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Research article

Global regularity of solutions to the 2D steady compressible Prandtl equations

Yonghui Zou*

School of Mathematical Sciences, Ocean University of China, Qingdao 266100, P.R.China

* **Correspondence:** Email: zouyonghuimath@163.com.

Abstract: In this paper, we study the global C^∞ regularity of solutions to the boundary layer equations for two-dimensional steady compressible flow under the favorable pressure gradient. To our knowledge, the difficulty of the proof is the degeneracy near the boundary. By using the regularity theory and maximum principles of parabolic equations together with the von Mises transformation, we give a positive answer to it. When the outer flow and the initial data satisfied appropriate conditions, we prove that Oleinik type solutions smooth up the boundary $y = 0$ for any $x > 0$.

Keywords: compressible Prandtl equations; global C^∞ regularity; favorable pressure
Mathematics Subject Classification: 35Q30, 76D10, 76N20

1. Introduction

In this paper, we study the 2D steady compressible Prandtl equations in $\{x > 0, y > 0\}$:

$$\begin{cases} u\partial_x u + v\partial_y u - \dfrac{1}{\rho}\partial_y^2 u = -\dfrac{\partial_x P(\rho)}{\rho}, \\ \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ u|_{x=0} = u_0(y), \quad \lim\limits_{y\to\infty} u = U(x), \\ u|_{y=0} = v|_{y=0} = 0, \end{cases} \tag{1.1}$$

where (u, v) is velocity field, $\rho(x)$ and $U(x)$ are the traces at the boundary $\{y = 0\}$ of the density and the tangential velocity of the outer Euler flow. The states ρ, U satisfy the Bernoulli law

$$U\partial_x U + \frac{\partial_x P(\rho)}{\rho} = 0. \tag{1.2}$$

The pressure $P(\rho)$ is a strictly increasing function of ρ with $0 < \rho_0 \le \rho \le \rho_1$ for some positive constants ρ_0 and ρ_1.

In this paper, we assume that the pressure satisfies the favorable pressure gradient $\partial_x P \leq 0$, which implies that

$$\partial_x \rho \leq 0.$$

The boundary layer is a very important branch in fluid mechanics. Ludwig Prandtl [14] first proposed the related theory of the boundary layer in 1904. Since then, many scholars have devoted themselves to studying the mathematical theory of the boundary layer [1,7–9,11,12,17–19,21–24,26,27]. For more complex fluids, such as compressible fluids, one can refer to [19,20,28] and the references therein for more details. Here, for our purposes, we only list some relevant works.

There are three very natural problems about the steady boundary layer: (i) Boundary layer separation, (ii) whether Oleinik's solutions are smooth up to the boundary for **any** $x > 0$ and (iii) vanishing viscosity limit of the steady Navier-Stokes system. Next, we will introduce the relevant research progress in these three aspects. The separation of the boundary layer is one of the very important research contents in the boundary layer theory. [17]. The earliest mathematical theory in this regard was proposed by Caffarelli and E in an unpublished paper [25]. Their results show that the existence time x^* of the solutions to the steady Prandtl equations in the sense of Oleinik is finite under the adverse pressure gradient. Moreover, the family $u_\mu(x, y) = \mu^{-\frac{1}{2}} u(x^* - \mu x, \mu^{\frac{1}{4}} y)$ is compact in $C^0(\mathbb{R}^2_+)$. Later, Dalibard and Masmoudi [4] proved the solution behaves near the separation as $\partial_y u(x, 0) \sim (x^* - x)^{\frac{1}{2}}$ for $x < x^*$. Shen, Wang and Zhang [18] found that the solution near the separation point behaves like $\partial_y u(x, y) \sim (x^* - x)^{\frac{1}{4}}$ for $x < x^*$. The above work further illustrates that the boundary layer separation is a very complex phenomenon. Recently, there were also some results about the steady compressible boundary layer separation [28]. The authors found that if the heat transfer in the boundary layer disappeared, then the singularity would be the same as that in the incompressible case. There is still relatively little mathematical theory on the separation of unsteady boundary layers. This is because back-flow and separation no longer occur simultaneously. When the boundary layer back-flow occurs, the characteristics of the boundary layer will continue to maintain for a period of time. Therefore, it is very important to study the back-flow point for further research on separation. Recently, Wang and Zhu [21] studied the back-flow problem of the two-dimensional unsteady boundary layer, which is a important work. It is very interesting to further establish the mathematical theory of the unsteady boundary layer separation.

Due to degenerate near the boundary, the high regularity of the solution of the boundary layer equation is a very difficult and meaningful work. In a local time $0 < x < x^* \ll 1$, Guo and Iyer [6] studied the high regularity of of the Prandtl equations. Oleinik and Samokhin [13] studied the existence of solutions of steady Prandtl equations and Wang and Zhang [23] proved that Oleinik's solutions are smooth up to the boundary $y = 0$ for **any** $x > 0$. The goal of this paper is to prove the global C^∞ regularity of the two-dimensional steady compressible Prandtl equations. Recently, Wang and Zhang [24] found the explicit decay for general initial data with exponential decay by using the maximum principle.

In addition, in order to better understand the relevant background knowledge, we will introduce some other related work. As the viscosity goes to zero, the solutions of the three-dimensional evolutionary Navier-Stokes equations to the solutions of the Euler equations are an interesting problem. Beirão da Veiga and Crispo [2] proved that in the presence of flat boundaries convergence holds uniformly in time with respect to the initial data's norm. For the non-stationary Navier-Stokes equations in the 2D power cusp domain, the formal asymptotic expansion of the solution near the singular point is constructed and the constructed asymptotic decomposition is justified in [15, 16] by Pileckas and Raciene.

Before introducing the main theorem, we introduce some preliminary knowledge. To use the von Mises transformation, we set

$$\tilde{u}(x,y) = \rho(x)u(x,y), \quad \tilde{v}(x,y) = \rho(x)v(x,y), \quad \tilde{u}_0(y) = \rho(0)u_0(y),$$

then we find that $(\tilde{u}, \tilde{v})$ satisfies:

$$\begin{cases} \tilde{u}\partial_x\tilde{u} + \tilde{v}\partial_y\tilde{u} - \partial_y^2\tilde{u} - \dfrac{\partial_x\rho}{\rho}\tilde{u}^2 = -\rho\partial_x P(\rho), \\ \partial_x\tilde{u} + \partial_y\tilde{v} = 0, \\ \tilde{u}|_{x=0} = \tilde{u}_0(y), \quad \lim\limits_{y\to\infty}\tilde{u} = \rho(x)U(x), \\ \tilde{u}|_{y=0} = \tilde{v}|_{y=0} = 0. \end{cases} \tag{1.3}$$

By the von Mises transformation

$$x = x, \quad \psi(x,y) = \int_0^y \tilde{u}(x,z)dz, \quad w = \tilde{u}^2, \tag{1.4}$$

$$\partial_x\tilde{u} = \frac{\partial_x\omega}{2\sqrt{\omega}} + \frac{\partial_\psi\omega\partial_x\psi}{2\sqrt{\omega}}, \quad \partial_y\tilde{u} = \frac{\partial_\psi\omega}{2}, \quad \partial_y^2\tilde{u} = \frac{\sqrt{\omega}\partial_\psi^2\omega}{2}, \tag{1.5}$$

and (1.3)–(1.5), we know that $w(x,\psi)$ satisfies:

$$\partial_x w - \sqrt{w}\partial_\psi^2 w - 2\frac{\partial_x\rho}{\rho}w = -2\rho\partial_x P(\rho), \tag{1.6}$$

with

$$w(x,0) = 0, \quad w(0,\psi) = w_0(\psi), \quad \lim_{\psi\to+\infty} w = (\rho(x)U(x))^2. \tag{1.7}$$

In addition, we have

$$2\partial_y\tilde{u} = \partial_\psi w, \qquad 2\partial_y^2\tilde{u} = \sqrt{w}\partial_\psi^2 w. \tag{1.8}$$

In [5], Gong, Guo and Wang studied the existence of the solutions of system (1.1) by using the von Mises transformation and the maximal principle proposed by Oleinik and Samokhin in [13]. Specifically, they proved that:

