha\delta^2 + \frac{1}{2}\delta^2 - 2\alpha^2\delta\lambda_0 - \beta^2\delta\lambda_0 + \beta\gamma\delta - \frac{1}{2}\alpha\beta\gamma - \beta\delta\gamma\lambda_0 + \delta c = 0, \\ \frac{1}{2}\delta^3 + \frac{1}{2}\alpha\beta^2 + \frac{1}{2}\beta^2\delta - \alpha^2\delta + \frac{1}{2}\delta^2 + 2\alpha^2\delta\lambda_0 + \beta^2\delta\lambda_0 + \beta\delta + \frac{1}{2}\alpha\delta + \alpha\beta\gamma - \frac{1}{2}\beta\gamma\delta + \beta\gamma\delta\lambda_0 - \delta c = 0. \end{cases} \tag{4.11}$$

Because $\alpha + \delta = 0$ as well $\alpha\gamma = 0$. Let's first suppose $\alpha = 0$. On this basis, (4.11) reduces to

$$\begin{cases} \frac{3}{2}\beta^2\delta + \beta\delta\gamma = 0, \\ \beta^3\lambda_0 + \beta^2\gamma\lambda_0 - \beta\delta^2 - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 - \beta^2\gamma\lambda_0 - \beta\delta + \beta c = 0, \\ \frac{3}{2}\beta^2\delta - \frac{1}{2}\beta\delta\gamma = 0, \\ \beta^3 + \beta^3\lambda_0 - \frac{1}{2}\delta^2\beta + \beta^2\gamma\lambda_0 - \beta c = 0, \\ \beta^3\lambda_0 + \frac{1}{2}\beta\delta^2 + \beta^2\gamma\lambda_0 - \beta c = 0, \\ \beta^2\gamma + \beta\gamma^2 + \frac{3}{2}\beta\delta^2 - \beta^2\gamma\lambda_0 - \beta\gamma^2\lambda_0 + \gamma c = 0, \\ \frac{1}{2}\delta^3 - \frac{3}{2}\beta^2\delta + \frac{1}{2}\delta^2 - \beta^2\delta\lambda_0 + \beta\gamma\delta - \beta\delta\gamma\lambda_0 + \delta c = 0, \\ \frac{1}{2}\delta^3 + \frac{1}{2}\beta^2\delta + \frac{1}{2}\delta^2 + \beta^2\delta\lambda_0 + \beta\delta - \frac{1}{2}\beta\gamma\delta + \beta\gamma\delta\lambda_0 - \delta c = 0. \end{cases} \tag{4.12}$$

If $\gamma \neq 0$ as well $\delta \neq 0$, on this basis, the first and fourth equations of (4.12) can be simplified to

$$\beta\delta\gamma = 0,$$

we get $\beta = 0$. The seventh equation of (4.12) reduces to

$$\gamma c = 0,$$

we obtain $c = 0$. The eighth and ninth equations of (4.12) can be simplified to

$$\delta^3 - \delta^2 - 2\delta c = 0,$$

we have $c = \frac{1}{2}(\delta^2 + \delta)$, and there is a contradiction. If $\gamma = 0$ as well $\delta \neq 0$, on this basis, we calculate that

$$\begin{cases} \frac{3}{2}\beta^2\delta = 0, \\ \beta^3\lambda_0 - \beta\delta^2 - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 - \beta\delta + \beta c = 0, \\ \beta^3 + \beta^3\lambda_0 - \frac{1}{2}\delta^2\beta - \beta c = 0, \\ \beta^3\lambda_0 + \frac{1}{2}\beta\delta^2 - \beta c = 0, \\ \frac{3}{2}\beta\delta^2 = 0, \\ \frac{1}{2}\delta^3 - \frac{3}{2}\beta^2\delta + \frac{1}{2}\delta^2 - \beta^2\delta\lambda_0 + \delta c = 0, \\ \frac{1}{2}\delta^3 + \frac{1}{2}\beta^2\delta + \frac{1}{2}\delta^2 + \beta^2\delta\lambda_0 + \beta\delta - \delta c = 0. \end{cases} \tag{4.13}$$

