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Research article

Global existence and uniform boundedness to a bi-attraction chemotaxis system with nonlinear indirect signal mechanisms

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Abstract: In this paper, we study the following quasilinear chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(\varphi(u)\nabla v) - \xi\nabla\cdot(\psi(u)\nabla w) + f(u), & x\in\Omega,\ t>0,\\ 0 = \Delta v - v + v_1^{\gamma_1},\ 0 = \Delta v_1 - v_1 + u^{\gamma_2}, & x\in\Omega,\ t>0,\\ 0 = \Delta w - w + w_1^{\gamma_3},\ 0 = \Delta w_1 - w_1 + u^{\gamma_4}, & x\in\Omega,\ t>0, \end{cases}$$

in a smoothly bounded domain $\Omega\subset\mathbb{R}^n (n\geq 1)$ with homogeneous Neumann boundary conditions, where $\varphi(\varrho)\leq\varrho(\varrho+1)^{\theta-1}$, $\psi(\varrho)\leq\varrho(\varrho+1)^{l-1}$ and $f(\varrho)\leq a\varrho-b\varrho^s$ for all $\varrho\geq 0$, and the parameters satisfy $a,b,\chi,\xi,\gamma_2,\gamma_4>0$, $s>1$, $\gamma_1,\gamma_3\geq 1$ and $\theta,l\in\mathbb{R}$. It has been proven that if $s\geq\max\{\gamma_1\gamma_2+\theta,\gamma_3\gamma_4+l\}$, then the system has a nonnegative classical solution that is globally bounded. The boundedness condition obtained in this paper relies only on the power exponents of the system, which is independent of the coefficients of the system and space dimension n. In this work, we generalize the results established by previous researchers.

Keywords: bi-attraction chemotaxis system; nonlinear indirect signal; global boundedness
Mathematics Subject Classification: 35K55, 35Q92, 35A01, 92C17

1. Introduction

Chemotaxis is a physiological phenomenon of organisms seeking benefits and avoiding harm, which has been widely concerned in the fields of both mathematics and biology. In order to depict such phenomena, in 1970, Keller and Segel [1] established the first mathematical model (also called the

Keller-Segel model). The general form of this model is described as follows

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + g(u), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, the value of τ can be chosen by 0 or 1 and the chemotaxis sensitivity coefficient $\chi > 0$. Here, u is the density of cell or bacteria and v stands for the concentration of chemical signal secreted by cell or bacteria. The functions $f(u)$ and $g(u)$ are used to characterize the growth and death of cells or bacteria and production of chemical signals, respectively.

Over the past serval decades, considerable efforts have been done on the dynamical behavior (including the global existence and boundedness, the convergence as well as the existence of blow-up solutions) of the solutions to system (1.1) (see [2–7]). Let us briefly recall some contributions among them in this direction. For example, assume that $f(u) = 0$ and $g(u) = u$. For $\tau = 1$, it has been shown that the classical solutions to system (1.1) always remain globally bounded when $n = 1$ [8]. Additionally, there will be a critical mass phenomenon to system (1.1) when $n = 2$, namely, if the initial data u_0 fulfill $\int_\Omega u_0 dx < \frac{4\pi}{\chi}$, the classical solutions are globally bounded [9]; and if $\int_\Omega u_0 dx > \frac{4\pi}{\chi}$, the solutions will blow up in finite time [10,11]. However, when $n \geq 3$, Winkler [12,13] showed that though the initial data satisfy some smallness conditions, the solutions will blow up either in finite or infinite time. Assume that the system (1.1) involves a non-trivial logistic source and $g(u) = u$. For $\tau = 0$ and $f(u) \leq a - \mu u^2$ with $a \geq 0$ and $\mu > 0$, Tello and Winkler [14] obtained that there exists a unique global classical solution for system (1.1) provided that $n \leq 2, \mu > 0$ or $n \geq 3$ and suitably large $\mu > 0$. Furthermore, for $\tau > 0$ and $n \geq 1$, suppose that Ω is a bounded convex domain. Winkler [15] proved that the system (1.1) has global classical solutions under the restriction that $\mu > 0$ is sufficiently large. When $\tau = 1$ and $f(u) = u - \mu u^2$, Winkler [16] showed that nontrivial spatially homogeneous equilibrium $(\frac{1}{\mu}, \frac{1}{\mu})$ is globally asymptotically stable provided that the ratio $\frac{\mu}{\chi}$ is sufficiently large and Ω is a convex domain. Later, based on maximal Sobolev regularity, Cao [17] also obtained the similar convergence results by removing the restrictions $\tau = 1$ and the convexity of Ω required in [16]. In addition, for the more related works in this direction, we mention that some variants of system (1.1), such as the attraction-repulsion systems (see [18–21]), the chemotaxis-haptotaxis models (see [22–24]), the Keller-Segel-Navier-Stokes systems (see [25–30]) and the pursuit-evasion models (see [31–33]), have been deeply investigated.

Recently, the Keller-Segel model with nonlinear production mechanism of the signal (i.e. $g(u)$ is a nonlinear function with respect to u) has attracted widespread attention from scholars. For instance, when the second equation in (1.1) satisfies $v_t = \Delta v - v + g(u)$ with $0 \leq g(u) \leq K u^\alpha$ for $K, \alpha > 0$, Liu and Tao [34] obtained the global existence of classical solutions under the condition that $0 < \alpha < \frac{2}{n}$. When $f(u) \leq u(a - bu^s)$ and the second equation becomes $0 = \Delta v - v + u^k$ with $k, s > 0$, Wang and Xiang [35] showed that if either $s > k$ or $s = k$ with $\frac{kn-2}{kn}\chi < b$, the system (1.2) has global classical solutions. When the second equation in (1.1) turns into $0 = \Delta v - \frac{1}{|\Omega|}\int_\Omega g(u) + g(u)$ for $g(u) = u^\kappa$ with $\kappa > 0$, Winkler [36] showed that the system has a critical exponent $\frac{2}{n}$ such that if $\kappa > \frac{2}{n}$, the solution blows up in finite time; conversely, if $\kappa < \frac{2}{n}$, the solution is globally bounded with respect to t. More results on Keller-Segel model with logistic source can be found in [6, 37–40].

In addition, previous contributions also imply that diffusion functions may lead to colorful dynamic behaviors. The corresponding model can be given by

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $D(u)$ and $S(u)$ are positive functions that are used to characterize the strength of diffusion and chemoattractants, respectively. When $D(u)$ and $S(u)$ are nonlinear functions of u, Tao-Winkler [41] and Winkler [42] proved that the existence of global classical solutions or blow-up solutions depend on the value of $\frac{S(u)}{D(u)}$. Namely, if $\frac{S(u)}{D(u)} \geq cu^{\alpha}$ with $\alpha > \frac{2}{n}, n \geq 2$ and $c > 0$ for all $u > 1$, then for any $M > 0$ there exist solutions that blow up in either finite or infinite time with mass $\int_{\Omega} u_0 = M$ in [42]. Later, Tao and Winkler [41] showed that such a result is optimal, i.e., if $\frac{S(u)}{D(u)} \leq cu^{\alpha}$ with $\alpha < \frac{2}{n}, n \geq 1$ and $c > 0$ for all $u > 1$, then the system (1.2) possesses global classical solutions, which are bounded in $\Omega \times (0, \infty)$. Furthermore, Zheng [43] studied a logistic-type parabolic-elliptic system with $u_t = \nabla \cdot ((u+1)^{m-1}\nabla u) - \chi \nabla(u(u+1)^{q-1}\nabla v) + au - bu^r$ and $0 = \Delta v - v + u$ for $m \geq 1, r > 1, a \geq 0, b, q, \chi > 0$. It is shown that when $q+1 < \max\{r, m+\frac{2}{n}\}$, or $b > b_0 = \frac{n(r-m)-2}{(r-m)n+2(r-2)}\chi$ if $q+1 = r$, then for any sufficiently smooth initial data there exists a classical solution that is global in time and bounded. For more relevant results, please refer to [38, 44–46].