Theorem 1.1. *If the initial data u_0 satisfies the following conditions:*

$$\begin{aligned} &u \in C_b^{2,\alpha}([0,+\infty))(\alpha > 0), \quad u(0) = 0, \quad \partial_y u(0) > 0, \quad \partial_y u(y) \geq 0 \quad \textit{for} \quad y \in [0,+\infty), \\ &\lim_{y\to+\infty} u(y) = U(0) > 0, \quad \rho^{-1}(0)\partial_y^2 u(y) - \rho^{-1}(0)\partial_x P(0) = O(y^2) \end{aligned} \tag{1.9}$$

and $\rho \in C^2([0,X_0])$, then there exists $0 < X \leq X_0$ such that system (1.1) *admits a solution $u \in C^1([0,X)\times\mathbb{R}_+)$. The solution has the following properties:*
(i) *u is continuous and bounded in $[0,X]\times\mathbb{R}_+$; $\partial_y u, \partial_y^2 u$ are continuous and bounded in $[0,X)\times\mathbb{R}_+$;*

$v, \partial_y v, \partial_x u$ are locally bounded in $[0, X) \times \mathbb{R}_+$.
(ii) $u(x, y) > 0$ *in* $[0, X) \times \mathbb{R}_+$ *and for any* $\bar{x} < X$*, there exists* $y_0, m > 0$ *such that for all* $(x, y) \in [0, \bar{x}] \times [0, y_0]$,

$$\partial_y u(x, y) \geq m > 0.$$

(iii) if $\partial_x P \leq 0 (\partial_x \rho \leq 0)$*, then*

$$X = +\infty.$$

Remarks 1.2. $u \in C_b^{2,\alpha}([0, +\infty))(\alpha > 0)$ *means that u is Hölder continuity and bounded.*

This theorem shows that under the favorable pressure gradient, the solution is global-in-x. However, if the pressure is an adverse pressure gradient, then boundary layer separation will occur. Xin and Zhang [26] studied the global existence of weak solutions of unsteady Prandtl equations under the favorable pressure gradient. For the unsteady compressible Prandtl equation, similar results are obtained in [3]. Recently, Xin, Zhang and Zhao [27] proposed a direct proof of the existence of global weak solutions of the Prandtl equation. The key content of this paper is that they have studied the uniqueness and regularity of weak solutions. This method can be applied to the compressible Prandtl equation.

Our main results are as follows:

Theorem 1.3. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume* u_0 *satisfies the condition* (1.9) *and the known function* ρ *and* $\partial_x P$ *are smooth. Then, there exists a constant* $C > 0$ *depending only on* $\varepsilon, X, u_0, P(\rho), k, m$ *such that for any* $(x, y) \in [\varepsilon, X] \times [0, +\infty)$,

$$|\partial_x^k \partial_y^m u(x, y)| \leq C,$$

where X, ε *are positive constants with* $\varepsilon < X$ *and* m, k *are any positive integers.*

Remarks 1.4. *Our methods may be used to other related models. There are similar results for the magnetohydrodynamics boundary layer and the thermal boundary layer. This work will be more difficult due to the influence of temperature and the magnetic field.*

Due to the degeneracy near the boundary $\psi = 0$, the proof of the main result is divided into two parts, Theorem 1.5 and Theorem 1.6. This is similar to the result of the incompressible boundary layer, despite the fluid being compressible and the degeneracy near the boundary. Different from the incompressible case [23], we have no divergence-free conditions, which will bring new terms. It is one of the difficulties in this paper to deal with these terms. Now, we will briefly introduce our proof framework. First, we prove the following theorem in the domain $[\varepsilon, X] \times [0, Y_1]$ for a small Y_1. The key ingredients of proof is that we employ interior priori estimates and the maximum principle developed by Krylov [10].

Theorem 1.5. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume* u_0 *satisfies the condition* (1.9) *and the known function* ρ *and* $\partial_x P$ *are smooth. Then, there exists a small constant* $Y_1 > 0$ *and a large constant* $C > 0$ *depending only on* $\varepsilon, X, Y_1, u_0, P(\rho), k, m$ *such that for any* $(x, y) \in [\varepsilon, X] \times [0, Y_1]$,

$$|\partial_x^k \partial_y^m u(x, y)| \leq C,$$

where X, ε *are positive constants with* $\varepsilon < X$ *and* m, k *are any positive integers.*

Next, we prove the following theorem in the domain $[\varepsilon, X]\times[Y_2, +\infty)$ for a small positive constant Y_2. The key of proof is that we prove (1.6) is a uniform parabolic equation in the domain $[\varepsilon, X]\times[Y_2, +\infty)$ in Section 4. Once we have (1.6) is a uniform parabolic equation, the global C^∞ regularity of the solution is a direct result of interior Schauder estimates and classical parabolic regularity theory. The proof can be given similarly to the steady incompressible boundary layer. For the sake of simplicity of the paper, more details can be found in [23] and we omit it here.

Theorem 1.6. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and $\partial_x P$ are smooth. Then, there exists a constant $Y_0 > 0$ such that for any constant $Y_2 \in (0, Y_0)$, there exists a constant $C > 0$ depending only on $\varepsilon, X, Y_2, u_0, P(\rho), k, m$ such that for any $(x, y) \in [\varepsilon, X] \times [Y_2, +\infty)$,*

$$|\partial_x^k \partial_y^m u(x, y)| \le C,$$

where X, ε are positive constants with $\varepsilon < X$ and m, k are any positive integers.

Therefore, Theorem 1.3 can be directly proven by combining Theorem 1.5 with Theorem 1.6.

The organization of this paper is as follows. In Section 2, we study lower order and higher order regularity estimates. In Section 3, we prove Theorem 1.5 in the domain near $y = 0$ by transforming back to the original coordinates (x, y). In Section 4, we prove (1.6) is a uniform parabolic equation by using the maximum principle and we also prove the Theorem 1.3.

2. Lower order and higher order regularity estimates

2.1. Lower order regularity estimates

In this subsection, we study the lower order regularity estimates using the standard interior a priori estimates developed by Krylov [10].

Lemma 2.1. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and $\partial_x P$ are smooth. Assume $0 < \varepsilon < X$, then there exists some positive constants $\delta_1 > 0$ and C independent of ψ such that for any $(x, \psi) \in [\varepsilon, X] \times [0, \delta_1]$,*

$$|\partial_x w(x, \psi)| \le C\psi.$$

Proof. Due to Lemma 2.1 in [5] (or Theorem 2.1.14 in [13]), there exists $\delta_1 > 0$ for any $(x, \psi) \in [0, X] \times [0, \delta_1]$, such that for some $\alpha \in (0, \frac{1}{2})$ and positive constants m, M (we assume $\delta_1 < 1$),

$$|\partial_x w| \le C\psi^{\frac{1}{2}+\alpha}, \quad 0 < m < \partial_\psi w < M, \quad m\psi < w < M\psi. \tag{2.1}$$

By (1.6), we obtain

$$\partial_x \partial_x w - \sqrt{w}\partial_\psi^2 \partial_x w = \frac{(\partial_x w)^2}{2w} + 2\frac{\rho\partial_x P \partial_x w}{2w} + \frac{\partial_x \rho}{\rho}\partial_x w + 2\partial_x\left(\frac{\partial_x \rho}{\rho}\right) w - 2\partial_x[\rho\partial_x P].$$

Take a smooth cutoff function $0 \le \phi(x) \le 1$ in $[0, X]$ such that

$$\phi(x) = 1, x \in [\varepsilon, X], \quad \phi(x) = 0, x \in [0, \frac{\varepsilon}{2}],$$

then

$$
\begin{aligned}
\partial_x \left[\partial_x w\phi(x)\right] - &\sqrt{w}\partial_\psi^2 \left[\partial_x w\phi(x)\right] \\
&= \frac{(\partial_x w)^2}{2w}\phi(x) + 2\frac{\rho\partial_x P\partial_x w}{2w}\phi(x) + \frac{\partial_x \rho}{\rho}\partial_x w\phi(x) \\
&+ 2\partial_x\left(\frac{\partial_x\rho}{\rho}\right)w\phi(x) - 2\partial_x(\rho\partial_x P)\phi(x) + \partial_x w\partial_x\phi(x) := \mathcal{W}.
\end{aligned}
$$

Combining with (2.1), we know

$$
|\mathcal{W}| \le C\psi^{2\alpha} + C\psi^{\alpha-\frac{1}{2}} + C\psi^{\alpha+\frac{1}{2}} + C\psi + C \le C\psi^{\alpha-\frac{1}{2}}. \tag{2.2}
$$

