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Research article

Global dynamical behavior of solutions for finite degenerate fourth-order parabolic equations with mean curvature nonlinearity

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Abstract: In this work, the initial-boundary value problem for the global dynamical properties of solutions to a class of finite degenerate fourth-order parabolic equations with mean curvature nonlinearity is studied. With the help of the Nehari flow and Levine's concavity method, we establish some sharp-like threshold classifications of the initial data under sub-critical, critical and supercritical initial energy levels, that is, we describe the size of an initial data set. It requires the presumption that the initial data starting from one region of phase space have uniform global dynamical behavior, which means that the solution exists globally and decays via energy estimates that ultimately result in the solution tending to zero in the forward time. For the case in which the initial data corresponds to another region, we prove that the solutions related to these initial data are subject to blow-up phenomena in a finite time. In addition, we estimate the corresponding upper bound of the lifespan of the blow-up solution.

Keywords: global existence; finite time blow up; ground state solution; degenerate parabolic equation
Mathematics Subject Classification: 35K35, 35K65, 35A01, 35D30

1. Introduction

1.1. Setting of the problem

This paper focuses on the global dynamical behavior of solutions to a class of finite degenerate fourth-order parabolic equations with mean curvature nonlinearity on $\Omega \subset \mathbb{R}^2$, i.e.,

$$u_t + \Delta_X^2 u - \nabla_X \cdot \left(\frac{\nabla_X u}{\sqrt{1 + |\nabla_X u|^2}} \right) = |u|^{p-2} u, \qquad x \in \Omega, t > 0, \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \qquad x \in \Omega, t > 0, \tag{1.2}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where Ω is a bounded open domain with C^4 boundary $\partial\Omega$ or a bounded convex polygonal, which satisfies that $\Omega \subset\subset \Omega'$ and Ω' is an open bounded domain of $\mathbb{R}^2$. ν is the unit outward normal on $\partial\Omega$ and $u_0 \in H^2_{X,0}(\Omega)$. $X = (X_1, X_2)$ is a system of C^∞ smooth linearly independent vector fields defined on Ω' with $X_j^* = -X_j$, and $\Delta_X := \sum_{j=1}^2 X_j^2$ is a subelliptic operator. Moreover, $1 < p < \frac{2}{\mu-2}$ and μ is the generalized Métivier index of X on Ω. In addition, we give the following assumptions:

(H1) There exists a family of (non-isotropic) dilations $\{\delta_\lambda\}_{\lambda>0}$ of the form

$$\delta_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, \quad \delta_\lambda(x) = (\lambda^{\sigma_1} x_1, \lambda^{\sigma_2} x_2),$$

where $1 = \sigma_1 \le \sigma_2$ represents integers satisfying that X_i is δ_λ-homogeneous of degree 1* :

$$X_j(f \circ \delta_\lambda) = \lambda(X_j f) \circ \delta_\lambda, \quad \forall \lambda > 0, f \in C^\infty(\mathbb{R}^2), j = 1, 2.$$

Next, we define by $\mu := \sum_{j=1}^{2} \sigma_j$ the so-called pointwise homogeneous dimension or non-isotropic dimension of $(\mathbb{R}^2, \delta_\lambda)$.

(H2) X fulfills Hörmander's condition [1] in Ω', i.e., X that is commuted as follows:

$$X_j = [X_{j1}, [X_{j2}, ...[X_{j_{k-1}}, X_{jk}]...]], \quad 1 \le j_i \le 2,$$

up to a certain fixed length $k \le Q$, span the tangent space at each point of Ω' and X_1, X_2 satisfy Hörmander's rank condition at 0, that is, $\dim\{Y(0) \mid Y \in \mathrm{Lie}(x)\} = 2$, where $\mathrm{Lie}(x)$ is the smallest Lie sub-algebra of the Lie algebra of the smooth vector fields containing X on $\mathbb{R}^2$. Note that $Q > 1$ is called the Hörmander's index of X on Ω', which is regarded as the smallest positive integer for the Hörmander's condition being fulfilled.

The global well-posedness and finite time blow-up of the solution to a nonlinear fourth order parabolic equation with mean curvature nonlinearity have been garnering widespread interest and arise in various applications in many fields, especially in mathematical biology and fluid dynamics [2, 3]. Indeed, many of the studies on epitaxial film growth have shown that it is characterized with mean curvature nonlinearities that occur mainly on the surfaces of materials, including studies on crystal growth, catalytic reactions and the production of nanostructures. One of the earliest models to apply a fourth-order parabolic differential equation to describe the growth of epitaxial films is given by

$$u_t = -K_1 \Delta^2 u + K_2 \det(D^2 u) + \xi(x, t), \tag{1.4}$$

where $u = u(x, t)$ describes the height of the growth interface at the spatial point $x \in \Omega$ at time $t \ge 0$, K_1 and K_2 are positive constants and the term $-K_1 \Delta^2 u$ is used to describe the random adatom† dispersal which attempts to minimize the system's chemical potential. One of the main reasons for this interest is that the suppression of the growth of epitaxial films of materials has a wide range of applications in material process manufacturing. For example, during the design of semiconductor-coated films (see [4] and the references therein), epitaxial film growth of some compositions reduces superconductivity at high

*The linear independence of the X_i is meant with respect to the vector space of the smooth vector fields on $\mathbb{R}^2$; this must not be confused with the linear independence of the vectors $X_1(x), X_2(x)$ in $\mathbb{R}^2$.