In the Keller-Segel model mentioned above, the chemical signals are secreted by cell population, directly. Nevertheless, in reality, the production of chemical signals may go through very complex processes. For example, signal substance is not secreted directly by cell population but is produced by some other signal substance. Such a process may be described as the following system involving an indirect signal mechanism

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + w, \tau w_t = \Delta w - w + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where u represents the density of cell, v and w denote the concentration of chemical signal and indirect chemical signal, respectively. For $\tau = 1$, assume that $f(u) = \mu(u - u^{\gamma})$ with $\mu, \gamma > 0$, Zhang-Niu-Liu [47] showed that the system has global classical solutions under the condition that $\gamma > \frac{n}{4} + \frac{1}{2}$ with $n \geq 2$. Such a boundedness result was also extended to a quasilinear system in [48, 49]. Ren [50] studied system (1.3) and obtained the global existence and asymptotic behavior of generalized solutions. For $\tau = 0$, Li and Li [51] investigated the global existence and long time behavior of classical solutions for a quasilinear version of system (1.3). In [52], we extended Li and Li's results to a quasilinear system with a nonlinear indirect signal mechanism. More relevant results involving indirect signal mechanisms can be found in [53–56].

In the existing literatures, the indirect signal secretion mechanism is usually a linear function of u. However, there are very few papers that study the chemotaxis system, where chemical signal production is not only indirect but also nonlinear. Considering the complexity of biological processes, such signal production mechanisms may be more in line with the actual situation. Thus, in this paper, we study the following chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(\varphi(u)\nabla v) - \xi\nabla\cdot(\psi(u)\nabla w) + f(u), & x\in\Omega,\ t>0,\\ 0 = \Delta v - v + v_1^{\gamma_1},\ 0 = \Delta v_1 - v_1 + u^{\gamma_2}, & x\in\Omega,\ t>0,\\ 0 = \Delta w - w + w_1^{\gamma_3},\ 0 = \Delta w_1 - w_1 + u^{\gamma_4}, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial v_1}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \end{cases} \tag{1.4}$$

where $\Omega\subset\mathbb{R}^n(n\geq 1)$ is a smoothly bounded domain and ν denotes the outward unit normal vector on $\partial\Omega$, the parameters $\chi,\xi,\gamma_2,\gamma_4>0$, and $\gamma_1,\gamma_3\geq 1$. The initial data $u(x,0)=u_0(x)$ satisfy some smooth conditions. Here, the nonlinear functions are assumed to satisfy

$$\varphi,\psi\in C^2([0,\infty)),\ \varphi(\varrho)\leq\varrho(\varrho+1)^{\theta-1} \text{ and } \psi(\varrho)\leq\varrho(\varrho+1)^{l-1} \text{ for all } \varrho\geq 0, \tag{1.5}$$

with $\theta,l\in\mathbb{R}$. The logistic source $f\in C^\infty([0,\infty))$ is supposed to satisfy

$$f(0)\geq 0 \text{ and } f(\varrho)\leq a\varrho - b\varrho^s \text{ for all } \varrho\geq 0, \tag{1.6}$$

with $a,b>0$ and $s>1$. The purpose of this paper is to detect the influence of power exponents (instead of the coefficients and space dimension n) of the system (1.4) on the existence and boundedness of global classical solutions.

We state our main result as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^n(n\geq 1)$ be a bounded domain with smooth boundary and the parameters fulfill $\xi,\chi,\gamma_2,\gamma_4>0$ and $\gamma_1,\gamma_3\geq 1$. Assume that the nonlinear functions φ,ψ and f satisfy the conditions* (1.5) *and* (1.6) *with $a,b>0$, $s>1$ and $\theta,l\in\mathbb{R}$. If $s\geq\max\{\gamma_1\gamma_2+\theta,\gamma_3\gamma_4+l\}$, then for any nonnegative initial data $u_0\in W^{1,\infty}(\Omega)$, the system* (1.4) *has a nonnegative global classical solution*

$$(u,v,v_1,w,w_1)\in(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))\times(C^{2,0}(\overline{\Omega}\times(0,\infty)))^4.$$

Furthermore, this solution is bounded in $\Omega\times(0,\infty)$, in other words, there exists a constant $C>0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|(v(\cdot,t),v_1(\cdot,t),w(\cdot,t),w_1(\cdot,t))\|_{W^{1,\infty}(\Omega)} < C$$

for all $t>0$.

The system (1.4) is a bi-attraction chemotaxis model, which can somewhat be seen as a variant of the classical attraction-repulsion system proposed by Luca [57]. In [58], Hong-Tian-Zheng studied an attraction-repulsion model with nonlinear productions and obtained the buondedness conditions which not only depend on the power exponents of the system, but also rely on the coefficients of the system as well as space dimension n. Based on [58], Zhou-Li-Zhao [59] further improved such boundedness results to some critical conditions. Compared to [58] and [59], the boundedness condition developed in Theorem 1.1 relies only on the power exponents of the system, which removes restrictions on the coefficients of the system and space dimension n. The main difficulties in the proof of Theorem 1.1 are how to reasonably deal with the integrals with power exponents in obtaining the estimate of $\int_\Omega(u+1)^p$ in Lemma 3.1. Based on a prior estimates of solutions (Lemma 2.2) and some scaling techniques of inequalities, we can overcome these difficulties and then establish the conditions of global boundedness.

The rest of this paper is arranged as follows. In Sec.2, we give a result on local existence of classical solutions and get some estimates of solutions. In Sec.3, we first prove the boundedness of $\int_\Omega(u+1)^p$ and then complete the proof of Theorem 1.1 based on the Moser iteration [41, Lemma A.1].

2. Preliminaries

To begin with, we state a lemma involving the local existence of classical solutions and get some estimates on the solutions of system (1.4).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters fulfill $\xi, \chi, \gamma_2, \gamma_4 > 0$ and $\gamma_1, \gamma_3 \geq 1$. Assume that the nonlinear functions φ, ψ and f satisfy the conditions* (1.5) *and* (1.6) *with $a, b > 0$, $s > 1$ and $\theta, l \in \mathbb{R}$. For any nonnegative initial data $u_0 \in W^{1,\infty}(\Omega)$, there exists $T_{\max} \in (0, \infty]$ and nonnegative functions*

$$(u, v, v_1, w, w_1) \in (C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))) \times (C^{2,0}(\overline{\Omega} \times (0, T_{\max})))^4,$$

which solve system (1.4) *in classical sense. Furthermore,*

$$\text{if } T_{\max} < \infty, \text{ then } \lim_{t \nearrow T_{\max}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