We take $\varphi(\psi) = \mu_1\psi - \mu_2\psi^{1+\beta}$ with constants μ_1, μ_2, then by (2.1) and (2.2), we get

$$
\begin{aligned}
\partial_x \left[\partial_x w\phi(x) - \varphi\right] - \sqrt{w}\partial_\psi^2 \left[\partial_x w\phi(x) - \varphi\right] &\le |\mathcal{W}| - \mu_2\sqrt{w}\beta(1+\beta)\psi^{\beta-1} \\
&\le C\psi^{\alpha-\frac{1}{2}} - \mu_2\sqrt{m}\beta(1+\beta)\psi^{\beta-\frac{1}{2}}.
\end{aligned}
$$

By taking μ_2 sufficiently large and $\alpha = \beta$, for $(x,\psi) \in (0,X] \times (0,\delta_1)$, we have

$$
\partial_x \left[\partial_x w\phi(x) - \varphi\right] - \sqrt{w}\partial_\psi^2 \left[\partial_x w\phi(x) - \varphi\right] < 0.
$$

For any $\psi \in [0,\delta_1]$, let $\mu_1 \ge \mu_2$, and we have

$$
\left(\partial_x w\phi - \varphi\right)(0,\psi) \le 0,
$$

and take μ_1 large enough depending on M, δ_1, μ_2 such that

$$
\left(\partial_x w\phi - \varphi\right)(x,\delta_1) \le M\delta_1^{\frac{1}{2}+\alpha} - \mu_1\delta_1 + \mu_2\delta_1^{1+\beta} \le 0.
$$

Since $w(x,0) = 0$, we know that for any $x \in [0,X]$,

$$
\left(\partial_x w\phi - \varphi\right)(x,0) = 0.
$$

By the maximum principle, it holds in $[0,X] \times [0,\delta_1]$ that

$$
\left(\partial_x w\phi - \varphi\right)(x,\psi) \le 0.
$$

Let δ_1 be chosen suitably small, for $(x,\psi) \in [\varepsilon,X] \times [0,\delta_1]$, and we obtain

$$
\partial_x w(x,\psi) \le \mu_1\psi - \mu_2\psi^{1+\beta} \le \frac{\mu_1}{2}\psi.
$$

Considering $-\partial_x w\phi - \varphi$, the result $-\partial_x w \le \frac{\mu_1}{2}\psi$ in $[\varepsilon,X] \times [0,\delta_1]$ can be proved similarly. This completes the proof of the lemma.

Lemma 2.2. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and $\partial_x P$ are smooth. Assume $0 < \varepsilon < X$, then there exists some positive constants $\delta_2 > 0$ and C independent of ψ such that for any $(x,\psi) \in [\varepsilon,X] \times [0,\delta_2]$,*

$$
|\partial_\psi\partial_x w(x,\psi)| \le C, \quad |\partial_x^2 w(x,\psi)| \le C\psi^{-\frac{1}{2}}, \quad |\partial_\psi^2\partial_x w(x,\psi)| \le C\psi^{-1}.
$$

Proof. From Lemma 2.1, there exists $\delta_1 > 0$ such that for any $(x,\psi) \in [\frac{\varepsilon}{2}, X] \times [0, \delta_1]$,

$$|\partial_x w(x,\psi)| \leq C\psi.$$

Let $\Psi_0 = \min\{\frac{2}{3}\delta_1, \frac{\varepsilon}{2}\}$, for any $(x_0,\psi_0) \in [\varepsilon, X] \times (0, \Psi_0]$, and we denote

$$\Omega = \{(x,\psi)| x_0 - \psi_0^{\frac{3}{2}} \leq x \leq x_0, \frac{1}{2}\psi_0 \leq \psi \leq \frac{3}{2}\psi_0\}.$$

By the definition of Ψ_0, we know $\Omega \subseteq [\frac{\varepsilon}{2}, X] \times [0, \delta_1]$, then it holds in Ω that

$$|\partial_x w| \leq C\psi. \tag{2.3}$$

The following transformation f is defined:

$$\Omega \to \widetilde{\Omega} := [-1, 0]_{\tilde{x}} \times [-\frac{1}{2}, \frac{1}{2}]_{\tilde{\psi}}, \quad (x,\psi) \mapsto (\tilde{x}, \tilde{\psi}),$$

where $x - x_0 = \psi_0^{\frac{3}{2}}\tilde{x}, \psi - \psi_0 = \psi_0\tilde{\psi}$.

Since $\partial_{\tilde{x}} = \psi_0^{\frac{3}{2}}\partial_x, \partial_{\tilde{\psi}} = \psi_0\partial_\psi$, it holds in Ω that

$$\partial_{\tilde{x}}\left(\psi_0^{-1}w\right) - \psi_0^{-\frac{1}{2}}\sqrt{w}\partial_{\tilde{\psi}}^2\left(\psi_0^{-1}w\right) - 2\frac{\partial_{\tilde{x}}\rho}{\rho}(\psi_0^{-1}w) = -2\rho\partial_{\tilde{x}}P\psi_0^{-1}.$$

Combining with (2.1), we get $0 < c \leq \psi_0^{-\frac{1}{2}}\sqrt{w} \leq C, |\psi_0^{-1}w| \leq C$, and for any $\widetilde{z_1}, \widetilde{z_2} \in \widetilde{\Omega}$,

$$|\psi_0^{-\frac{1}{2}}\sqrt{w}(\tilde{z}_1) - \psi_0^{-\frac{1}{2}}\sqrt{w}(\tilde{z}_2)| = \psi_0^{-\frac{1}{2}}\frac{|w(\tilde{z}_1) - w(\tilde{z}_2)|}{\sqrt{w}(\tilde{z}_1) + \sqrt{w}(\tilde{z}_2)} \leq C\frac{\psi_0|\tilde{z}_1 - \tilde{z}_2|}{\psi_0} = C|\tilde{z}_1 - \tilde{z}_2|.$$

This means that for any $\alpha \in (0,1)$, we have

$$|\psi_0^{-\frac{1}{2}}\sqrt{w}|_{C^\alpha(\widetilde{\Omega})} \leq C.$$

Since P and ρ are smooth, we have

$$|\rho^{-1}\partial_{\tilde{x}}\rho|_{C^{0,1}([-1,0]_{\tilde{x}})} + |\rho\partial_{\tilde{x}}P\psi_0^{-1}|_{C^{0,1}([-1,0]_{\tilde{x}})} \leq C.$$

By standard interior priori estimates (see Theorem 8.11.1 in [10] or Proposition 2.3 in [23]), we have

$$|w\psi_0^{-1}|_{C^\alpha([-\frac{1}{2},0]_{\tilde{x}}\times[-\frac{1}{4},\frac{1}{4}]_{\tilde{\psi}})} + |\partial_{\tilde{\psi}}^2 w\psi_0^{-1}|_{C^\alpha([-\frac{1}{2},0]_{\tilde{x}}\times[-\frac{1}{4},\frac{1}{4}]_{\tilde{\psi}})} \leq C. \tag{2.4}$$

Let $f := \partial_x w\psi_0^{-1}$, which satisfies

$$\partial_{\tilde{x}}f - \frac{\sqrt{w}}{\psi_0^{\frac{1}{2}}}\partial_{\tilde{\psi}}^2 f - \frac{\partial_{\tilde{\psi}}^2 w}{2\sqrt{w}\psi_0^{\frac{1}{2}}}f - 2\frac{\partial_{\tilde{x}}\rho}{\rho}f = -2\partial_x[\rho\partial_{\tilde{x}}P]\psi_0^{-1} + 2\partial_x\left(\frac{\partial_{\tilde{x}}\rho}{\rho}\right)(\psi_0^{-1}w).$$

By (2.3), we have $|f| \leq C$ in $\widetilde{\Omega}$. Due to

$$\left|\psi_0^{\frac{1}{2}}w^{-\frac{1}{2}}(\tilde{z}_1) - \psi_0^{\frac{1}{2}}w^{-\frac{1}{2}}(\tilde{z}_2)\right| = \psi_0^{\frac{1}{2}}\frac{\left|\frac{w(\tilde{z}_1)-w(\tilde{z}_2)}{w(\tilde{z}_1)w(\tilde{z}_2)}\right|}{w^{-\frac{1}{2}}(\tilde{z}_1) + w^{-\frac{1}{2}}(\tilde{z}_2)} \leq C|\tilde{z}_1 - \tilde{z}_2|,$$

we have

$$\left|\psi_0^{\frac{1}{2}} w^{-\frac{1}{2}}\right|_{C^{\alpha}(\widetilde{\Omega})} \le C. \tag{2.5}$$