By solving (4.13), we obtain $\beta = c = 0$. Suppose second that $\alpha \neq 0$, $\alpha + \delta \neq 0$ as well $\gamma = 0$. In this case, (4.11) can be simplified to

$$\begin{cases} \alpha^3 - 2\alpha^3\lambda_0 + \frac{3}{2}\alpha\beta^2 - \frac{1}{2}\alpha\delta^2 - \frac{1}{2}\alpha^2\delta + \frac{3}{2}\beta^2\delta - \alpha\beta^2\lambda_0 - \frac{1}{2}\alpha\delta + \alpha c = 0, \\ \beta^3\lambda_0 + 2\alpha^2\beta\lambda_0 - \frac{1}{2}\alpha^2\beta - \beta\delta^2 - \frac{3}{2}\alpha\beta\delta - \beta c = 0, \\ \beta^3 - \beta^3\lambda_0 + \alpha^2\beta - 2\alpha^2\beta\lambda_0 - \alpha\beta\delta - \beta\delta + \beta c = 0, \\ 2\alpha^3\lambda_0 - \frac{1}{2}\alpha^2\delta - \frac{1}{2}\alpha\delta^2 - \frac{3}{2}\beta^2\delta - \frac{1}{2}\alpha\beta^2 + \alpha\beta^2\lambda_0 - \frac{1}{2}\alpha\delta - \alpha c = 0, \\ \beta^3 + \beta^3\lambda_0 + \frac{1}{2}\alpha^2\beta - \frac{1}{2}\delta^2\beta + 2\alpha^2\beta\lambda_0 - 2\alpha\beta\delta - \beta c = 0, \\ \beta^3\lambda_0 + \frac{1}{2}\beta\delta^2 + 2\alpha^2\beta\lambda_0 + \frac{1}{2}\alpha\beta\delta - \beta c = 0, \\ \frac{1}{2}\alpha^2\beta - \frac{3}{2}\beta\delta^2 + \alpha\beta\delta = 0, \\ \frac{1}{2}\delta^3 - \frac{1}{2}\alpha\beta^2 - \frac{3}{2}\beta^2\delta + \frac{1}{2}\alpha\delta^2 + \frac{1}{2}\delta^2 - 2\alpha^2\delta\lambda_0 - \beta^2\delta\lambda_0 + \delta c = 0, \\ \frac{1}{2}\delta^3 + \frac{1}{2}\alpha\beta^2 + \frac{1}{2}\beta^2\delta - \alpha^2\delta + \frac{1}{2}\delta^2 + 2\alpha^2\delta\lambda_0 + \beta^2\delta\lambda_0 + \beta\delta + \frac{1}{2}\alpha\delta - \delta c = 0. \end{cases}$$

Next suppose that $\beta = 0$, we have

$$\begin{cases} \alpha^3 - 2\alpha^3\lambda_0 - \frac{1}{2}\alpha\delta^2 - \frac{1}{2}\alpha^2\delta - \frac{1}{2}\alpha\delta + \alpha c = 0, \\ 2\alpha^3\lambda_0 - \frac{1}{2}\alpha^2\delta - \frac{1}{2}\alpha\delta^2 - \frac{1}{2}\alpha\delta - \alpha c = 0, \\ \frac{1}{2}\delta^3 + \frac{1}{2}\alpha\delta^2 + \frac{1}{2}\delta^2 - 2\alpha^2\delta\lambda_0 + \delta c = 0, \\ \frac{1}{2}\delta^3 - \alpha^2\delta + \frac{1}{2}\delta^2 + 2\alpha^2\delta\lambda_0 + \frac{1}{2}\alpha\delta - \delta c = 0. \end{cases} \tag{4.14}$$

Then we get

$$\alpha^3 + \delta^3 - \frac{1}{2}\alpha\delta^2 + \delta^2 - \alpha^2\delta - \frac{1}{2}\alpha\delta = 0.$$

Let $\alpha = \lambda\delta$, $\lambda \neq 0$, it becomes

$$(\lambda^3 - \lambda^2 - \frac{1}{2}\lambda + 1)\delta^3 + (1 - \frac{1}{2}\lambda)\delta^2 = 0,$$

we have $\delta = \dfrac{\lambda - 2}{2\lambda^3 - 2\lambda^2 - \lambda + 2}$. For $\alpha = \lambda\delta$, (4.14) now reduces to

$$\begin{cases} \lambda^2\delta^2 - \frac{1}{2}\delta^2 - \frac{1}{2}\delta + \frac{1}{2}\lambda\delta^2 - 2\lambda^2\delta^2\lambda_0 + c = 0, \\ \frac{1}{2}\delta^2 + \frac{1}{2}\delta + \frac{1}{2}\lambda\delta^2 - 2\lambda^2\delta^2\lambda_0 + c = 0, \\ \lambda^2\delta^2 - \frac{1}{2}\delta^2 - \frac{1}{2}\delta - \frac{1}{2}\lambda\delta - 2\lambda^2\delta^2\lambda_0 + c = 0. \end{cases}$$