Proof. The proof relies on the Schauder fixed point theorem and partial differential regularity theory, which is similar to [60, Lemma 2.1]. For convenience, we give a proof here. For any $T \in (0, 1)$ and the nonnegative initial data $u_0 \in W^{1,\infty}$, we set

$$X := C^0(\overline{\Omega} \times [0, T]) \text{ and } S := \left\{ u \in X \middle| \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq R \text{ for all } t \in [0, T] \right\},$$

where $R := \|u_0\|_{L^\infty(\Omega)} + 1$. We can pick smooth functions φ_R, ψ_R on $[0, \infty)$ such that $\varphi_R \equiv \varphi$ and $\psi_R \equiv \psi$ when $0 \leq \varrho \leq R$ and $\varphi_R \equiv R$ and $\psi_R \equiv R$ when $\varrho \geq R$. It is easy to see that S is a bounded closed convex subset of X. For any $\hat{u} \in S$, let v, v_1, w and w_1 solve

$$\begin{cases} -\Delta v + v = v_1^{\gamma_1}, & x \in \Omega,\ t \in (0, T), \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v_1 + v_1 = \hat{u}^{\gamma_2}, & x \in \Omega,\ t \in (0, T), \\ \frac{\partial v_1}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \end{cases} \tag{2.2}$$

as well as

$$\begin{cases} -\Delta w + w = w_1^{\gamma_3}, & x \in \Omega,\ t \in (0, T), \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta w_1 + w_1 = \hat{u}^{\gamma_4}, & x \in \Omega,\ t \in (0, T), \\ \frac{\partial w_1}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \end{cases} \tag{2.3}$$

respectively, in turn, let u be a solution of

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\varphi_R(u) \nabla v) - \xi \nabla \cdot (\psi_R(u) \nabla w) + f(u), & x \in \Omega,\ t \in (0, T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial v_1}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.4}$$

Thus, we introduce a map $\Phi : \hat{u}(\in S) \mapsto u$ defined by $\Phi(\hat{u}) = u$. We shall show that for any $T > 0$ sufficiently small, Φ has a fixed point in S. Using the elliptic regularity [61, Theorem 8.34] and Morrey's theorem [62], for a certain fixed $\hat{u} \in S$, we conclude that the solutions to (2.2) satisfy $v_1(\cdot, t) \in C^{1+\delta}(\Omega)$ and $v(\cdot, t) \in C^{3+\delta}(\Omega)$ for all $\delta \in (0, 1)$, as well as the solutions to (2.3) satisfy $w_1(\cdot, t) \in C^{1+\delta}(\Omega)$ and

$w(\cdot,t) \in C^{3+\delta}(\Omega)$ for all $\delta \in (0,1)$. From the Sobolev embedding theorem and L^p-estimate, there exist $m_i > 0, i = 1, ..., 4$ such that

$$\|\nabla v_1\|_{L^\infty\left((0,T);C^\delta(\Omega)\right)} \le m_1\|v_1\|_{L^\infty\left((0,T);W^{2,p}(\Omega)\right)} \le m_2\|\hat{u}^{\gamma_2}\|_{L^\infty\left((0,T)\times\Omega\right)}$$

and

$$\|\nabla w_1\|_{L^\infty\left((0,T);C^\delta(\Omega)\right)} \le m_3\|w_1\|_{L^\infty\left((0,T);W^{2,p}(\Omega)\right)} \le m_4\|\hat{u}^{\gamma_4}\|_{L^\infty\left((0,T)\times\Omega\right)}$$

for $p > \max\{1, n\gamma_1\gamma_2, n\gamma_3\gamma_4\}$. Furthermore, we can also find $m_i > 0, i = 5, ..., 10$ such that

$$\|\nabla v\|_{L^\infty\left((0,T);C^\delta(\Omega)\right)} \le m_5\|v\|_{L^\infty\left((0,T);W^{2,p}(\Omega)\right)} \le m_6\|v_1^{\gamma_1}\|_{L^\infty\left((0,T)\times\Omega\right)} \le m_7\|\hat{u}^{\gamma_1\gamma_2}\|_{L^\infty\left((0,T)\times\Omega\right)}$$

and

$$\|\nabla w\|_{L^\infty\left((0,T);C^\delta(\Omega)\right)} \le m_8\|w\|_{L^\infty\left((0,T);W^{2,p}(\Omega)\right)} \le m_9\|w_1^{\gamma_3}\|_{L^\infty\left((0,T)\times\Omega\right)} \le m_{10}\|\hat{u}^{\gamma_3\gamma_4}\|_{L^\infty\left((0,T)\times\Omega\right)}$$

for $p > \max\{1, n\gamma_1\gamma_2, n\gamma_3\gamma_4\}$. Since $\nabla v, \nabla w \in L^\infty((0,T)\times\Omega)$ and $u_0 \in C^\delta(\overline{\Omega})$ for all $\delta \in (0,1)$ due to the Sobolev embedding $W^{1,\infty}(\Omega) \hookrightarrow C^\delta(\Omega)$, we can infer from [63, Theorem V1.1] that $u \in C^{\delta,\frac{\delta}{2}}(\overline{\Omega}\times[0,T])$ and

$$\|u\|_{C^{\delta,\frac{\delta}{2}}}(\overline{\Omega}\times[0,T]) \le m_{11} \text{ for all } \delta \in (0,1), \tag{2.5}$$

with some $m_{11} > 0$ depending only on $\|\nabla v\|_{L^\infty\left((0,T);C^\delta(\Omega)\right)}, \|\nabla w\|_{L^\infty\left((0,T);C^\delta(\Omega)\right)}$ and R. Thus, we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \|u(\cdot,t) - u_0\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + m_{11}t^{\frac{\delta}{2}}. \tag{2.6}$$

Hence if $T < (\frac{1}{m_{11}})^{\frac{2}{\delta}}$, we can obtain

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + 1 \tag{2.7}$$

for all $t \in [0,T]$, which implies that $u \in S$. Thus, we derive that Φ maps S into itself. We deduce that Φ is continuous. Moreover, we get from (2.5) that Φ is a compact map. Hence, by the Schauder fixed point theorem there exists a fixed point $u \in S$ such that $\Phi(u) = u$.

Applying the regularity theory of elliptic equations, we derive that $v_1(\cdot,t) \in C^{2+\delta}(\Omega)$, $v(\cdot,t) \in C^{4+\delta}(\Omega)$ and $w_1(\cdot,t) \in C^{2+\delta}(\Omega)$, $w(\cdot,t) \in C^{4+\delta}(\Omega)$ for all $\delta \in (0,1)$. Recalling (2.5), we get $v_1(x,t) \in C^{2+\delta,\frac{\delta}{2}}(\Omega\times[\iota,T])$, $v(x,t) \in C^{4+\delta,\frac{\delta}{2}}(\Omega\times[\iota,T])$ and $w_1(x,t) \in C^{2+\delta,\frac{\delta}{2}}(\Omega\times[\iota,T])$, $w(x,t) \in C^{4+\delta,\frac{\delta}{2}}(\Omega\times[\iota,T])$ for all $\delta \in (0,1)$ and $\iota \in (0,T)$. We use the regularity theory of parabolic equation [63, Theorem V6.1] to get

$$u(x,t) \in C^{2+\delta,1+\frac{\delta}{2}}(\overline{\Omega}\times[\iota,T])$$

for all $\iota \in (0,T)$. The solution may be prolonged in the interval $[0,T_{\max})$ with either $T_{\max} = \infty$ or $T_{\max} < \infty$, where in the later case