Since

$$\frac{\partial_{\tilde\psi}^2 w}{2\sqrt{w}\psi_0^{\frac{1}{2}}} = \partial_{\tilde\psi}^2 w \psi_0^{-1} \frac{\psi_0^{\frac{1}{2}}}{2\sqrt{w}},$$

which along with (2.4) and (2.5) gives

$$\left|\frac{\partial_{\tilde\psi}^2 w}{2\sqrt{w}\psi_0^{\frac{1}{2}}}\right|_{C^{\alpha}([-\frac{1}{2},0]_{\tilde x}\times[-\frac{1}{4},\frac{1}{4}]_{\tilde\psi})} \le C.$$

As before, by (2.4) and the density ρ and P are smooth, via the standard interior a priori estimates, it yield that

$$|\partial_{\tilde x} f|_{L^\infty([-\frac{1}{4},0]_{\tilde x}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde\psi})} + |\partial_{\tilde\psi} f|_{L^\infty([-\frac{1}{4},0]_{\tilde x}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde\psi})} + |\partial_{\tilde\psi}^2 f|_{L^\infty([-\frac{1}{4},0]_{\tilde x}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde\psi})} \le C.$$

Therefore, we obtain

$$|\partial_x^2 w\,(x_0,\psi_0)\,| \le C\psi_0^{-\frac{1}{2}}, \quad |\partial_\psi \partial_x w\,(x_0,\psi_0)\,| \le C, \quad |\partial_\psi^2 \partial_x w\,(x_0,\psi_0)\,| \le C\psi_0^{-1}.$$

This completes the proof of the lemma.

2.2. Higher order regularity estimates

In this subsection, we study the higher order regularity estimates using the maximum principle. The two main results of this subsection are Lemma 2.3 and Lemma 2.7.

Lemma 2.3. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and $\partial_x P$ are smooth. Assume $0 < \varepsilon < X$ and $k \ge 2$, then there exists some positive constants $\delta > 0$ and C independent of ψ such that for any $(x,\psi) \in [\varepsilon, X] \times [0,\delta]$,*

$$|\partial_x^k w| \le C\psi, \quad |\partial_\psi \partial_x^k w| \le C, \quad |\partial_\psi^2 \partial_x^k w| \le C\psi^{-1}.$$

Proof. By Lemma 2.1 and Lemma 2.2, we may inductively assume that for $0 \le j \le k-1$, there holds that in $\left[\frac{\varepsilon}{2}, X\right] \times [0,\delta_3]$ (assume $\delta_3 \ll 1$),

$$|\partial_\psi \partial_x^j w| \le C, \quad |\partial_\psi^2 \partial_x^j w| \le C\psi^{-1}, \quad |\partial_x^j w| \le C\psi, \quad |\partial_x^j \sqrt{w}| \le C\psi^{\frac{1}{2}}, \quad |\partial_x^k w| \le C\psi^{-\frac{1}{2}}. \tag{2.6}$$

We will prove that there exists $\delta_4 < \delta_3$ so that in $[\varepsilon, X] \times [0,\delta_4]$,

$$|\partial_\psi \partial_x^k w| \le C, \quad |\partial_\psi^2 \partial_x^k w| \le C\psi^{-1}, \quad |\partial_x^k w| \le C\psi, \quad |\partial_x^k \sqrt{w}| \le C\psi^{\frac{1}{2}}, \quad |\partial_x^{k+1} w| \le C\psi^{-\frac{1}{2}}. \tag{2.7}$$

The above results are deduced from the following Lemma 2.4, Lemma 2.5 and Lemma 2.6.

Lemma 2.4. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and $\partial_x P$ are smooth. Assume that* (2.6) *holds, then there is a positive constant M_1 for any $(x,\psi) \in \left[\frac{7\varepsilon}{8}, X\right] \times [0,\delta_3]$ and $0 < \beta \ll 1$,*

$$|\partial_x^k w| < M_1 \psi^{1-\beta}, \quad |\partial_x^k \sqrt{w}| \le M_1 \psi^{\frac{1}{2}-\beta}.$$

Proof. Take a smooth cutoff function $0 \le \phi(x) \le 1$ in $[0, X]$ such that

$$\phi(x) = 1, x \in [\frac{7\varepsilon}{8}, X], \quad \phi(x) = 0, x \in [0, \frac{5\varepsilon}{8}].$$

As in [23], fix any $h < \frac{\varepsilon}{8}$. Set

$$\Omega = \{(x,\psi) | 0 < x \le X, 0 < \psi < \delta_3\},$$

and let

(i) $f = \frac{\partial_x^{k-1} w(x-h,\psi) - \partial_x^{k-1} w(x,\psi)}{-h}\phi + M\psi \ln \psi, \qquad (x,\psi) \in [\frac{5\varepsilon}{8}, X] \times [0, +\infty),$

(ii) $f = M\psi \ln \psi, \qquad (x,\psi) \in [0, \frac{5\varepsilon}{8}) \times [\psi, +\infty),$

so we get $f(x,0) = 0$, $f(0,\psi) \le 0$. We know

$$f(x,\delta_3) \le C(\delta_3)^{-\frac{1}{2}} + M\delta_3 \ln \delta_3 \le 0,$$

where M is large enough. Then, by choosing the appropriate M, we know that the positive maximum of f cannot be achieved in the interior. Finally, the lemma can be proven by the arbitrariness of h.

Assume that there exists a point

$$p_0 = (x_0, \psi_0) \in \Omega,$$

such that

$$f(p_0) = \max_{\bar{\Omega}} f > 0.$$

It is easy to know that

$$x_0 > \frac{5\varepsilon}{8}, \qquad \partial_x^{k-1} w(x_0 - h, \psi_0) < \partial_x^{k-1} w(x_0, \psi_0).$$

By (2.1), denote $\xi = \sqrt{m}$, we have

$$-\sqrt{w}\partial_\psi^2 (M\psi \ln \psi) = -M\sqrt{w}\psi^{-1} \le -\xi M \psi^{-\frac{1}{2}}. \tag{2.8}$$

By (1.6), a direct calculation gives

$$
\begin{aligned}
\partial_x\partial_x^{k-1}w - &\sqrt{w}\partial_\psi^2\partial_x^{k-1}w \\
&= -2\partial_x^{k-1}(\rho\partial_x P) + \sum_{m=1}^{k-2} C_{k-1}^m \left(\partial_x^{k-1-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w + \left(\partial_x^{k-1}\sqrt{w}\right)\partial_\psi^2 w \\
&\quad + 2\sum_{m=0}^{k-1} C_{k-1}^m \partial_x^{k-1-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w \\
&= -2\partial_x^{k-1}(\rho\partial_x P) + \sum_{m=1}^{k-2} C_{k-1}^m \left(\partial_x^{k-1-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w + \frac{\partial_x^{k-1}w}{2\sqrt{w}}\frac{\partial_x w}{\sqrt{w}} \\
&\quad + \left(\frac{\partial_x^{k-1}w}{2\sqrt{w}}\right)\frac{2\rho\partial_x P}{\sqrt{w}} - \left(\frac{\partial_x^{k-1}w}{2\sqrt{w}}\right)\frac{2\frac{\partial_x\rho}{\rho}w}{\sqrt{w}} + 2\sum_{m=0}^{k-1} C_{k-1}^m \partial_x^{k-1-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w \\
&\quad + \sum_{m=0}^{k-2} C_{k-2}^m \partial_\psi^2 w\partial_x^{m+1}w\partial_x^{k-2-m}\frac{1}{2\sqrt{w}} \\
&:= \sum_{i=1}^{4} I_i
\end{aligned}
$$

and

$$
\begin{aligned}
I_1 &= -2\partial_x^{k-1}(\rho\partial_x P) + \sum_{m=1}^{k-2} C_{k-1}^m \left(\partial_x^{k-1-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w + \frac{\partial_x^{k-1}w}{2\sqrt{w}}\frac{\partial_x w}{\sqrt{w}}, \\
I_2 &= \frac{\rho\partial_x P}{w}\partial_x^{k-1}w, \\
I_3 &= -\frac{\partial_x\rho}{\rho}\partial_x^{k-1}w + 2\sum_{m=0}^{k-1} C_{k-1}^m \partial_x^{k-1-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w, \\
I_4 &= \sum_{m=0}^{k-2} C_{k-2}^m \partial_\psi^2 w\partial_x^{m+1}w\partial_x^{k-2-m}\frac{1}{2\sqrt{w}}.
\end{aligned}
$$

For $x \geq \frac{5\varepsilon}{8}$, we consider the following equality

$$
\partial_x f_1 - \sqrt{w(p_1)}\partial_\psi^2 f_1 = \frac{\sqrt{w(p_1)} - \sqrt{w(p)}}{-h}\partial_\psi^2\partial_x^{k-1}w(p) + \sum_{i=1}^{4}\frac{1}{-h}\left(I_i(p_1) - I_i(p)\right), \tag{2.9}
$$

where

$$
f_1 = \frac{1}{-h}(\partial_x^{k-1}w(p_1) - \partial_x^{k-1}w(p)),
$$

with $p_1 = (x-h,\psi)$, $p = (x,\psi)$.