†adatom-atoms that are adsorbed onto the surface but have not yet become part of the crystal; they diffuse on a terrace and likely hit a terrace boundary.

temperatures. Therefore, the material's service life can be effectively increased by constructing a film on the substrate via chemical vapor deposition. Although the process is very complex, several descriptions and simulations of atomic models and continuum models exist for such problems [5]. A significant challenge in the construction of these models is understanding these growth processes qualitatively and quantitatively to formulate control laws that optimize specific properties of the films, such as flatness and electrical conductivity. Based on dynamical mean-field theory and phenomenological theory [5, 6], another model that can describe epitaxial growth related to problem (1.1) is given by

$$u_t = -\Delta^2 u - \lambda \nabla \cdot (|\nabla u|^{p-2} \nabla u) + g(x, t, u), \tag{1.5}$$

where $u : \Omega \times [0, \infty) \to \mathbb{R}$ describes the height of the growing interface at the spatial location $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ at the temporal instant $t \in [0, \infty)$, under the conditions that $\lambda = 1$ or -1, $p > 2$. The model focuses on describing the particle dynamics on the thin film at the microscopic level [6, 7]. Specifically, each term in the model reflects the corresponding physical significance of the particle, i.e., the term $-\Delta^2 u$ represents the Euler-Lagrange equation corresponding to the Willmore functional, which is subject to the fact that, with the increase of chemical bonds between atoms and the crystal structure, the random dispersion of atoms will minimize the chemical potential of the system. The term $-\lambda \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ describes the jumping of atoms, and the sign of λ represents the jumping direction (upward or downward) of atoms; and $g(x, t, u)$ represents the linear or the nonlinear noise. Among the many models describing epitaxial growth, there is a model containing mean curvature that can be used to describe the crystal surface growth process given by

$$u_t = -\Delta^2 u - \lambda \nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right) \tag{1.6}$$

in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$. This model is based on the BCF theory proposed by Burton, Cabrera and Frank [8]. And, it is primarily established on the two-dimensional bounded field $\Omega \subset \mathbb{R}^2$, where the mean curvature term is regarded as a degenerate operator to describe the mean curvature. It plays a central role in the analysis of the fundamental mathematics of a few physical and geometric issues, including the described mean curvature questions for Cartesian surfaces in the Euclidean space [9, 10], the capillarity phenomenon for incompressible fluids [11] and reaction-diffusion processes where the flux features saturation at the interface regimes. In particular, $u : \Omega \times [0, \infty) \to \mathbb{R}$ describes the height of the growing interface at the spatial location $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ at the temporal instant $t \in [0, \infty)$. The term $-\Delta^2 u$ denotes the surface diffusion, which is caused by the difference of the chemical potential that is proportional to the curvature of the surface. In the meantime, $-\lambda \nabla \cdot (\frac{\nabla u}{1+|\nabla u|^2})$ denotes the effect of surface roughening. Such roughening is caused by Schwoebel barriers [12].

The finite degenerate elliptic operators have been an active field of many investigations since the celebrated Hörmander's work on hypoellipticity [1, 13]. Nonlinear equations and systems involving degenerate vector fields have been widely applied in many different areas, such as Lewy's example [14], the $\bar{\partial}$-Neumann problem in complex geometry [15], the stochastic differential equations [16], the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$ in quantum mechanics [17] and nonholonomic mechanics, etc., many of which can be written as a specific mathematical models that include the form of the Hörmander vector field sum of squares operators [18, 19]. As a pioneer in micro-local theory of partial differential equations, Hörmander [1] has proven a fundamental conclusion about vector fields: if a family of smooth vector fields satisfies the finite rank condition, i.e., $X_1, X_2, ..., X_m$, and its commutators up to order r have a great rank at every point on the manifold, then the corresponding sum of squares operator is hypoelliptic and the associated subelliptic estimates have been admitted. Hence, Δ_X is regarded as the subelliptic operator, and the corresponding Harnack inequality and maximum principle of Δ_X have been investigated in [20]. Later, with the help of the celebrated lifting and approximating theory, Rothschild and Stein [21] obtained the sharp regularity estimates of the subelliptic operator Δ_X. At the same time, by applying Métivier's condition, Métivier [22] studied the eigenvalue problem for the subelliptic operator Δ_X. Furthermore, in order to define a Carnot-Carathéodory metric related to vector fields, which appears both in PDEs and sub-Riemannian geometry [23], Hörmander's condition plays an indispensable and important role. In addition, hypoelliptic structures are very common in degenerate equations with geometric problems. For instance, in typical cases of Heisenberg groups [24], groups of the Heisenberg type and even general Carnot groups [25], their Lie algebras have a family of smooth vector fields that satisfy Hörmander's rank condition; thus, the sub-Laplacian constructed by those vector fields is hypoelliptic. In particular, as long as X satisfies Hörmander's condition, one can claim that X and the corresponding subelliptic operator Δ_X represent the finite degenerate vector fields and the finitely degenerate operator, respectively.