$$\|u(\cdot.t)\|_{L^\infty(\Omega)} \to \infty \ \ as \ \ t \to T_{\max}.$$

Additionally, since $f(0) \ge 0$, we thus get from the parabolic comparison principle that u is nonnegative. By employing the elliptic comparison principle to the second, the third, the fourth and the fifth equations in (1.4), we conclude that v, v_1, w, w_1 are also nonnegative. Thus, we complete the proof of Lemma 2.1.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters fulfill $\xi, \chi, \gamma_2, \gamma_4 > 0$ and $\gamma_1, \gamma_3 \geq 1$. Assume that the nonlinear functions φ, ψ and f satisfy the conditions* (1.5) *and* (1.6) *with $a, b > 0, s > 1$ and $\theta, l \in \mathbb{R}$. For any $\eta_1, \eta_2, \eta_3, \eta_4 > 0$ and $\tau > 1$, we can find $c_1, c_2, c_3, c_4 > 0$ which depend only on $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \eta_1, \eta_2, \eta_3, \eta_4, \tau$, such that*

$$\int_\Omega w_1^\tau \leq \eta_2 \int_\Omega (u+1)^{\gamma_4 \tau} + c_1 \text{ and } \int_\Omega w^\tau \leq \eta_1 \eta_2 \int_\Omega (u+1)^{\gamma_3 \gamma_4 \tau} + c_2, \tag{2.8}$$

as well as

$$\int_\Omega v_1^\tau \leq \eta_4 \int_\Omega (u+1)^{\gamma_2 \tau} + c_3 \text{ and } \int_\Omega v^\tau \leq \eta_3 \eta_4 \int_\Omega (u+1)^{\gamma_1 \gamma_2 \tau} + c_4 \tag{2.9}$$

for all $t \in (0, T_{\max})$.

Proof. Integrating the first equation of system (1.4) over Ω and using Hölder's inequality, it is easy to get that

$$\frac{d}{dt} \int_\Omega u dx \leq \int_\Omega au - bu^s \leq a \int_\Omega u - \frac{b}{|\Omega|^{s-1}} \left(\int_\Omega u \right)^s \quad \text{for all } t \in (0, T_{\max}). \tag{2.10}$$

Employing the standard ODE comparison theory, we conclude

$$\int_\Omega u \leq \max \left\{ \int_\Omega u_0, \left(\frac{a}{b} \right)^{\frac{1}{s-1}} |\Omega| \right\} \quad \text{for all } t \in (0, T_{\max}). \tag{2.11}$$

Moreover, integrating the fifth equation of system (1.4) over Ω, one may get

$$\|w_1\|_{L^1(\Omega)} = \|u^{\gamma_4}\|_{L^1(\Omega)} \leq \|(u+1)^{\gamma_4}\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \tag{2.12}$$

For any $\tau > 1$, multiplying the fifth equation of system(1.4) with $w_1^{\tau-1}$, we can get by integration by parts that

$$\frac{4(\tau-1)}{\tau^2} \int_\Omega |\nabla w_1^{\frac{\tau}{2}}|^2 + \int_\Omega w_1^\tau = \int_\Omega u^{\gamma_4} w_1^{\tau-1} \leq \frac{\tau-1}{\tau} \int_\Omega w_1^\tau + \frac{1}{\tau} \int_\Omega u^{\gamma_4 \tau}, \tag{2.13}$$

where Young's inequality has been used. Thus, we deduce

$$\|w_1\|_{L^\tau(\Omega)} \leq \|u^{\gamma_4}\|_{L^\tau(\Omega)} \leq \|(u+1)^{\gamma_4}\|_{L^\tau(\Omega)} \quad \text{for all } t \in (0, T_{\max}), \tag{2.14}$$

and

$$\frac{4(\tau-1)}{\tau} \int_\Omega |\nabla w_1^{\frac{\tau}{2}}|^2 \leq \int_\Omega u^{\gamma_4 \tau} \leq \int_\Omega (u+1)^{\gamma_4 \tau} \quad \text{for all } t \in (0, T_{\max}). \tag{2.15}$$

Using Ehrling's lemma, we know that for any $\eta_2 > 0$ and $\tau > 1$, there exists $c_5 = c_5(\eta_2, \tau) > 0$ such that

$$\|\phi\|^2_{L^2(\Omega)} \leq \eta_2 \|\phi\|^2_{W^{1,2}(\Omega)} + c_5 \|\phi\|^2_{L^{\frac{2}{\tau}}(\Omega)} \quad \text{for all } \phi \in W^{1,2}(\Omega). \tag{2.16}$$

Let $\phi = w_1^{\frac{\tau}{2}}$. Combining (2.12) with (2.14), (2.15), there exists $c_6 = c_6(\eta_2, \gamma_4, \tau) > 0$ such that

$$\int_\Omega w_1^\tau \leq \eta_2 \int_\Omega (u+1)^{\gamma_4 \tau} + c_6 \|(u+1)^{\gamma_4}\|^\tau_{L^1(\Omega)}. \tag{2.17}$$

If $\gamma_4 \in (0,1]$, by (2.11) and Hölder's inequality, we can derive

$$\|(u+1)^{\gamma_4}\|_{L^1(\Omega)}^{\tau} \le c_7, \tag{2.18}$$

with $c_7 = c_7(\eta_2, \tau, \gamma_4) > 0$. If $\gamma_4 \in (1,\infty)$, invoking interpolation inequality and Young's inequality, we can get from (2.11) that

$$\|(u+1)^{\gamma_4}\|_{L^1(\Omega)}^{\tau} \le \|(u+1)^{\gamma_4}\|_{L^\tau(\Omega)}^{\tau\rho}\|(u+1)^{\gamma_4}\|_{L^{\frac{1}{\gamma_4}}(\Omega)}^{\tau(1-\rho)} \le \eta_2\int_\Omega (u+1)^{\gamma_4\tau} + c_8, \tag{2.19}$$

where $\rho = \frac{\gamma_4-1}{\gamma_4-\frac{1}{\tau}} \in (0,1)$ and $c_8 = c_8(\eta_2,\tau,\gamma_4) > 0$. Collecting (2.17)–(2.19), we can directly infer that the first inequality of (2.8) holds. Integrating the fourth equation of system (1.4) over Ω, we have $\|w\|_{L^1(\Omega)} = \|w_1^{\gamma_3}\|_{L^1(\Omega)}$ for all $t \in (0, T_{\max})$. Due to $\gamma_3 \ge 1$, from the first inequality of (2.8), it is easy to see that

$$\|w\|_{L^1(\Omega)} = \int_\Omega w_1^{\gamma_3} \le \eta_2\int_\Omega (u+1)^{\gamma_3\gamma_4} + \tilde{c}_1 \tag{2.20}$$

for all $t \in (0, T_{\max})$, where $\tilde{c}_1 = \tilde{c}_1(\eta_2,\gamma_3,\gamma_4) > 0$. By the same procedures as in (2.13)-(2.19) , we thus can obtain for any $\eta_1 > 0$ and $\tau > 1$ that

$$\int_\Omega w^\tau \le \eta_1\int_\Omega w_1^{\gamma_3\tau} + c_9 \quad \text{for all } t \in (0, T_{\max}), \tag{2.21}$$

where $c_9 = c_9(\eta_1, \tau, \gamma_3) > 0$. Recalling $\gamma_3 \ge 1$ and using the first inequality of (2.8) again, we get that

$$\int_\Omega w_1^{\gamma_3\tau} \le \eta_2\int_\Omega (u+1)^{\gamma_3\gamma_4\tau} + c_{10} \quad \text{for all } t \in (0, T_{\max}), \tag{2.22}$$

with $c_{10} > 0$. Hence, the second inequality of (2.8) can be obtained from (2.21) and (2.22). In addition, we can employ the same processes as above to prove (2.9). Here, we omit the detailed proof. Thus, the proof of Lemma 2.2 is complete.