For any $x \geq \frac{5\varepsilon}{8}$, by (2.6), it is easy to conclude that

$$\begin{aligned}
&\left|\frac{1}{-h}(\sqrt{w}(p_1) - \sqrt{w}(p))\partial_\psi^2\partial_x^{k-1}w(p)\right| \leq C\psi^{-\frac{1}{2}},\\
&\left|\frac{1}{-h}(I_1(p_1) - I_1(p))\right| \leq C\psi^{-\frac{1}{2}},\\
&\left|\sum_{i=3}^{4}\frac{1}{-h}(I_i(p_1) - I_i(p))\right| \leq C\psi^{-\frac{1}{2}},
\end{aligned} \tag{2.10}$$

where C is dependent on the parameter h.

Since

$$\frac{1}{-h}(I_2(p_1) - I_2(p)) = f_1 \cdot [\frac{\rho\partial_x P}{w}(p_1)] + \partial_x^{k-1}w(p)\frac{1}{-h}[\frac{\rho\partial_x P}{w}(p_1) - \frac{\rho\partial_x P}{w}(p)],$$

combining with (2.6), $f_1(p_0) > 0$ and $\partial_x P \leq 0$, it holds at $p = p_0$ that

$$\frac{1}{-h}(I_2\,(p_1) - I_2(p_0)) \leq C. \tag{2.11}$$

Summing up (2.10) and (2.11), we conclude that at $p = p_0$,

$$\partial_x f_1 - \sqrt{w}\partial_\psi^2 f_1 \leq C_0\psi^{-\frac{1}{2}}.$$

This along with (2.8) shows that for $x \geq \frac{5\varepsilon}{8}$, it holds at $p = p_0$ that

$$\partial_x f - \sqrt{w}\partial_\psi^2 f \leq C\psi^{-\frac{1}{2}} - \xi M\psi^{-\frac{1}{2}}. \tag{2.12}$$

By taking M large enough, we have $\partial_x f\,(p_0) - \sqrt{w}\partial_\psi^2 f\,(p_0) < 0$. By the definition of p_0, we obtain

$$\partial_x f\,(p_0) - \sqrt{w}\partial_\psi^2 f\,(p_0) \geq 0,$$

which leads to a contradiction. Therefore, for M chosen as above and independent of h, we have

$$\max_{\bar{\Omega}} f \leq 0.$$

We can similarly prove that $\min_{\bar{\Omega}} f \geq 0$ by replacing $M\psi\ln\psi$ in f with $-M\psi\ln\psi$. By the arbitrariness of h, for any $(x,\psi) \in (\frac{7\varepsilon}{8}, X] \times (0,\delta_3]$ we have

$$|\partial_x^k w| \leq -M\psi\ln\psi.$$

Due to

$$2\sqrt{w}\partial_x^k\sqrt{w} + \sum_{m=1}^{k-1} C_k^m(\partial_x^m\sqrt{w}\partial_x^{k-m}\sqrt{w}) = \partial_x^k(\sqrt{w}\sqrt{w}) = \partial_x^k w, \tag{2.13}$$

which along with (2.6) shows that in $(\frac{7}{8}\varepsilon, X] \times (0,\delta_3]$,

$$|\sqrt{w}\partial_x^k\sqrt{w}| \leq -C\psi\ln\psi.$$

Lemma 2.5. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and $\partial_x P$ are smooth. Assume that* (2.6) *holds, then for any $(x,\psi) \in [\frac{15}{16}\varepsilon, X] \times [0,\delta_3]$,*

$$|\partial_x^k w| \le C\psi, \quad |\partial_x^k \sqrt{w}| \le C\psi^{\frac{1}{2}}.$$

Proof. Take a smooth cutoff function $\phi(x)$ so that

$$\phi(x) = 1, x \in [\frac{15\varepsilon}{16}, X], \quad \phi(x) = 0, x \in [0, \frac{7\varepsilon}{8}].$$

Set

$$f = \partial_x^k w\phi - \mu_1\psi + \mu_2\psi^{\frac{3}{2}-\beta}$$

with constants μ_1, μ_2. Let β be small enough in Lemma 2.4. Then it holds in $[\frac{7\varepsilon}{8}, X] \times [0,\delta_3]$ that

$$|\partial_x^k w| \le C\psi^{1-\beta}, \quad |\partial_x^k \sqrt{w}| \le C\psi^{\frac{1}{2}-\beta}. \tag{2.14}$$

We denote

$$\Omega = \{(x,\psi)|0 < x \le X, 0 < \psi < \delta_3\}.$$

As in [23], we have $f(x,0) = 0$, $f(0,\psi) \le 0$ and $f(x,\delta_3) \le 0$ by taking μ_1 large depending on μ_2. We claim that the maximum of f cannot be achieved in the interior.

By (1.6), we have

$$\begin{aligned}\partial_x\partial_x^k w - &\sqrt{w}\partial_\psi^2\partial_x^k w \\ &= -2\partial_x^k(\rho\partial_x P) + \sum_{m=0}^{k-1} C_k^m(\partial_x^{k-m}\sqrt{w})\partial_\psi^2\partial_x^m w + 2\sum_{m=0}^{k} C_k^m \partial_x^{k-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w,\end{aligned}$$

and

$$\partial_\psi^2\partial_x^m w = \partial_x^m\partial_\psi^2 w = \partial_x^m\left(\frac{\partial_x w}{\sqrt{w}} + \frac{2\rho\partial_x P}{\sqrt{w}} - \frac{2\partial_x\rho}{\rho}\sqrt{w}\right).$$

For any $x \ge \frac{7\varepsilon}{8}$, $0 \le j \le k-1$ and $0 \le m \le k-1$, from (2.6) and (2.14), we get

$$|\partial_x^j w| \le C\psi, \quad |\partial_x^k w| \le C\psi^{1-\beta}, \quad |\partial_x^{k-m}\sqrt{w}| \le C\psi^{\frac{1}{2}-\beta}.$$

Then let $\beta \ll \frac{1}{2}$, for $0 \le m \le k-1$ and $x \ge \frac{7\varepsilon}{8}$, we obtain

$$|\partial_\psi^2\partial_x^m w| \le C\psi^{\frac{1}{2}-\beta} + C\psi^{-\frac{1}{2}} + C\psi^{\frac{1}{2}-\beta} \le C\psi^{-\frac{1}{2}}.$$

Therefore, we conclude that for $x \ge \frac{7\varepsilon}{8}$,

$$\partial_x\partial_x^k w - \sqrt{w}\partial_\psi^2\partial_x^k w \le C + C\psi^{-\beta} + C\psi^{1-\beta} \le C\psi^{-\beta}.$$

By the above inequality and (2.1), it holds at $p = p_0$ that

$$\begin{aligned}\partial_x f - \sqrt{w}\partial_\psi^2 f &= \partial_x\partial_x^k w - \sqrt{w}\partial_\psi^2\partial_x^k w + \partial_x^k w\partial_x\phi - \sqrt{w}\partial_\psi^2(-\mu_1\psi + \mu_2\psi^{\frac{3}{2}-\beta}) \\ &\le C_2\psi^{-\beta} - \xi\mu_2\psi^{-\beta},\end{aligned}$$

where $\xi = \left(\frac{3}{2} - \beta\right)\left(\frac{1}{2} - \beta\right)\sqrt{m} > 0$. Then we have $\partial_x f - \sqrt{w}\partial_\psi^2 f < 0$ in Ω by taking μ_2 large depending on C_2. This means that the maximum of f cannot be achieved in the interior. Therefore, we have

$$\max_{\bar{\Omega}} f \leq 0.$$

In the same way, we can prove that

$$\max_{\bar{\Omega}} -\partial_x^k w\phi - \mu_1\psi + \mu_2\psi^{\frac{3}{2}-\beta} \leq 0.$$

So, for any $(x,\psi) \in [\frac{15}{16}\varepsilon, X] \times [0, \delta_3]$, we have

$$|\partial_x^k w| \leq \mu_1\psi - \mu_2\psi^{\frac{3}{2}-\beta} \leq \mu_1\psi.$$

Combining with (2.6) and (2.13), it holds in $[\frac{15}{16}\varepsilon, X] \times [0, \delta_3]$ that

$$|\partial_x^k \sqrt{w}| \leq C\psi^{\frac{1}{2}}.$$

This completes the proof of the lemma.