For the vector fields $X = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$, Δ_X is exactly the usual Laplacian operator Δ, and the corresponding equation (1.1) has received a lot of attention in this case; for instance, King et al. [26] investigated the growth of nanoscale thin films, which is modeled by equations of the following form

$$u_t + \Delta^2 - \nabla \cdot (f \nabla u)) = g,$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $g \in L^2((0, T) \times \Omega)$, and they displayed the existence, uniqueness, positivity and asymptotic behavior of solutions in an appropriate functional space. In fact, there are many more equations induced by operator Δ_X to describe the epitaxial thin film model with a degenerate fourth-order parabolic equation. In [27], by means of energy and entropy estimates, the authors obtained some results on well-posedness in higher spatial dimensions for degenerate parabolic equations of fourth order with nonnegative initial data; they also give the positivity and asymptotic behavior of solutions for equations of the form

$$u_t + \Delta^2 + \nabla \cdot (m(u) \nabla u)) = g,$$

where m is a specific function. Furthermore, when the diffusion operator contains exponential nonlinearity, the fourth-order equation has more degeneration; in [28], they obtained the existence of the solution for the equation

$$u_t + \nabla \cdot \left(|\nabla \Delta u|^{p(x)-2} \nabla \Delta u \right) = f(x, u),$$

where p and f are specific functions. Such a model may describe some properties of medical magnetic resonance images in space and time. In the particular case in which the nonlinear source is given by $f(x, u) = u(x, t) - a(x)$, the function u and $a(x)$ represent a digital image and its observation, respectively. Recently, Zhang and Zhou [29] studied the well-posedness and dynamic properties of solutions to the initial-boundary value problem of the following form of fourth-order parabolic equations with mean curvature nonlinearity:

$$u_t + \Delta^2 u - \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = |u|^{p-2} u.$$

With the help of the semigroup method and potential well method, they showed the existence and uniqueness of strong solutions, and then they obtained several conditions for the global existence and finite time blow-up of the solution with different initial energies. Meanwhile, for the global solution, they proved the exponential decay and ω-limit set of the global solutions, while, for the blow-up solution, they give the estimate of the blow-up time. For further research results on the global dynamical behavior of solutions related to the fourth-order parabolic equations for the epitaxial thin film model, one can see [30, 31] and the references therein. Regarding the vector fields $X = (X_1, X_2, \cdots, X_n)$ with a general finite degenerate characteristic, Chen and Xu [32] carried out an earlier study on degenerate parabolic equations with finite degenerate characteristic; through the use of the potential well method, they showed the existence theorem for global solutions with exponential decay. Moreover, the finite time blow-up of solutions for the finite degenerate parabolic problem

$$u_t - \Delta_X u = |u|^{p-1}u$$

was obtained. Recently, for a class of the following finite degenerate semilinear parabolic equations with a singular potential term:

$$u_t - \Delta_X u - \mu V(x)u = g(x)|u|^{p-1}u,$$

Xu [33] established the local existence and uniqueness of the weak solution via the Galerkin method and Banach fixed-point theorem. Moreover, by constructing a family of potential wells, the global existence, the decay estimate and the finite time blow-up of solutions with subcritical or critical initial energy were also obtained. Indeed, the vector fields $X = (X_1, X_2, \cdots, X_n)$ can also encompass many degenerate cases, such as the gradient operator $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$ with singularity on a manifold with conical singularities. Studies of nonlinear parabolic equations corresponding to such degenerate vector fields can be found in [34–37].

A novelty of this paper is that the concerned system is assumed to be a system with dynamic Hörmander-type diffusion with mean curvature nonlinearity, so the differential operators generated in the system are degenerate, i.e., the second-order operators are not elliptic, but subelliptic. Roughly speaking, this means that the operator is elliptical only in certain directions of the derivative. Nevertheless, the Hörmander condition guarantees that the Laplacian function resulting from these selected derivatives is quasi-elliptic. From the point of view of a single particle, this means that the state cannot change in all directions, and that the particle can only move in the allowed direction, which is the subspace of the tangential space. This subspace depends on the state (position) of the particle. Similarly, the growth conditions of the Hamiltonian are restricted to some selected direction of the derivative. This extension is not trivial, and it relies on recent profound achievements in the development of Hörmander operators and the theory of subellipsoidal quasi-linear equations. When the known regularity of the result is not enough to proceed, we shall use the method of energy estimation to overcome the problem. In addition, the technique used here differs from the standard elliptic case and can be used in other situations to obtain similar existential results.