3. Global existence and boundedness

In order to prove the global existence and uniform boundedness of classical solutions to system (1.4), we established the following L^p−estimate for component u.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary and the parameters fulfill $\xi, \chi, \gamma_2, \gamma_4 > 0$ and $\gamma_1, \gamma_3 \ge 1$. Assume that the nonlinear functions φ, ψ and f satisfy the conditions* (1.5) *and* (1.6) *with $a, b > 0, s > 1$ and $\theta, l \in \mathbb{R}$. If $s \ge \max\{\gamma_1\gamma_2 + \theta, \gamma_3\gamma_4 + l\}$, for any $p > \max\{1, 1-\theta, 1-l, \gamma_1\gamma_2 - s + 1, \gamma_3\gamma_4 - s + 1\}$, there exists $C > 0$ such that*

$$\int_\Omega (u+1)^p \le C \tag{3.1}$$

for all $t \in (0, T_{\max})$.

Proof. For any $p > 1$, we multiply the first equation of system (1.4) with $(u + 1)^{p-1}$ and use integration by parts over Ω to obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \leq & -\frac{4(p-1)}{p^2}\int_\Omega |\nabla (u+1)^{\frac{p}{2}}|^2 + \chi(p-1)\int_\Omega (u+1)^{p-2}\varphi(u)\nabla u\cdot\nabla v \\ & + \xi(p-1)\int_\Omega (u+1)^{p-2}\psi(u)\nabla u\cdot\nabla w + a\int_\Omega u(u+1)^{p-1} \\ & - b\int_\Omega u^s(u+1)^{p-1} \end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$. Let $\Psi_1(y) = \int_0^y (\zeta + 1)^{p-2}\psi(\zeta)d\zeta$ and $\Psi_2(y) = \int_0^y (\zeta + 1)^{p-2}\varphi(\zeta)d\zeta$. It is easy to get

$$\nabla\Psi_1(u) = (u+1)^{p-2}\psi(u)\nabla u \tag{3.3}$$

and

$$\nabla\Psi_2(u) = (u+1)^{p-2}\varphi(u)\nabla u \tag{3.4}$$

for all $t \in (0, T_{\max})$. Furthermore, by a simple calculation, one can get

$$|\Psi_1(u)| \leq \frac{1}{p+l-1}(u+1)^{p+l-1} \tag{3.5}$$

and

$$|\Psi_2(u)| \leq \frac{1}{p+\theta-1}(u+1)^{p+\theta-1} \tag{3.6}$$

for all $t \in (0, T_{\max})$. Thus, the second term on the right-hand side of (3.2) can be estimated as

$$\begin{aligned}\chi(p-1)\int_\Omega (u+1)^{p-2}\varphi(u)\nabla u\cdot\nabla v &= \chi(p-1)\int_\Omega \nabla\Psi_2(u)\cdot\nabla v \\ &\leq \chi(p-1)\int_\Omega \Psi_2(u)|\Delta v| \\ &\leq \frac{\chi(p-1)}{p+\theta-1}\int_\Omega (u+1)^{p+\theta-1}|\Delta v| \end{aligned} \tag{3.7}$$

for all $t \in (0, T_{\max})$. Similarly, we can deduce

$$\xi(p-1)\int_\Omega (u+1)^{p-2}\psi(u)\nabla u\cdot\nabla w \leq \frac{\xi(p-1)}{p+l-1}\int_\Omega (u+1)^{p+l-1}|\Delta w| \tag{3.8}$$

for all $t \in (0, T_{\max})$. From the basic inequality $(u + 1)^s \leq 2^s(u^s + 1)$ with $s > 0$ and $u \geq 0$, we can get

$$-b\int_\Omega u^s(u+1)^{p-1} \leq -\frac{b}{2^s}\int_\Omega (u+1)^{p+s-1} + b\int_\Omega (u+1)^{p-1} \tag{3.9}$$

for all $t \in (0, T_{\max})$. Set $m = \max\{a, b\}$. From (3.7)–(3.9), the (3.2) can be rewritten as

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \leq \frac{\chi(p-1)}{p+\theta-1}\int_\Omega (u+1)^{p+\theta-1}\cdot|v - v_1^{\gamma_1}| + \frac{\xi(p-1)}{p+l-1}\int_\Omega (u+1)^{p+l-1}\cdot|w - w_1^{\gamma_3}|$$

$$
\begin{aligned}
&+m\int_\Omega (u+1)^p-\frac{b}{2^s}\int_\Omega (u+1)^{p+s-1}\\
\le&\frac{\chi(p-1)}{p+\theta-1}\int_\Omega (u+1)^{p+\theta-1}v+\frac{\chi(p-1)}{p+\theta-1}\int_\Omega (u+1)^{p+\theta-1}v_1^{\gamma_1}\\
&+\frac{\xi(p-1)}{p+l-1}\int_\Omega (u+1)^{p+l-1}w+\frac{\xi(p-1)}{p+l-1}\int_\Omega (u+1)^{p+l-1}w_1^{\gamma_3}\\
&+m\int_\Omega (u+1)^p-\frac{b}{2^s}\int_\Omega (u+1)^{p+s-1} \qquad (3.10)
\end{aligned}
$$

for all $t\in(0,T_{\max})$, where we have used the equations $0=\Delta v-v+v_1^{\gamma_1}$ and $0=\Delta w_1-w_1+u^{\gamma_4}$ in system (1.4). In the following, we shall establish the L^p- estimate of component u.

Case (i) $s>\max\{\gamma_1\gamma_2+\theta,\gamma_3\gamma_4+l\}$.

It follows from Young's inequality that

$$
\int_\Omega (u+1)^{p+\theta-1}v_1^{\gamma_1}\le\frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)}\int_\Omega (u+1)^{p+s-1}+c_{11}\int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}} \qquad (3.11)
$$

for all $t\in(0,T_{\max})$, with $c_{11}=\left(\frac{2^{s+4}\chi(p-1)}{b(p+\theta-1)}\right)^{\frac{p+\theta-1}{s-\theta}}>0$. Due to $s-\theta>\gamma_1\gamma_2$, we infer from Young's inequality and Lemma 2.2 by choosing $\tau=\frac{p+s-1}{\gamma_2}$ that

$$
\begin{aligned}
\int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}}&\le\frac{b(p+\theta-1)}{2^{s+4}\chi\eta_4(p-1)c_{11}}\int_\Omega v_1^{\frac{p+s-1}{\gamma_2}}+c_{12}\\
&\le\frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)c_{11}}\int_\Omega (u+1)^{p+s-1}+c_{13} \qquad (3.12)
\end{aligned}
$$

for all $t\in(0,T_{\max})$, with $c_{12}=\left(\frac{2^{s+4}\chi\eta_4(p-1)c_{11}}{b(p+\theta-1)}\right)^{\frac{\gamma_1\gamma_2}{s-\theta-\gamma_1\gamma_2}}|\Omega|$ and $c_{13}=c_{12}+c_3$. According to Young's inequality, we can find $c_{14}=\left(\frac{2^{s+4}\chi(p-1)}{b(p+\theta-1)}\right)^{\frac{p+\theta-1}{s-\theta}}>0$ such that

$$
\int_\Omega (u+1)^{p+\theta-1}v\le\frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)}\int_\Omega (u+1)^{p+s-1}+c_{14}\int_\Omega v^{\frac{p+s-1}{s-\theta}} \qquad (3.13)
$$