Lemma 2.6. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and $\partial_x P$ are smooth. Assume that* (2.6) *holds, then for any $(x,\psi) \in [\varepsilon, X] \times [0, \delta_4]$,*

$$|\partial_\psi\partial_x^k w| \leq C, \quad |\partial_\psi^2\partial_x^k w| \leq C\psi^{-1}, \quad |\partial_x^{k+1} w| \leq C\psi^{-\frac{1}{2}}.$$

Proof. By Lemma 2.5 and (2.6), for any $(x,\psi) \in [\frac{15}{16}\varepsilon, X] \times [0, \delta_3]$,

$$|\partial_x^j w| \leq C\psi, \quad |\partial_x^j \sqrt{w}| \leq C\psi^{\frac{1}{2}}, \quad 0 \leq j \leq k. \tag{2.15}$$

Set $\Psi_0 = \min\{\frac{2}{3}\delta_3, \frac{\varepsilon}{16}\}$, for $(x_0, \psi_0) \in [\varepsilon, X] \times (0, \Psi_0]$, we denote

$$\Omega = \{(x,\psi)|x_0 - \psi_0^{\frac{3}{2}} \leq x \leq x_0, \frac{1}{2}\psi_0 \leq \psi \leq \frac{3}{2}\psi_0\}.$$

A direct calculation gives

$$\begin{aligned}\partial_x\partial_x^k w - \sqrt{w}\partial_\psi^2\partial_x^k w = &-2\partial_x^k(\rho\partial_x P) + \partial_x^k \sqrt{w}\partial_\psi^2 w + \sum_{m=1}^{k-2} C_k^m \left(\partial_x^{k-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w \\ &+ C_k^{k-1}\frac{\partial_x w}{2\sqrt{w}}\partial_\psi^2\partial_x^{k-1} w + 2\sum_{m=0}^{k} C_k^m \partial_x^{k-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w.\end{aligned}$$

By (1.6), we obtain

$$\begin{aligned}\partial_\psi^2\partial_x^m w &= \partial_x^m\partial_\psi^2 w \\ &= \partial_x^m\left(\frac{\partial_x w}{\sqrt{w}} + \frac{2\rho\partial_x P}{\sqrt{w}} - 2\frac{\partial_x\rho}{\rho}\sqrt{w}\right) \\ &= \frac{\partial_x^{m+1} w}{\sqrt{w}} + \sum_{l=1}^{m} C_m^l \partial_x^{m-l+1} w\partial_x^l \frac{1}{\sqrt{w}} + \partial_x^m\left(\frac{2\rho\partial_x P}{\sqrt{w}}\right) - \partial_x^m\left(2\frac{\partial_x\rho}{\rho}\sqrt{w}\right),\end{aligned}$$

and

$$\partial_x^k \sqrt{w} = \partial_x^{k-1} \frac{\partial_x w}{2\sqrt{w}} = \frac{\partial_x^k w}{2\sqrt{w}} + \sum_{l=1}^{k-1} C_{k-1}^l \partial_x^{k-1-l+1} w \partial_x^l \frac{1}{2\sqrt{w}},$$

then

$$\begin{aligned}
\partial_x \partial_x^k w - &\sqrt{w} \partial_\psi^2 \partial_x^k w \\
=& -2\partial_x^k(\rho \partial_x P) + \sum_{m=1}^{k-2} C_k^m (\partial_x^{k-m} \sqrt{w}) \partial_\psi^2 \partial_x^m w \\
&+ \frac{\partial_x^k w}{2\sqrt{w}} \partial_\psi^2 w + \sum_{l=1}^{k-1} C_{k-1}^l \partial_x^{k-1-l+1} w \partial_x^l \left(\frac{1}{2\sqrt{w}} \right) \partial_\psi^2 w \\
&+ C_k^{k-1} \frac{\partial_x w \partial_x^k w}{2w} + 2 \sum_{m=0}^{k} C_k^m \partial_x^{k-m} \left(\frac{\partial_x \rho}{\rho} \right) \partial_x^m w \\
&+ C_k^{k-1} \frac{\partial_x w}{2\sqrt{w}} \left[\sum_{l=1}^{k-1} C_{k-1}^l \partial_x^{k-l} w \partial_x^l \frac{1}{\sqrt{w}} + \partial_x^{k-1} \left(\frac{2\rho \partial_x P}{\sqrt{w}} \right) - \partial_x^{k-1} \left(2 \frac{\partial_x \rho}{\rho} \sqrt{w} \right) \right].
\end{aligned}$$

The following transformation f is defined:

$$\Omega \to \widetilde{\Omega} := [-1, 0]_{\tilde{x}} \times [-\frac{1}{2}, \frac{1}{2}]_{\tilde{\psi}}, \quad (x, \psi) \mapsto (\tilde{x}, \tilde{\psi}),$$

where $x - x_0 = \psi_0^{\frac{3}{2}} \tilde{x}, \psi - \psi_0 = \psi_0 \tilde{\psi}$.

Let $f = \partial_x^k w \psi_0^{-1}$, we get

$$\begin{aligned}
\partial_{\tilde{x}} f - &\frac{\sqrt{w}}{\psi_0^{\frac{1}{2}}} \partial_{\tilde{\psi}}^2 f - \frac{1}{2\sqrt{w}} \partial_\psi^2 w \psi_0^{\frac{3}{2}} f - \frac{\partial_x w}{2w} \psi_0^{\frac{3}{2}} f \\
=& -2\psi_0^{\frac{1}{2}} \partial_x^k(\rho \partial_x P) + \psi_0^{\frac{1}{2}} \sum_{m=1}^{k-2} C_k^m (\partial_x^{k-m} \sqrt{w}) \partial_\psi^2 \partial_x^m w \\
&+ \psi_0^{\frac{1}{2}} \sum_{l=1}^{k-1} C_{k-1}^l \partial_x^{k-l} w \left(\partial_x^l \frac{1}{2\sqrt{w}} \right) \partial_\psi^2 w \\
&+ 2\psi_0^{\frac{1}{2}} \sum_{m=0}^{k} C_k^m \partial_x^{k-m} \left(\frac{\partial_x \rho}{\rho} \right) \partial_x^m w \\
&+ \psi_0^{\frac{1}{2}} \frac{\partial_x w}{2\sqrt{w}} \left[\sum_{l=1}^{k-1} C_{k-1}^l \partial_x^{k-l} w \partial_x^l \frac{1}{\sqrt{w}} + \partial_x^{k-1} \left(\frac{2\rho \partial_x P}{\sqrt{w}} \right) - \partial_x^{k-1} \left(2 \frac{\partial_x \rho}{\rho} \sqrt{w} \right) \right] \\
:=& F.
\end{aligned}$$

From the proof of Lemma 2.2 and Lemma 2.6, we know that in $\widetilde{\Omega}$ for $\alpha \in (0, 1)$,

$$|f| \le C, \quad 0 < c \le \psi_0^{-\frac{1}{2}} \sqrt{w} \le C, \quad |\psi_0^{-\frac{1}{2}} \sqrt{w}|_{C^\alpha(\widetilde{\Omega})} \le C.$$

By (2.6), (2.15) and the equality

$$\partial_\psi\left(\partial_\psi^2\partial_x^m w\right)=\frac{\partial_\psi\partial_x^{m+1}w}{\sqrt{w}}-\frac{\partial_\psi w\partial_x^{m+1}w}{2(\sqrt{w})^3}+\sum_{l=1}^{m}C_m^l\partial_x^{m-l+1}\partial_\psi w\partial_x^l\frac{1}{\sqrt{w}}$$
$$+\sum_{l=1}^{m}C_m^l\partial_x^{m-l+1}w\partial_x^l\frac{\partial_\psi w}{-2(\sqrt{w})^3}+\partial_x^m\left(\frac{\rho\partial_xP\partial_\psi w}{-(\sqrt{w})^3}\right)-\partial_x^m\left(\frac{\partial_x\rho}{\rho}\frac{\partial_\psi w}{\sqrt{w}}\right),$$

we can conclude that for $j\le k-1$ and $m\le k-2$,

$$\left|\nabla_{\tilde{x},\tilde{\psi}}\partial_x^j\sqrt{w}\right|\le C\psi_0^{\frac{1}{2}},\quad \left|\nabla_{\tilde{x},\tilde{\psi}}\partial_x^j(\frac{1}{\sqrt{w}})\right|\le C\psi_0^{-\frac{1}{2}},\quad \left|\nabla_{\tilde{x},\tilde{\psi}}\partial_\psi^2\partial_x^m w\right|\le C\psi_0^{-\frac{1}{2}}.$$