In particular, if we pay attention to the influence of the initial data on the long-term dynamical behavior of the solution, we can observe that a large body of papers often make assumptions about initial data with different scales being applied in different situations for the purpose of a discussion of the global existence and nonexistence of solutions for the nonlinear fourth-order parabolic equations. Specifically, when studying the global existence of solutions to nonlinear fourth-order parabolic equations, the initial data are often restricted to be sufficiently small under the defocusing setting; yet, the discussion of the finite time blow-up of solutions often requires the initial data to be sufficiently large, which means that the initial data corresponding to the negative initial energy can be arbitrarily large in this direction. These phenomena that exert vastly different constraints on the scale of initial data are of great interest to us. More precisely, the question we should now focus on is what is the size requirement for the so-called large initial data to ensure that the blow-up phenomenon of the solution occurs? Or , what is the size requirement of the small initial data that can ensure that the solution still exists as a global one? Hence, a more detailed division of initial data is necessary to clearly understand the role that initial data play in the study of the global existence and nonexistence of solutions for the nonlinear fourth-order parabolic equations. Indeed, we know that the potential well method may be a powerful technique for treating the problem (1.1), which was first established by Payne and Sattinger [38] in 1975; also see [39–41]. Therefore, our main task in this paper is to clarify which mechanism of influence exists between the initial data and the global dynamical behavior of the solution. Further, we expect to find some threshold conditions about the initial data that can portray the existence and nonexistence of the solution as a whole picture, as well as clearly elucidate the part of the set of the initial data that corresponds to the global solutions and the part that corresponds to the blow-up solutions.

1.2. Statement of the $H^s_{X,0}(\Omega)$ space theory

Our main task is to reveal the relation between the dynamical behavior of solutions for finite degenerate fourth-order parabolic equations with mean curvature nonlinearity based on the initial data. However, before this, we need to clarify some necessary theories about the phase space, and this preparatory work plays an important role in our subsequent research and demonstration.

Definition 1.1. *[Métivier condition [13]] Let the system of vector fields X satisfy Hömander's condition on Ω' with the Hömander index Q. Suppose that $V_j(x)$ $(1 \leq j \leq Q)$, spanned by all commutators of X_1, X_2 with length $\leq j$, denotes be the subspaces of the tangent space at each $x \in \Omega'$. If $\mu_j = \dim V_j(x)$ is constant in a neighborhood of each $x \in \bar{\Omega} \subset \Omega'$, then we claim that the vector field X satisfies the Métivier condition on Ω. We call the Métivier index*

$$\mu := \sum_{j=1}^{Q} (\mu_j - \mu_{j-1}),\ \mu_0 := 0,$$

the Hausdorff dimension or homogeneous dimension of Ω, as related to the subelliptic metric, which is induced by the vector field X.

The Métivier condition is a crucial condition on the research on finite degenerate elliptic operators. However, there is a large number of vector fields that do not satisfy Métivier's condition, e.g., Grushin-type vector fields. Hence, one can give the general case below.

Definition 1.2. *[Generalized Métivier index [22]] For Definition 1.1, let*

$$\mu(x) := \sum_{j=1}^{Q} (\mu_j(x) - \mu_{j-1}(x)),\ \mu_0(x) := 0,$$

where $\mu_j(x)$ is the dimension of $V_j(x)$ for $x \in \Omega'$. Considering $\Omega \subset\subset \Omega'$, we set

$$\nu = \max_{x \in \Omega} \mu(x),$$

i.e., the non-isotropic dimension of Ω *related to* X*, which can be done as the generalized Métivier index of* Ω*; see [32]. In addition,* $\mu(x)$ *is regarded as the pointwise homogeneous dimension or non-isotropic dimension at* x*.*

In consideration of the system of vector fields $X = (X_1, X_2)$, we introduce the following weighted Sobolev space [42].

Definition 1.3 (The space $M^{k,p}(\Omega)$)**.** *For any integer* $k \geq 1$*,* $p \geq 1$ *and* $\Omega \subset \mathbb{R}^2$*, we denote*

$$M^{k,p}(\Omega) = \left\{ f \in L^p(\Omega) \mid X^J f \in L^p(\Omega), \forall J = (j_1, j_2), |J| \leq k \right\},$$

where $X^J f = X_{j_1} X_{j_2} f$*,* $|J| = s$ *and we define the norm in* $M^{k,p}(\Omega)$ *to be*

$$\|f\|_{M^{k,p}(\Omega)} = \left(\sum_{|J| \leq k} \|X^J f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} < \infty.$$

Definition 1.4 (The space $H^s_X(\Omega')$)**.** *For* $n = 2$*, the system of real smooth vector fields* $X = (X_1, X_2)$ *is defined on an open domain* Ω *in* $\mathbb{R}^2$*, where* $X_k = \sum_{i=1}^2 a_{ki} \frac{\partial}{\partial x_i}$ $(1 \leq k \leq 2)$*. Assume that* $X_k^* = -X_k$*, where* X_k^* *is, formally, a skew-adjoint operator of* X_k*, i.e., for any* $u, v \in C_0^\infty(\Omega)$*,* $(X_k u, v) = -(u, X_k v)$*. We show the following weighted Sobolev space* $H^s_X(\Omega)$*,* $s \in \mathbb{N}^+$*:*

$$H^s_X(\Omega) = \left\{ u \in L^2(\Omega) \mid X^J u \in L^2(\Omega), |J| \leq s \right\},$$

where $J = (j_1, j_2)$ $(1 \leq j_i \leq 2)$ *is a multi-index, and* $|j| = l$ *denotes the length of* J*. We denote the vector field* X^J *as* $X^J = X_{j_1} X_{j_2}$*; thus, if* $|J| = 0$*, then* $X^J = id$*. It is well known that* $H^s_X(\Omega)$ *is a Hilbert space with the norm* $\|u\|^2_{H^s_X(\Omega)} = \sum_{|J| \leq s} \|X^J u\|^2_{L^2(\Omega)}$*.*