for all $t\in(0,T_{\max})$. For $s-\theta>\gamma_1\gamma_2$, we use Lemma 2.2 with $\tau=\frac{p+s-1}{s-\theta}$ and Young's inequality to get

$$
\int_\Omega v^{\frac{p+s-1}{s-\theta}}\le\eta_3\eta_4\int_\Omega (u+1)^{\frac{\gamma_1\gamma_2(p+s-1)}{s-\theta}}+c_4\le\frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)c_{14}}\int_\Omega (u+1)^{p+s-1}+c_{15} \qquad (3.14)
$$

for all $t\in(0,T_{\max})$, with $c_{15}=(\eta_3\eta_4)^{\frac{s-\theta}{s-\theta-\gamma_1\gamma_2}}\cdot\left(\frac{2^{s+4}\chi(p-1)c_{14}}{b(p+\theta-1)}\right)^{\frac{\gamma_1\gamma_2}{s-\theta-\gamma_1\gamma_2}}+c_4$. Analogously, we have

$$
\int_\Omega (u+1)^{p+l-1}w_1^{\gamma_3}\le\frac{b(p+l-1)}{2^{s+4}\xi(p-1)}\int_\Omega (u+1)^{p+s-1}+c_{16}\int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}} \qquad (3.15)
$$

for all $t\in(0,T_{\max})$, where $c_{16}=\left(\frac{2^{s+4}\xi(p-1)}{b(p+l-1)}\right)^{\frac{p+l-1}{s-l}}$. Since $s-l>\gamma_3\gamma_4$, it follows from Young's inequality and Lemma 2.2 with $\tau=\frac{p+s-1}{\gamma_4}$ that

$$
\int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}}\le\frac{b(p+l-1)}{2^{s+4}\xi(p-1)c_{16}\eta_2}\int_\Omega w_1^{\frac{p+s-1}{\gamma_4}}+c_{17}\le\frac{b(p+l-1)}{2^{s+4}\xi(p-1)c_{16}}\int_\Omega (u+1)^{p+s-1}+c_{18} \qquad (3.16)
$$

for all $t \in (0, T_{\max})$, where $c_{17} = \left(\frac{2^{s+4}\xi(p-1)c_{16}\eta_2}{b(p+l-1)}\right)^{\frac{\gamma_3\gamma_4}{s-l-\gamma_3\gamma_4}} |\Omega|$ and $c_{18} = c_{17} + c_1$. Similarly, there exists $c_{19} = \left(\frac{2^{s+4}\xi(p-1)}{b(p+l-1)}\right)^{\frac{p+l-1}{s-l}} > 0$ such that

$$\int_\Omega (u+1)^{p+l-1} w \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{19} \int_\Omega w^{\frac{p+s-1}{s-l}} \tag{3.17}$$

for all $t \in (0, T_{\max})$. Using Lemma 2.2 once more, one may obtain

$$\int_\Omega w^{\frac{p+s-1}{s-l}} \le \eta_1\eta_2 \int_\Omega (u+1)^{\frac{\gamma_3\gamma_4(p+s-1)}{s-l}} + c_2 \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)c_{19}} \int_\Omega (u+1)^{p+s-1} + c_{20} \tag{3.18}$$

for all $t \in (0, T_{\max})$, where $c_{20} = (\eta_1\eta_2)^{\frac{s-l}{s-l-\gamma_3\gamma_4}} \cdot \left(\frac{2^{s+4}\xi(p-1)c_{19}}{b(p+l-1)}\right)^{\frac{\gamma_3\gamma_4}{s-l-\gamma_3\gamma_4}} + c_2$. Due to $s > 1$, we thus have

$$\int_\Omega (u+1)^p \le c_{21} \int_\Omega (u+1)^{p+s-1} + c_{22} \tag{3.19}$$

for all $t \in (0, T_{\max})$, where $c_{21} = \frac{b}{2^{s+2}(m+1)}$ and $c_{22} = \left(\frac{2^{s+2}(m+1)}{b}\right)^{\frac{p}{s-1}} |\Omega|$. From (3.11)-(3.19), the inequality (3.10) can be estimated as

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + \int_\Omega (u+1)^p \le & \frac{\chi(p-1)}{p+\theta-1}\left[\frac{b(p+\theta-1)}{2^{s+2}\chi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{11}c_{13} + +c_{14}c_{15}\right] \\
& + \frac{\xi(p-1)}{p+l-1}\left[\frac{b(p+l-1)}{2^{s+2}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{16}c_{18} + c_{19}c_{20}\right] \\
& - \frac{b}{2^s}\int_\Omega (u+1)^{p+s-1} + \frac{b}{2^{s+2}}\int_\Omega (u+1)^{p+s-1} + c_{22}(m+1) \\
\le & -\frac{b}{2^{s+2}}\int_\Omega (u+1)^{p+s-1} + c_{23}
\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{\max})$, where $c_{23} = (c_{11}c_{13} + c_{14}c_{15}) \cdot \frac{\chi(p-1)}{p+\theta-1} + (c_{16}c_{18} + c_{19}c_{20}) \cdot \frac{\xi(p-1)}{p+l-1} + c_{22}(m+1)$. Hence, we can derive (3.1) easily by using the ODE comparison principle.

Case (ii) $s = \max\{\gamma_1\gamma_2 + \theta, \gamma_3\gamma_4 + l\}$.

(a) $s = \gamma_1\gamma_2 + \theta = \gamma_3\gamma_4 + l$. Recalling (3.11), (3.13), (3.15) and (3.17), there hold

$$\int_\Omega (u+1)^{p+\theta-1} v_1^{\gamma_1} \le \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{11} \int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}} \tag{3.21}$$

and

$$\int_\Omega (u+1)^{p+\theta-1} v \le \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{14} \int_\Omega v^{\frac{p+s-1}{s-\theta}} \tag{3.22}$$

and

$$\int_\Omega (u+1)^{p+l-1} w_1^{\gamma_3} \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{16} \int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}} \tag{3.23}$$

as well as

$$\int_\Omega (u+1)^{p+l-1} w \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{19} \int_\Omega w^{\frac{p+s-1}{s-l}} \tag{3.24}$$

for all $t \in (0, T_{\max})$.

Since $s-\theta = \gamma_1\gamma_2$ and $s-l = \gamma_3\gamma_4$. Thus, we can directly get from Lemma 2.2 that

$$\int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}} = \int_\Omega w_1^{\frac{p+s-1}{\gamma_4}} \le \eta_2 \int_\Omega (u+1)^{p+s-1} + c_1 \tag{3.25}$$

and

$$\int_\Omega w^{\frac{p+s-1}{s-l}} = \int_\Omega w^{\frac{p+s-1}{\gamma_3\gamma_4}} \le \eta_1\eta_2 \int_\Omega (u+1)^{p+s-1} + c_2, \tag{3.26}$$

and

$$\int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}} = \int_\Omega v_1^{\frac{p+s-1}{\gamma_2}} \le \eta_4 \int_\Omega (u+1)^{p+s-1} + c_3 \tag{3.27}$$

as well as

$$\int_\Omega v^{\frac{p+s-1}{s-\theta}} = \int_\Omega v^{\frac{p+s-1}{\gamma_1\gamma_2}} \le \eta_3\eta_4 \int_\Omega (u+1)^{p+s-1} + c_4 \tag{3.28}$$

for all $t \in (0, T_{\max})$. For the arbitrariness of $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, we choose $\eta_2 = \frac{b(p+l-1)}{2^{s+4}c_{16}\xi(p-1)}$, $\eta_1\eta_2 = \frac{b(p+l-1)}{2^{s+4}c_{19}\xi(p-1)}$, $\eta_4 = \frac{b(p+\theta-1)}{2^{s+4}c_{11}\chi(p-1)}$ and $\eta_3\eta_4 = \frac{b(p+\theta-1)}{2^{s+4}c_{14}\chi(p-1)}$ in(3.25)-(3.28), respectively. Combining (3.19) with (3.21)-(3.28), the inequality (3.10) can be rewritten as