A simple computation demonstrates that the result is $\delta = -1$, then we get $\lambda = 1$, $\alpha = -1$. In this case, we have $c = -\frac{3}{2} + 2\lambda_0$ and $c = -\frac{1}{2} + 2\lambda_0$, so there is a contradiction. Thus it turns out Theorem 4.8.

5. Conlusions

We focus on the existence conditions of ASS related to YC in the context of three-dimensional LLG. We classify those ASS in three-dimensional LLG. The major results demonstrate that ASS related to YC are present in G_1, G_2, G_3, G_5, G_6 and G_7, while they are not identifiable in G_4. Based on this research, we will explore gradient Schouten solitons associated with YC using the theories in [31–33].

Use of AI tools declaration

During writing this work, the authors confirm that they are not using any AI techniques.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. R. S. Hamilton, Three Manifold with positive Ricci curvature, *J. Differential Geom.*, **17** (1982), 255–306. https://doi.org/10.4310/jdg/1214436922
2. R. S. Hamilton, The Ricci flow on surfaces, *Contemp. Math.*, **71** (1988), 237–262. https://doi.org/10.1090/conm/071/954419
3. V. Rovenski, Generalized Ricci solitons and Einstein metrics on weak κ-contact manifolds, *Commun. Anal. Mech.*, **15** (2023), 177–188. https://doi.org/10.3934/cam.2023010
4. A. Arfah, Geometry of semi-Riemannian group manifold and its applications in spacetime admitting Ricci solitons, *Intern. J. Geom. Methods Modern Phys.*, **18** (2021), 2150233. https://doi.org/10.1142/S0219887821502339
5. T. Wu, Y. Wang, Affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups, *Turk. J. Math.*, **45** (2021), 2773–2816. https://doi.org/10.3906/mat-2105-49
6. K. Onda, Examples of algebraic Ricci solitons in the pseudo-Riemannian case, *Acta Math. Hung.*, **144** (2014), 247–265. https://doi.org/10.1007/s10474-014-0426-0
7. S. Azami, Affine Generalized Ricci Solitons of Three-Dimensional Lorentzian Lie Groups Associated to Yano Connection, *J. Nonlinear Math. Phys.*, (2023), 1–24. https://doi.org/10.1007/s44198-022-00104-2
8. P. N. Klepikov, D. N. Oskorbin, E. D. Rodionov, Homogeneous Ricci solitons on four-dimensional Lie groups with a left-invariant Riemannian metric, *Dokl Math.*, **92** (2015), 701–703. https://doi.org/10.1134/S1064562415060150
9. J. Lauret, Ricci soliton homogeneous nilmanifolds, *Math. Ann.*, **319** (2001), 715–733. https://doi.org/10.1007/PL00004456
10. W. Batat, K. Onda, Algebraic Ricci solitons of three-dimensional Lorentzian Lie groups, *J. Geom. Phys.*, **114** (2017), 138–152. https://doi.org/10.1016/j.geomphys.2016.11.018
11. Y. Wang, Affine connections and Gauss-Bonnet theorems in the Heisenberg group, *arXiv preprint*, arXiv:2021.01907 (2021). https://doi.org/10.48550/arXiv.2102.01907

12. Z. Balogh, J. Tyson, E. Vecchi, Intrinsic curvature of curves and surfaces and a Gauss-Bonnet theorem in the Heisenberg group, *Math. Z.*, **287** (2017), 1–38. https://doi.org/10.1007/s00209-016-1815-6

13. Y. Wang, Canonical connections and algebraic Ricci solitons of three-dimensional Lorentzian Lie groups, *Chin. Ann. Math. Ser. B.*, **43** (2022), 443–458. https://doi.org/10.1007/s11401-022-0334-5

14. M. Brozos-Vazquez, E. Garcia-Rio, S. Gavino-Fernandez, Locally conformally flat Lorentzian gradient Ricci solitons, *J. Geom. Anal.*, **23** (2013), 1196–1212. https://doi.org/10.1007/s12220-011-9283-z

15. Y. Wang, Affine Ricci soliton of three-dimensional Lorentzian Lie groups, *J. Nonlinear Math. Phys.*, **28** (2021), 277–291.