Proposition 1 (Weighted Poincaré inequality [43])**.** *Let the system of vector fields* X *satisfy Hörmander's condition on* Ω *and* $\partial\Omega$ *be* C^∞*-smooth and non-characteristic for* X*. Then, the first eigenvalue* λ_1^* *of the operator* $-\Delta_X$ *is strictly positive and satisfies*

$$\lambda_1^* \|u\|_2 \leq \|Xu\|_2, \text{ for } u \in H^1_{X,0}(\Omega).$$

In consideration of Proposition 1, we employ $\|Xu\|_2 = \left(\sum_{j=1}^2 \|X_j u\|_2^2 \right)^{\frac{1}{2}}$ as the an equivalent of $H^1_{X,0}(\Omega)$.

Proposition 2 (Weighted Sobolev embedding theorem [42])**.** *Suppose that the system of vector fields* X *satisfies Hörmander's condition on* Ω *with the Hörmander index* $Q > 1$*; also,* $\partial\Omega$ *is* C^∞*-smooth and non-characteristic for* X*. Thus, for all* $u \in C^\infty(\bar{\Omega})$*, we claim that*

$$\|u\|_{p^*} \leq C(\|Xu\|_p + \|u\|_p),$$

where $C > 0$ *is constant,* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\nu}$ *for* $p \in [1, \nu)$ *and* $\nu \geq 1 + Q > 2$ *is the generalized Métivier index of* X *on* Ω*.*

This paper is organized as follows. In Section 2, we consider the local existence and uniqueness of the solution to the initial-boundary value problem (1.1) by using the semigroup method and give the proof of Theorem 2.2. In Section 3, we collect some well-known properties of the ground state solution and show the existence of the ground state solution for problem (3.3) in Theorem 3.2. In Section 4, we give the results on the global existence and finite time blow-up of solutions with subcritical initial energy (i.e., $E(0) < d$) and give the proofs of Theorem 4.3. In Section 5, we discuss the ground state solutions for problem (1.1) and study the elements of ω-limit set $\omega(u_0)$; we then display the proofs of Theorem 5.1 and Theorem 5.2. Moreover, we present the conclusions on the global existence and blow-up with critical initial energy (i.e., $E(0) = d$) and give the proofs of Theorem 5.3 and Theorem 5.4. Finally, in Section 6, we investigate the global existence and nonexistence of the solution to problem (1.1) with supercritical initial energy (i.e., $E(0) > d$) and successively outline the proofs of Theorem 6.1, Theorem 6.2 and Theorem 6.4 (for $E(0) > 0$).

1.3. Notation conventions

We denote the standard $L^q(\Omega)$ norm by $\|\cdot\|_q$ for $1 \le q \le \infty$ and abbreviate the $L^2(\Omega)$ norm to $\|\cdot\|$. The inner product in $L^2(\Omega)$ is denoted by $(v, w) := \int_\Omega vw\mathrm{d}x$ for all $v, w \in L^2(\Omega)$. $H^2_{X,0}(\Omega)$ is a Hilbert space with the inner product $(\cdot,\cdot)_{H^2}$ given by $(w, v)_{H^2} = (X^2 w, X^2 v)$ for all $w, v \in H^2_{X,0}(\Omega)$. For simplicity, we denote by S the optimal constant of $H^2_{X,0}(\Omega) \hookrightarrow L^q(\Omega)$; then, for any $v \in H^2_{X,0}(\Omega)$, it follows that

$$\|v\|_q \le S\|X^2 v\|. \tag{1.7}$$

For simplicity, the duality pairing between $H^{-2}_X(\Omega)$ and $H^2_{X,0}(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$.

We define the energy functional $E : H^2_{X,0}(\Omega) \to \mathbb{R}$ by

$$E(v) = \frac{1}{2}\|X^2 v\|^2 + \int_\Omega \left(\sqrt{1 + |Xv|^2} - 1\right)\mathrm{d}x - \frac{1}{p}\|v\|_p^p, \tag{1.8}$$

and the Lyapunov function $E(t) := E(u(t))$ is applied for any solution u to problem (1.1).