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + \int_\Omega (u+1)^p \le & \frac{\chi(p-1)}{p+\theta-1}\Big[\frac{b(p+\theta-1)}{2^{s+2}\chi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_3c_{11} + c_4c_{14}\Big] \\
& + \frac{\xi(p-1)}{p+l-1}\Big[\frac{b(p+l-1)}{2^{s+2}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_1c_{16} + c_2c_{19}\Big] \\
& - \frac{b}{2^s}\int_\Omega (u+1)^{p+s-1} + \frac{b}{2^{s+2}}\int_\Omega (u+1)^{p+s-1} + c_{22}(m+1) \\
\le & -\frac{b}{2^{s+2}}\int_\Omega (u+1)^{p+s-1} + c_{24}
\end{aligned} \tag{3.29}$$

for all $t \in (0, T_{\max})$, where $c_{24} = (c_3c_{11} + c_4c_{14}) \cdot \frac{\chi(p-1)}{p+\theta-1} + (c_1c_{16} + c_2c_{19}) \cdot \frac{\xi(p-1)}{p+l-1} + c_{22}(m+1)$. From the ODE comparison principle, we can get the desired conclusion (3.1).

(b) $s = \gamma_1\gamma_2 + \theta > \gamma_3\gamma_4 + l$. Recalling (3.11) and (3.13) again, there hold

$$\int_\Omega (u+1)^{p+\theta-1} v_1^{\gamma_1} \le \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{11} \int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}} \tag{3.30}$$

and

$$\int_\Omega (u+1)^{p+\theta-1} v \le \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{14} \int_\Omega v^{\frac{p+s-1}{s-\theta}}. \tag{3.31}$$

For $s = \gamma_1\gamma_2 + \theta$, we can get from Lemma 2.2 that

$$\int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}} = \int_\Omega v_1^{\frac{p+s-1}{\gamma_2}} \le \eta_4 \int_\Omega (u+1)^{p+s-1} + c_3 \tag{3.32}$$

and

$$\int_\Omega v^{\frac{p+s-1}{s-\theta}} = \int_\Omega v^{\frac{p+s-1}{\gamma_1\gamma_2}} \le \eta_3\eta_4 \int_\Omega (u+1)^{p+s-1} + c_4 \tag{3.33}$$

for all $t \in (0, T_{\max})$. Here, we also choose $\eta_4 = \frac{b(p+\theta-1)}{2^{s+4}c_{11}\chi(p-1)}$ in (3.32) and $\eta_3\eta_4 = \frac{b(p+\theta-1)}{2^{s+4}c_{14}\chi(p-1)}$ in (3.33). For $s > \gamma_3\gamma_4 + l$, we can conclude from (3.15)-(3.18) that

$$\int_\Omega (u+1)^{p+l-1} w_1^{\gamma_3} \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{16} \int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}} \tag{3.34}$$

for all $t \in (0, T_{\max})$, with $c_{16} = \left(\frac{2^{s+4}\xi(p-1)}{b(p+l-1)}\right)^{\frac{p+l-1}{s-l}}$. Moreover, using Lemma 2.2, it is easy to get

$$\int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}} \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)c_{16}\eta_2} \int_\Omega w_1^{\frac{p+s-1}{\gamma_4}} + c_{17} \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)c_{16}} \int_\Omega (u+1)^{p+s-1} + c_{18} \tag{3.35}$$

for all $t \in (0, T_{\max})$, where $c_{17} = \left(\frac{2^{s+4}\xi(p-1)c_{16}\eta_2}{b(p+l-1)}\right)^{\frac{\gamma_3\gamma_4}{s-l-\gamma_3\gamma_4}} |\Omega|$ and $c_{18} = c_{17} + c_1$. By a simple calculation, we know

$$\int_\Omega (u+1)^{p+l-1} w \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{19} \int_\Omega w^{\frac{p+s-1}{s-l}} \tag{3.36}$$

and

$$\int_\Omega w^{\frac{p+s-1}{s-l}} \le \eta_1\eta_2 \int_\Omega (u+1)^{\frac{\gamma_3\gamma_4(p+s-1)}{s-l}} + c_2 \le \frac{b(p+l-1)}{2^{s+4}\xi(p-1)c_{19}} \int_\Omega (u+1)^{p+s-1} + c_{20} \tag{3.37}$$

for all $t \in (0, T_{\max})$, where $c_{19} = \left(\frac{2^{s+4}\xi(p-1)}{b(p+l-1)}\right)^{\frac{p+l-1}{s-l}}$ and $c_{20} = (\eta_1\eta_2)^{\frac{s-l}{s-l-\gamma_3\gamma_4}} \cdot \left(\frac{2^{s+4}\xi(p-1)c_{19}}{b(p+l-1)}\right)^{\frac{\gamma_3\gamma_4}{s-l-\gamma_3\gamma_4}} + c_2$. Recalling (3.19), for $s > 1$, we have

$$\int_\Omega (u+1)^p \le c_{21} \int_\Omega (u+1)^{p+s-1} + c_{22} \tag{3.38}$$

for all $t \in (0, T_{\max})$, where $c_{21} = \frac{b}{2^{s+2}(m+1)}$ and $c_{22} = \left(\frac{2^{s+2}(m+1)}{b}\right)^{\frac{p}{s-1}} |\Omega|$. Collecting (3.30)-(3.38), it can be deduced from (3.10) that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + \int_\Omega (u+1)^p \le & \frac{\chi(p-1)}{p+\theta-1}\left[\frac{b(p+\theta-1)}{2^{s+2}\chi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_3c_{11} + c_4c_{14}\right] \\
& + \frac{\xi(p-1)}{p+l-1}\left[\frac{b(p+l-1)}{2^{s+2}\xi(p-1)} \int_\Omega (u+1)^{p+s-1} + c_{16}c_{18} + c_{19}c_{20}\right] \\
& - \frac{b}{2^s}\int_\Omega (u+1)^{p+s-1} + \frac{b}{2^{s+2}}\int_\Omega (u+1)^{p+s-1} + c_{22}(m+1)
\end{aligned}$$

$$\leq -\frac{b}{2^{s+2}}\int_\Omega (u+1)^{p+s-1} + c_{25} \tag{3.39}$$

for all $t \in (0, T_{\max})$, where $c_{25} = (c_3c_{11} + c_4c_{14}) \cdot \frac{\chi(p-1)}{p+\theta-1} + (c_{16}c_{18} + c_{19}c_{20}) \cdot \frac{\xi(p-1)}{p+l-1} + c_{22}(m+1)$. In view of the ODE comparison principle, we conclude (3.1), directly.