16. Y. L. Li, S. Dey, S. Pahan, A. Ali, Geometry of conformal η-Ricci solitons and conformal η-Ricci almost solitons on Paracontact geometry, *Open Math.*, **20** (2022), 574–589. https://doi.org/10.1515/math-2022-0048

17. G. Calvaruso, Three-dimensional homogeneous generalized Ricci solitons, *Mediterr. J. Math.*, **14** (2017), 216. https://doi.org/10.1007/s00009-017-1019-2

18. S. Azami, Generalized Ricci solitons of three-dimensional Lorentzian Lie groups associated canonical connections and Kobayashi-Nomizu connections, *J. Nonlinear Math. Phys.*, **30** (2023), 1–33. https://doi.org/10.1007/s44198-022-00069-2

19. E. Calvino-Louzao, L. M. Hervella, J. Seoane-Bascoy, R. Vazquez-Lorenzo, Homogeneous Cotton solitons, *J. Phys. A: Math. Theor.*, **46** (2013), 285204. https://doi.org/10.1088/1751-8113/46/28/285204

20. T. H. Wears, Homogeneous Cotton solitons,On algebraic solitons for geometric evolution equations on three-dimensional Lie groups, *Tbil Math. J.*, **9** (2016), 33–58. https://doi.org/10.1515/tmj-2016-0018

21. F. Etayo, R. Santamaría, Distinguished connections on $(J^2 = \pm 1)$-metric manifolds, *Arch. Math.*, **52** (2016), 159–203. https://doi.org/10.5817/AM2016-3-159

22. T. Wu, S. Wei, Y. Wang, Gauss-Bonnet theorems and the Lorentzian Heisenberg group, *Turk. J. Math.*, **45** (2021), 718–741. https://doi.org/10.3906/mat-2011-19

23. Z. Chen, Y. Li, S. Sarkar, S. Dey, A. Bhattacharyya, Ricci Soliton and Certain Related Metrics on a Three-Dimensional Trans-Sasakian Manifold, *Universe*, **8** (2022), 595. https://doi.org/10.3390/universe8110595

24. Y. Wang, X. Liu, Ricci solitons on three-dimensional η-Einstein almost Kenmotsu manifolds, *Taiwanese J. Math.*, **19** (2015), 91–100. https://doi.org/10.11650/tjm.19.2015.4094

25. X. M. Chen, Almost Quasi-Yamabe Solitons on Almost Cosymplectic Manifolds, *Intern. J. Geom. Methods Modern Phys.*, **17** (2020), 2050070. https://doi.org/10.1142/S021988782050070X

26. X. M. Chen, The κ-almost Yamabe solitons and almost coKähler manifolds, *Intern. J. Geom. Methods Modern Phys.*, **18** (2021), 2150179. https://doi.org/10.1142/S0219887821501796

27. G. Calvaruso, Homogeneous structures on three-dimensional Lorentzian manifolds, *J. Geom. Phys.*, **57** (2007), 1279–1291. https://doi.org/10.1016/j.geomphys.2006.10.005

28. L. A. Cordero, P. E. Parker, Left-invariant Lorentzian metrics on 3-dimensional Lie groups, *Rend. Lincei-Mat Appl.*, **17** (1997), 129–155.

29. H. R. Salimi, On the geometry of some para-hypercomplex Lie groups, *Arch. Math.*, **45** (2009), 159–170. https://doi.org/10.48550/arXiv.1305.2855

30. G. Calvaruso, Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds, *Geometriae Dedicata.*, **127** (2007), 99–119. https://doi.org/10.1007/s10711-007-9163-7

31. V. Borges, On complete gradient Schouten solitons, *Nonlinear Anal.*, **221** (2022), 112883. https://doi.org/10.1016/j.na.2022.112883

32. L. Ju, J. Zhou, Y. Zhang, Conservation laws analysis of nonlinear partial differential equations and their linear soliton solutions and Hamiltonian structures, *Commun. Anal. Mech.*, **15** (2023), 24–49. https://doi.org/10.3934/cam.2023002

33. J. Zhang, S. Zhu, On criticality coupled sub-Laplacian systems with Hardy type potentials on Stratified Lie groups, *Commun. Anal. Mech.*, **15** (2023), 70–90. https://doi.org/10.3934/cam.2023005