Combining (2.4) with (2.5), we can obtain

$$|\frac{1}{2\sqrt{w}}\partial_{\tilde{\psi}}^2 w\psi_0^{\frac{3}{2}}+\frac{\partial_x w}{2w}\psi_0^{\frac{3}{2}}|_{C^\alpha(\widetilde{\Omega})}+|F|_{C^\alpha(\widetilde{\Omega})}\le C.$$

By the standard interior priori estimates, we obtain

$$|\partial_{\tilde{x}}f|_{L^\infty([-\frac{1}{4},0]_{\tilde{x}}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde{\psi}})}+|\partial_{\tilde{\psi}}f|_{L^\infty([-\frac{1}{4},0]_{\tilde{x}}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde{\psi}})}+|\partial_{\tilde{\psi}}^2f|_{L^\infty([-\frac{1}{4},0]_{\tilde{x}}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde{\psi}})}\le C.$$

Therefore, this means that

$$|\partial_x^{k+1}w\,(x_0,\psi_0)\,|\le C\psi_0^{-\frac{1}{2}},\quad |\partial_\psi\partial_x^k w\,(x_0,\psi_0)\,|\le C,\quad |\partial_\psi^2\partial_x^k w\,(x_0,\psi_0)\,|\le C\psi_0^{-1}.$$

Since (x_0,ψ_0) is arbitrary, this completes the proof of the lemma.

Lemma 2.7. *If u is a solution for equation* (1.1) *in Theorem 1.1, assume u_0 satisfies the condition* (1.9) *and the known function ρ and ∂_xP are smooth. Assume $0<\varepsilon<X$ and integer $m,k\ge 0$, then there exists a positive constant $\delta>0$ such that for any $(x,\psi)\in[\varepsilon,X]\times[0,\delta]$,*

$$|\partial_\psi^m\partial_x^k w|\le C\psi^{1-m}. \tag{2.16}$$

Proof. From Lemma 2.1, (2.1), Lemma 2.2 and Lemma 2.3, a direct calculation can prove that

$$\left|\partial_x^k\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac{1}{2}},\quad \left|\partial_x^k\partial_\psi\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac{3}{2}},\quad \left|\partial_x^k\partial_\psi^2\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac{5}{2}},$$

and (2.16) holds for $m=0,1,2$. Then for $0\le m\le j$ with $j\ge 1$, we inductively assume that

$$|\partial_\psi^m\partial_x^k w|\le C\psi^{1-m},\quad \left|\partial_x^k\partial_\psi^m\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac{1}{2}-m}. \tag{2.17}$$

In the next part, we will prove that (2.17) still holds for $m=j+1$.

By (1.6), we obtain

$$\begin{aligned}\partial_\psi^{j+1}\partial_x^k w&=\partial_\psi^{j-1}\partial_x^k\partial_\psi^2 w\\&=\partial_x^k\partial_\psi^{j-1}\left(\frac{\partial_x w}{\sqrt{w}}+\frac{2\rho\partial_xP}{\sqrt{w}}-2\frac{\partial_x\rho}{\rho}\frac{w}{\sqrt{w}}\right)\\&=\partial_x^k\left(\sum_{i=0}^{j-1}C_{j-1}^i\partial_\psi^{j-1-i}\partial_x w\partial_\psi^i\frac{1}{\sqrt{w}}+2\rho\partial_xP\partial_\psi^{j-1}\frac{1}{\sqrt{w}}-2\frac{\partial_x\rho}{\rho}\sum_{i=0}^{j-1}C_{j-1}^i\partial_\psi^{j-1-i}w\partial_\psi^i\frac{1}{\sqrt{w}}\right).\end{aligned}$$

Combining with (2.17), we get

$$|\partial_\psi^{j+1}\partial_x^k w| \le C\psi^{\frac{3}{2}-j} + C\psi^{\frac{1}{2}-j} + C\psi^{\frac{3}{2}-j} \le C\psi^{\frac{1}{2}-j}. \tag{2.18}$$

By straight calculations, we get

$$\begin{aligned}
0 &= \partial_x^k\partial_\psi^{j+1}\Big(\frac{1}{\sqrt{w}}\frac{1}{\sqrt{w}}w\Big)\\
&= \partial_x^k\Big[2\sqrt{w}\partial_\psi^{j+1}\frac{1}{\sqrt{w}} + \sum_{i=1}^{j}\sum_{l=0}^{j+1-i} C_{j+1}^i C_{j+1-i}^l \left(\partial_\psi^i \frac{1}{\sqrt{w}}\right)\left(\partial_\psi^l \frac{1}{\sqrt{w}}\right)\partial_\psi^{j+1-l-i}w\\
&\qquad + \sum_{l=0}^{j} C_{j+1}^l \frac{1}{\sqrt{w}}\left(\partial_\psi^l \frac{1}{\sqrt{w}}\right)\partial_\psi^{j+1-l}w\Big].
\end{aligned}$$

Combining the above equality with (2.17), we can conclude that

$$\Big|\partial_x^k\partial_\psi^{j+1}\frac{1}{\sqrt{w}}\Big| \le C\psi^{-\frac{3}{2}-j}.$$

This completes the proof of the lemma.

3. Proof of Theorem 1.5

In this section, we will prove the regularity of the solution u in the domain

$$\{(x,\psi)|\varepsilon \le x \le X, 0 \le y \le Y_1\}.$$

Proof of Theorem 1.5:

Proof. For the convenience of proof, we denote

$$(\tilde{x},\psi) = \left(x, \int_0^y \tilde{u}dy\right).$$

A direct calculation gives (see P13 in [23])

$$\partial_y = \sqrt{w}\partial_\psi, \quad \partial_x = \partial_{\tilde{x}} + \partial_x\psi(x,y)\partial_\psi, \quad \partial_x\psi = \frac{1}{2}\sqrt{w}\int_0^\psi w^{-\frac{3}{2}}\partial_{\tilde{x}}w d\psi.$$

By (2.1) and Lemma 2.3, we have $|\partial_x\psi| \le C\psi$. Due to $\partial_y = \sqrt{w}\partial_\psi$, we obtain

$$\begin{aligned}
\partial_x^k 2\partial_y\tilde{u} &= \left(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi\right)^k \partial_\psi w,\\
\partial_x^k 2\partial_y^2\tilde{u} &= \left(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi\right)^k \left(\partial_{\tilde{x}}w + 2\rho\partial_x P - 2\frac{\partial_x\rho}{\rho}w\right)\\
&= \left(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi\right)^k(\partial_{\tilde{x}}w) + 2\partial_{\tilde{x}}^k(\rho\partial_x P) - 2\left(\frac{\partial_x\rho}{\rho}\right)(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi)^k w - 2\partial_{\tilde{x}}^k\left(\frac{\partial_x\rho}{\rho}\right)w.
\end{aligned}$$

By $|\partial_x\psi| \le C\psi$ and Lemma 2.7, we obtain that Theorem 1.5 holds for $m = 0, 1, 2$,

$$|\partial_x^k\partial_y\tilde{u}| + |\partial_x^k\partial_y^2\tilde{u}| \le C. \tag{3.1}$$

We inductively assume that for any integer k and $m \ge 1$,

$$|\partial_x^k\partial_y^j\tilde{u}| \le C, \quad j \le m. \tag{3.2}$$

A direct calculation gives

$$\begin{aligned}
&\partial_x^k\partial_y^{m+1}\tilde{u}\\
&= \partial_x^k\partial_y^{m-1}\partial_y^2\tilde{u}\\
&= \partial_x^k\partial_y^{m-1}\left(\tilde{u}\partial_x\tilde{u} - \partial_y\tilde{u}\int_0^y \partial_x\tilde{u}dy - \frac{\partial_x\rho}{\rho}\tilde{u}^2\right)\\
&= \partial_x^k\Big(\sum_{i=0}^{m-1} C_{m-1}^i\partial_y^{m-1-i}\tilde{u}\partial_y^i\partial_x\tilde{u} - \sum_{i=0}^{m-2} C_{m-1}^{i+1}\partial_y^{m-1-i}\tilde{u}\partial_y^i\partial_x\tilde{u} - \partial_y^m\tilde{u}\int_0^y \partial_x\tilde{u}dy - \frac{\partial_x\rho}{\rho}\partial_y^{m-1}\tilde{u}^2\Big),
\end{aligned}$$

and we can deduce from (3.1) and (3.2) that

$$|\partial_x^k\partial_y^j\tilde{u}| \le C, \quad j \le m+1. \quad \Rightarrow \quad |\partial_x^k\partial_y^j u| \le C, \quad j \le m+1.$$

This completes the proof of the theorem.