(c) $s = \gamma_3\gamma_4 + l > \gamma_1\gamma_2 + \theta$. The proof of this case is similar to the case **(b)**. Using (3.15) and (3.17) again, we get

$$\int_\Omega (u+1)^{p+l-1} w_1^{\gamma_3} \leq \frac{b(p+l-1)}{2^{s+4}\xi(p-1)}\int_\Omega (u+1)^{p+s-1} + c_{16}\int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}} \tag{3.40}$$

and

$$\int_\Omega (u+1)^{p+l-1} w \leq \frac{b(p+l-1)}{2^{s+4}\xi(p-1)}\int_\Omega (u+1)^{p+s-1} + c_{19}\int_\Omega w^{\frac{p+s-1}{s-l}} \tag{3.41}$$

for all $t \in (0, T_{\max})$. Since $s = \gamma_3\gamma_4 + l$, it is easy to deduce from Lemma 2.2 that

$$\int_\Omega w_1^{\frac{(p+s-1)\gamma_3}{s-l}} = \int_\Omega w_1^{\frac{p+s-1}{\gamma_4}} \leq \eta_2\int_\Omega (u+1)^{p+s-1} + c_1 \tag{3.42}$$

and

$$\int_\Omega w^{\frac{p+s-1}{s-l}} = \int_\Omega v^{\frac{p+s-1}{\gamma_3\gamma_4}} \leq \eta_1\eta_2\int_\Omega (u+1)^{p+s-1} + c_2 \tag{3.43}$$

for all $t \in (0, T_{\max})$. Due to the arbitrariness of η_1 and η_2, here we let $\eta_2 = \frac{b(p+l-1)}{2^{s+4}c_{16}\xi(p-1)}$ in (3.42) and $\eta_1\eta_2 = \frac{b(p+l-1)}{2^{s+4}c_{19}\xi(p-1)}$ in (3.43). Since $s > \gamma_1\gamma_2 + \theta$, we can derive from (3.11)-(3.14) that

$$\int_\Omega (u+1)^{p+\theta-1} v_1^{\gamma_1} \leq \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)}\int_\Omega (u+1)^{p+s-1} + c_{11}\int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}} \tag{3.44}$$

for all $t \in (0, T_{\max})$, with $c_{11} = \left(\frac{2^{s+4}\chi(p-1)}{b(p+\theta-1)}\right)^{\frac{p+\theta-1}{s-\theta}} > 0$. Due to $s - \theta > \gamma_1\gamma_2$, from Young's inequality and Lemma 2.2, we can obtain

$$\int_\Omega v_1^{\frac{(p+s-1)\gamma_1}{s-\theta}} \leq \frac{b(p+\theta-1)}{2^{s+4}\chi\eta_4(p-1)c_{11}}\int_\Omega v_1^{\frac{p+s-1}{\gamma_2}} + c_{12} \leq \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)c_{11}}\int_\Omega (u+1)^{p+s-1} + c_{13} \tag{3.45}$$

for all $t \in (0, T_{\max})$, with $c_{12} = \left(\frac{2^{s+4}\chi\eta_4(p-1)c_{11}}{b(p+\theta-1)}\right)^{\frac{\gamma_1\gamma_2}{s-\theta-\gamma_1\gamma_2}}|\Omega|$ and $c_{13} = c_{12} + c_3$. In view of Young's inequality, it is easy to get

$$\int_\Omega (u+1)^{p+\theta-1} v \leq \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)}\int_\Omega (u+1)^{p+s-1} + c_{14}\int_\Omega v^{\frac{p+s-1}{s-\theta}} \tag{3.46}$$

for all $t \in (0, T_{\max})$, with $c_{14} = \left(\frac{2^{s+4}\chi(p-1)}{b(p+\theta-1)}\right)^{\frac{p+\theta-1}{s-\theta}} > 0$. Due to $s - \theta > \gamma_1\gamma_2$, thus we use Lemma 2.2 with $\tau = \frac{p+s-1}{s-\theta}$ and Young's inequality to obtain

$$\int_\Omega v^{\frac{p+s-1}{s-\theta}} \leq \eta_3\eta_4\int_\Omega (u+1)^{\frac{\gamma_1\gamma_2(p+s-1)}{s-\theta}} + c_4 \leq \frac{b(p+\theta-1)}{2^{s+4}\chi(p-1)c_{14}}\int_\Omega (u+1)^{p+s-1} + c_{15} \tag{3.47}$$

for all $t \in (0, T_{\max})$, with $c_{15} = (\eta_3\eta_4)^{\frac{s-\theta}{s-\theta-\gamma_1\gamma_2}} \cdot \left(\frac{2^{s+4}\chi(p-1)c_{14}}{b(p+\theta-1)}\right)^{\frac{\gamma_1\gamma_2}{s-\theta-\gamma_1\gamma_2}} + c_4$. For $s > 1$, we get from (3.19) that

$$\int_\Omega (u+1)^p \leq c_{21} \int_\Omega (u+1)^{p+s-1} + c_{22} \tag{3.48}$$

for all $t \in (0, T_{\max})$, where $c_{21} = \frac{b}{2^{s+2}(m+1)}$ and $c_{22} = \left(\frac{2^{s+2}(m+1)}{b}\right)^{\frac{p}{s-1}} |\Omega|$. Collecting (3.40)–(3.48), we can infer from (3.10) that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + \int_\Omega (u+1)^p \leq -\frac{b}{2^{s+2}}\int_\Omega (u+1)^{p+s-1} + c_{26} \tag{3.49}$$

for all $t \in (0, T_{\max})$, where $c_{26} = (c_{11}c_{13} + c_{14}c_{15}) \cdot \frac{\chi(p-1)}{p+\theta-1} + (c_1c_{16} + c_2c_{19}) \cdot \frac{\xi(p-1)}{p+l-1} + c_{22}(m+1)$. Hence, we can conclude (3.1) by using the ODE comparison principle. Thus, we complete the proof of Lemma 3.1.

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters fulfill $\xi, \chi, \gamma_2, \gamma_4 > 0$ and $\gamma_1, \gamma_3 \geq 1$. Assume that the nonlinear functions φ, ψ and f satisfy the conditions (1.5) and (1.6) with $a, b > 0$, $s > 1$ and $\theta, l \in \mathbb{R}$. According to Lemma 3.1, for any $p > \max\{1, 1-\theta, 1-l, n\gamma_1\gamma_2, n\gamma_3\gamma_4, \gamma_1\gamma_2 - s + 1, \gamma_3\gamma_4 - s + 1\}$, there exists $c_{27} > 0$ such that $\|u\|_{L^p(\Omega)} \leq c_{27}$ for all $t \in (0, T_{\max})$. We deal with the second, the third, the fourth and the fifth equations in system (1.4) by elliptic L^p−estimate to obtain

$$\|v(\cdot,t)\|_{W^{2,\frac{p}{\gamma_1\gamma_2}}(\Omega)} + \|v_1(\cdot,t)\|_{W^{2,\frac{p}{\gamma_2}}(\Omega)} + \|w(\cdot,t)\|_{W^{2,\frac{p}{\gamma_3\gamma_4}}(\Omega)} + \|w_1(\cdot,t)\|_{W^{2,\frac{p}{\gamma_4}}(\Omega)} \leq c_{28} \tag{3.50}$$

for all $t \in (0, T_{\max})$, with some $c_{28} > 0$. Applying the Sobolev imbedding theorem, we can infer that

$$\|v(\cdot,t)\|_{W^{1,\infty}} + \|v_1(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{W^{1,\infty}} + \|w_1(\cdot,t)\|_{W^{1,\infty}} \leq c_{29} \tag{3.51}$$

for all $t \in (0, T_{\max})$, with some $c_{29} > 0$. In view of Moser iteration [41, Lemma A.1], there exists $c_{30} > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq c_{30}$$

for all $t \in (0, T_{\max})$, which combining with Lemma 2.1 implies that $T_{\max} = \infty$. The proof of Theorem 1.1 is complete.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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