Definition 1.5. *The function $u = u(t, x)$ is said to be the solution for the problem* (1.1) *over* $[0, T]$, *provided that $u \in L^\infty(0, T; H^2_{X,0}(\Omega))$ with $u_t \in L^2(0, T; L^2(\Omega))$ such that $u(0) = u_0 \in H^2_{X,0}(\Omega)$ and it satisfies*

$$(u_t(t), v) + (X^2 u(t), X^2 v) + \left(\frac{Xu(t)}{\sqrt{1 + |Xu(t)|^2}}, Xv\right) = (|u(t)|^{p-2}u(t), v) \tag{1.9}$$

for any $v \in H^2_{X,0}(\Omega)$ and a.e. $t \ge 0$; also,

$$\int_s^t \|u_t(\tau)\|^2\mathrm{d}\tau + E(t) = E(s) \text{ for } 0 \le s < t < T. \tag{1.10}$$

According to the definition of $E(t)$ and $I(u)$ (see (3.2)), it follows that

$$E(t) = \frac{p-2}{2p}\|X^2 u(t)\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xu(t)|^2}{p\sqrt{1 + |Xu(t)|^2}} - 1\right)\mathrm{d}x + \frac{1}{p}I(u(t)). \tag{1.11}$$

Remark 1. *Since $p > 2$, it is easy to verify that $\frac{p+(p-1)|Xu(t)|^2}{p\sqrt{1+|Xu(t)|^2}} > 1$. Hence, we obtain from* (1.11) *that*

$$E(t) > \frac{p-2}{2p}\|X^2 u(t)\|^2 + \frac{1}{p}I(u(t)). \tag{1.12}$$

Finally, we set

$$\lambda_1 := \inf_{u\in H_0^2(\Omega)\backslash\{0\}} \frac{\|X^2u\|^2}{\|u\|^2}. \tag{1.13}$$

As shown in [43], λ_1 is the principal eigenvalue of the operator X^2 with homogeneous Dirichlet boundary conditions.

1.4. Potential well

Now, let us state the following lemmas concerning the properties of $E(t)$ and $I(u)$, which are useful for subsequent proofs of the global dynamical behavior of the solution for problem (1.1).

Lemma 1.6. *Assume that* $u \in H^2_{X,0}(\Omega)\backslash\{0\}$*; then,*

(i) $\lim_{\lambda\to0^+} E(\lambda u) = 0$ *and* $\lim_{\lambda\to+\infty} E(\lambda u) = -\infty$*;*
(ii) there exists a unique $\lambda^* = \lambda^*(u) > 0$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda u)\mid_{\lambda=\lambda^*} = 0$$

and $E(\lambda u)$ *is increasing for* $0 \le \lambda < \lambda^*$*, decreasing for* $\lambda > \lambda^*$ *and takes its maximum at* $\lambda = \lambda^*$*;*
(iii) $I(\lambda u) > 0$ *for* $0 \le \lambda < \lambda^*$*,* $I(\lambda u) < 0$ *for* $\lambda > \lambda^*$ *and* $I(\lambda^* u) = 0$*.*

Lemma 1.7. *Assume that* $u \in H^2_{X,0}(\Omega)\backslash\{0\}$ *and* $r = S^{\frac{p}{2-p}}$*; then,*

(i) if $0 \le \|X^2u\| \le r$*, then* $I(u) \ge 0$*;*
(ii) if $I(u) < 0$*, then* $\|X^2u\| > r$*;*
(iii) if $I(u) = 0$*, then* $\|X^2u\| = 0$ *or* $\|X^2u\| \ge r$*.*

Remark 2. *The proofs of Lemma 1.6 and Lemma 1.7 are similar, as shown in [39] with a simple modification; hence, we chose to omit it.*

Lemma 1.8. *Assume that* $u \in H^2_{X,0}(\Omega)$*; then,*

(i) it follows that $E(u) > 0$ *as long as* $u \in \mathcal{N}_+$*;*
(ii) for each $\kappa > 0$*, if* $u \in \mathcal{N}_+$ *satisfies that* $E(u) < \kappa$*, then* $\|X^2u\|$ *is bounded in* $H^2_{X,0}(\Omega)$ *and*

$$\sup_{u\in\{E(u)<\kappa\}\cap\mathcal{N}_+} \|X^2u\| \le \sqrt{\frac{2p\kappa}{p-2}};$$

(iii) for any $u \in \mathcal{N}_-$*, we conclude that* $\mathrm{dist}(0, \mathcal{N} \cup \mathcal{N}_-) := \inf_{u\in\mathcal{N}\cup\mathcal{N}_-} \|X^2u\| > 0$*.*

Proof. (i) Given that $u \in \mathcal{N}_+$, which means that $I(u) > 0$ and $\|X^2u\| \neq 0$, we obtain from (1.12) that

$$E(u) > \frac{p-2}{2p}\|X^2u\|^2 + \frac{1}{p}I(u) \ge \frac{p-2}{2p}\|X^2u\|^2 > 0.$$

(ii) Following from $I(u) > 0$, and by the proof of (i) with $E(u) < \kappa$, we know that $\kappa > E(u) \ge \frac{p-2}{2p}\|X^2u\|^2 > 0$, which yields $\|X^2u\|^2 < \frac{2\kappa p}{p-2}$.