4. Proof of Theorem 1.3 and 1.6

In this section, we prove our main theorem. The key point is to prove that (1.6) is a uniform parabolic equation. The proof is based on the classical parabolic maximum principle. The specific proof details are as follows.

Proof. By (1.2) and $\partial_x P \le 0$, we obtain

$$C \ge U^2(x) = U^2(0) - 2\int_0^x \frac{\partial_x P(\rho)}{\rho}dx \ge U^2(0).$$

By (1.7) and w increasing in ψ (see below), we know that there exists some positive constants Ψ and C_0 such that for any $(x, \psi) \in [0, X] \times [\Psi, +\infty)$,

$$w \ge C_0U^2(0). \tag{4.1}$$

From Theorem 1.1, we know that there exists positive constants y_0, M, m such that for any $(x, \psi) \in [0, X] \times [0, y_0]$ (we can take y_0 to be small enough),

$$M \ge \partial_y\tilde{u}(x, y) \ge m. \tag{4.2}$$

The fact that $\psi \sim y^2$ is near the boundary $y = 0$ (see Remark 4.1 in [23]), for some small positive constant $0 < \kappa < 1$, we get

$$\frac{\kappa}{2}y_0^2 \le \psi \le \kappa y_0^2 \Rightarrow \sigma y_0 \le y \le \frac{y_0}{2}, \tag{4.3}$$

for some constant $\sigma > 0$ depends on κ, m, M.

We denote

$$\Omega = \{(x,\psi)|0 \le x \le X, \frac{\kappa}{2}y_0^2 \le \psi \le +\infty\}.$$

By (4.2) and (4.3), we get $\tilde{u}(x, \sigma y_0) \ge m\sigma y_0$, then for any $x \in [0, X]$, we have

$$w(x, \frac{\kappa}{2}y_0^2) \ge m^2\sigma^2 y_0^2. \tag{4.4}$$

Since the initial data u_0 satisfies the condition (1.9) and $w = \tilde{u}^2$, we know $w(0,\psi) > 0$ for $\psi > 0$ and there exists a positive constant ζ, such that for $\psi \in [\frac{\kappa}{2}y_0^2, \Psi]$,

$$w(0,\psi) > \zeta. \tag{4.5}$$

Then, we only consider

$$\Omega_1 = \{(x,\psi)|0 \le x \le X, \frac{\kappa}{2}y_0^2 \le \psi \le \Psi\}.$$

We denote $H(x,\psi) := e^{-\lambda x}\partial_\psi w(x,\psi)$, which satisfies the following system in the region $\Omega_0 = \{(x,\psi)|0 \le x < X, 0 < \psi < +\infty\}$:

$$\begin{cases} \partial_x H - \dfrac{\partial_\psi w}{2\sqrt{w}}\partial_\psi H - \sqrt{w}\partial_\psi^2 H + (\lambda - 2\dfrac{\partial_x \rho}{\rho})H = 0, \\ H|_{x=0} = \partial_\psi w_0(\psi), \quad H|_{\psi=0} = 2e^{-\lambda x}\partial_y \tilde{u}|_{y=0}, \quad H|_{\psi=+\infty} = 0. \end{cases} \tag{4.6}$$

Then, we choose λ properly large such that $\lambda - 2\frac{\partial_x\rho}{\rho} \ge 0$. Due to

$$H|_{x=0} = \partial_\psi w_0(\psi) \ge 0, \quad H|_{\psi=0} = 2e^{-\lambda x}\partial_y \tilde{u}|_{y=0} > 0, \quad H|_{\psi=+\infty} = 0,$$

it follows that

$$H(x,\psi) = e^{-\lambda x}F(x,\psi) = e^{-\lambda x}\partial_\psi w \ge 0, \quad (x,\psi) \in [0, X^*) \times \mathbb{R}_+,$$

which means $\partial_\psi w \ge 0$ in $[0, X) \times \mathbb{R}_+$. Hence, w is increasing in ψ. Therefore, we know that there exists a positive constant $\lambda \ge m^2\sigma^2 y_0^2$ such that for any $x \in [0, X]$,

$$w(x,\Psi) \ge \lambda. \tag{4.7}$$

By (1.6), for any $\varepsilon > 0$, we know $W := w + \varepsilon x$ satisfies the following system in Ω_1 :

$$\begin{cases} \partial_x W - \sqrt{w}\partial_\psi^2 W - 2\dfrac{\partial_x \rho}{\rho}W = \mathcal{F}, \\ W|_{x=0} = W_0 > \zeta, \quad W|_{\psi=\frac{\kappa}{2}y_0^2} = W_1 \ge m^2\sigma^2 y_0^2, \quad W|_{\psi=\Psi} = W_2 \ge \lambda, \end{cases}$$

where

$$\mathcal{F} = -2\rho\partial_x P + \varepsilon - 2\varepsilon x\frac{\partial_x \rho}{\rho}.$$

Since $\partial_x P \le 0$, we know the diffusive term $\mathcal{F} > 0$. Therefore, the minimum cannot be reached inside Ω_1. Set

$$\eta_0 = \min\{W_0, W_1, W_2\},$$

then by the maximum principle, we obtain $W = w + \varepsilon x \ge \eta_0$. Let $\varepsilon \to 0$, we have $w \ge \eta_0$ in Ω_1. Then we denote

$$\eta = \min\left\{\eta_0, C_0 U^2(0)\right\} > 0,$$

combining with (4.1), we have $w \ge \eta$ in Ω. Therefore, there exists some positive constant c such that $c \le w$ in Ω. From Theorem 1.1, we have $w \le C$ in Ω. In sum, there exists positive constants c, C such that $c \le w \le C$ in Ω. This further means that

$$0 < \sqrt{c} \le \sqrt{w} \le \sqrt{C}, \tag{4.8}$$

where C depends on X. Therefore, we prove (1.6) is a uniform parabolic equation. Furthermore, by Theorem 1.1, we know $\partial_y \tilde{u}, \partial_y^2 \tilde{u}$ are continuous and bounded in $[0, X) \times \mathbb{R}_+$. Combining ρ, $\partial_x P$ are smooth, (4.8) with

$$2\partial_y \tilde{u} = \partial_\psi w, \quad 2\partial_y^2 \tilde{u} = \sqrt{w}\partial_\psi^2 w = \partial_x w - 2\frac{\partial_x \rho}{\rho} w + 2\rho \partial_x P(\rho),$$

we obtain

$$\|\sqrt{w}\|_{C^\alpha(\Omega)} \le C.$$

Once we have the above conclusion, the proof of Theorem 1.6 can be given in a similar fashion to [23]. Here, we provide a brief explanation for the reader's convenience. More details can be found in [23].

Step 1: For any $(x_1, \psi_1) \in [\varepsilon, X] \times [\kappa y_0^2, +\infty)$, we denote

$$\Omega_{x_1,\psi_1} = \{(x, \psi) | x_1 - \frac{\varepsilon}{2} \le x \le x_1, \psi_1 - \frac{\kappa}{2} y_0^2 \le \psi \le \psi_1 + \frac{\kappa}{2} y_0^2\}.$$

Step 2: Note that the known function ρ, $\partial_x P$ is smooth, we can repeat interior Schauder estimates in Ω_{x_1,ψ_1} to achieve uniform estimates independent of choice of (x_1, ψ_1) for any order derivatives of w. Since the width and the length of Ω_{x_1,ψ_1} are constants and the estimates employed are independent of (x_1, ψ_1), restricting the estimates to the point (x_1, ψ_1), we can get for any $m < +\infty, |\nabla^m w(x_1, \psi)| \le C_{X,m,y_0,\varepsilon}$.

Step 3: Since (x_1, ψ_1) is arbitrary, we have for any $m < +\infty, |\nabla^m w(x_1, \psi)| \le C_{X,m,y_0,\varepsilon}$ in $[\varepsilon, X] \times [\kappa y_0^2, +\infty)$. Then, as in Section 3, we can prove Theorem 1.6.

Finally, Theorem 1.3 is proven by combining Theorem 1.5 and Theorem 1.6.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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