(iii) As long as $u \in \mathcal{N}_-$, that is, $I(u) < 0$, we deduce that $\|X^2 u\| \neq 0$. From the definition of $\mathcal{N}_-$ and the Sobolev embedding inequality, we get

$$\|Xu\|^2 < \|u\|_p^p \leq S^p \|Xu\|^p, \tag{1.14}$$

which gives $\|X^2 u\| > S^{-\frac{p}{p-2}}$.

Lemma 1.9. *Assume that $u \in H^2_{X,0}(\Omega)$; then, $d \geq \frac{p-2}{2p} S^{-\frac{2p}{p-2}}$.*

Proof. For any $u \in \mathcal{N}$, we know that $I(u) = 0$. Then, from (1.12) and (iii) in Lemma 1.6, we have

$$E(u) \geq \frac{p-2}{2p} \|X^2 u\|^2 \geq \frac{p-2}{2p} \text{dist}(0, \mathcal{N}) \geq \frac{p-2}{2p} S^{-\frac{2p}{p-2}}.$$

Lemma 1.10. *Assume that $u \in H^2_{X,0}(\Omega)$ satisfies that $I(u) < 0$; then,*

(i) there exists a unique constant $\lambda^* \in (0,1)$ such that $\lambda^* u \in \mathcal{N}$;

(ii) $I(u) < p(E(t) - d)$.

Proof. (i) Fix any $u \in H^2_{X,0}(\Omega)$ satisfying that $I(u) < 0$, and for any $\lambda > 0$; we define

$$i(\lambda) := I(\lambda u) = \lambda^2 \|X^2 u\|^2 + \int_\Omega \frac{\lambda^2 |Xu|^2}{\sqrt{1 + \lambda^2 |Xu|^2}} \mathrm{d}x - \lambda^p \|u\|_p^p;$$

then, $i'(\lambda) = \lambda h(\lambda)$, where

$$\begin{aligned} h(\lambda) =& 2\|X^2 u\|^2 + \int_\Omega \frac{2|Xu|^2 + \lambda^2 |Xu|^4}{(1 + \lambda^2 |Xu|^2)^{\frac{3}{2}}} \mathrm{d}x - p\lambda^{p-2} \|u\|_p^p \\ =& 2\|X^2 u\|^2 + \int_\Omega \frac{|Xu|^2}{(1 + \lambda^2 |Xu|^2)^{\frac{3}{2}}} \mathrm{d}x + \int_\Omega \frac{|Xu|^2}{(1 + \lambda^2 |Xu|^2)^{\frac{1}{2}}} \mathrm{d}x - p\lambda^{p-2} \|u\|_p^p. \end{aligned}$$

One can find that $h(\lambda)$ is strictly decreasing in $\lambda > 0$; also, by (iii) in Lemma 1.8 and $p > 2$, we have

$$h(0) = I(\lambda u) \geq 2\|X^2 u\|^2 \geq 2S^{-\frac{2p}{p-2}} > 0 \text{ and } \lim_{\lambda \to \infty} h(\lambda) = -\infty.$$

Thus, there exists a unique $\lambda_0 > 0$ that yields $h(\lambda_0) = 0$, $h(\lambda) > 0$ for any $0 < \lambda < \lambda_0$ and $h(\lambda) < 0$ for any $\lambda > \lambda_0$. In addition, it infers that $h'(\lambda_0) = 0$, and $h(\lambda)$ is strictly increasing in $\lambda_0 > 0$ and strictly decreasing in $\lambda > \lambda_0$. Then, it follows that there exists a unique $\lambda^* > 0$ satisfying that $h(\lambda^*) = 0$. Moreover, since $h(1) = I(u) < 0$, it implies that $\lambda^* < 1$.

(ii) Set $g(\lambda) := pE(\lambda u) - I(\lambda u)$, $\lambda > 0$. By (1.12), we know that

$$g(\lambda) = \frac{p-2}{2} \lambda^2 \|X^2 u\|^2 + \int_\Omega \left(\frac{p + (p-1)\lambda^2 |Xu|^2}{\sqrt{1 + \lambda^2 |Xu|^2}} - p \right) \mathrm{d}x.$$

Thus, $g(\lambda)$ is strictly increasing for $\lambda_0 > 0$. Let $\lambda^* \in (0,1)$ be the constant given in (i); then, it follows from $\lambda^* u \in \mathcal{N}$ and the definition of d that

$$pE(u) - I(u) > pE(\lambda^* u) - I(\lambda^* u) = pE(\lambda^* u) \geq pd,$$

which confirms the result.

To discuss the finite time blow-up of solutions above the critical initial energy, we define

$$\lambda_\kappa = \inf\{\|u\|^2 \mid u \in \mathcal{N}, E(u) < \kappa\} \text{ and } \Lambda_\kappa = \sup\{\|u\|^2 \mid u \in \mathcal{N}, E(u) < \kappa\}$$

for all $\kappa > d$. Obviously, we derive the monotonicity properties as

$$\kappa \mapsto \lambda_\kappa \text{ is non-increasing and } \kappa \mapsto \Lambda_\kappa \text{ is non-decreasing.}$$

Furthermore, we can show the properties of λ_κ and Λ_κ as follows.

Lemma 1.11. *Assume that λ_κ and Λ_κ are two constants parameterized by κ; then,*

(i) $\lambda_\kappa \geq \theta^{-\frac{p}{2}} S^{-\frac{p(4-p)}{2(p-2)}}$ if $p \in (1, 3]$ and $\lambda_\kappa \geq \theta^{-\frac{p}{2}} \left(\frac{2p\kappa}{p-2}\right)^{-\frac{p(4-p)}{2(p-2)}}$ if $p \in (3, +\infty)$;

(ii) $\Lambda_\kappa \leq \left(\frac{2p\kappa}{\lambda_1(p-2)}\right)^{\frac{1}{2}}$,

where S and λ_1 are the constants given in (1.7) *and* (1.13)*, respectively. And, θ is the best constant of the Galiardo-Nirenberg's inequality $\|v\|_p \leq \theta \|X^2 v\|^{\frac{p-2}{p}} \|v\|^{\frac{2}{p}}$.*

Proof. To begin, we estimate the lower bound of λ_κ. Fix any $u \in \mathcal{N} \cap \{E(u) < \kappa\}$; we deduce that $u \in \mathcal{N}$, which means that

$$I(u) = \|X^2 u\|^2 + \int_\Omega \frac{|Xu|^2}{\sqrt{1+|Xu|^2}} \mathrm{d}x - \|u\|_p^p = 0;$$

together with Galiardo-Nirenberg's inequality, one can infer that $\|X^2 u\|^2 \leq \|u\|_p^p \leq \theta^p \|X^2 u\|^{p-2} \|u\|^2$. Thus, we obtain that $\|u\| \geq \theta^{-\frac{p}{2}} \|X^2 u\|^{\frac{4-p}{2}}$. If $p \leq 4$, then it is easy to confirm from the definition of λ_κ and Lemma 1.7 that

$$\lambda_\kappa = \inf_{u \in \mathcal{N} \cap \{E(u)<\kappa\}} \|u\| \geq \inf_{u \in \mathcal{N}} \|u\| \geq \theta^{-\frac{p}{2}} \left(\inf_{u \in \mathcal{N}} \|X^2 u\|\right)^{\frac{4-p}{2}} \geq \theta^{-\frac{p}{2}} S^{-\frac{p(4-p)}{2(p-2)}}.$$

Besides that, for $p > 4$, one can infer from the definition of λ_κ and Lemma 1.8 that

$$\lambda_\kappa = \inf_{u \in \mathcal{N} \cap \{E(u)<\kappa\}} \|u\| \geq \theta^{-\frac{p}{2}} \left(\sup_{u \in \mathcal{N}} \|X^2 u\|\right)^{\frac{4-p}{2}} \geq \theta^{-\frac{p}{2}} \theta^{-\frac{p}{2}} \left(\frac{2p\kappa}{p-2}\right)^{-\frac{p(4-p)}{2(p-2)}}.$$

Next, we estimate the upper bound of Λ_κ. For every $u \in \mathcal{N} \cap \{E(u) < \kappa\}$, by (1.13) and the inequality $\frac{p+(p-1)|Xu(t)|^2}{p\sqrt{1+|Xu(t)|^2}} > 1$, we get

$$\|u\| \leq \lambda_1^{-\frac{1}{2}} \|X^2 u\| \leq \left(\frac{2p\kappa}{\lambda_1(p-2)}\right)^{\frac{1}{2}};$$

then, (ii) follows from the definition of Λ_κ.

Lemma 1.12. *Assume that a positive and twice-differentiable function $\Phi(t)$ satisfies the following inequality:*

$$\Phi''(t)\Phi(t) - (1+\delta)(\Phi'(t))^2 \geq 0 \textit{ for all } t > 0,$$

where $\delta > 0$ is a positive constant, together with $\Phi(0) > 0$ and $\Phi'(0) > 0$; then, it follows that there exists a time $0 < t_* \leq \frac{\Phi(0)}{\delta\Phi'(0)}$ such that $\Phi(t)$ approaches infinity as $t \to t_*$.

2. Local existence and uniqueness

In this section, we describe problem (1.1) as a class of abstract Cauchy problems by citing some relevant important conclusions. In consideration of the classical semigroup method, we briefly show the local existence and uniqueness of the solution to problem (1.1).

Definition 2.1 (Sectorial operators)**.** *Assume that $\mathcal{P}(M,\theta)$ with $M \geq 1$ and $\theta \in [0,\pi)$ is the class of all closed densely defined linear operators A in X_0, which satisfies the following:*

(i) $S_\theta = \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| \leq \theta\} \cup \{0\} \subset \rho(-A)$,
(ii) $(1+|\lambda|)\|(A+\lambda)^{-1}\|_{\mathcal{L}(X_0)} \leq M$ *for* $\lambda \in S_\theta$.

The elements in $\mathcal{P}(\theta) = \cup_{M\geq 1}\mathcal{P}(M,\theta)$ are called sectorial operators of the angle θ. If $A \in \mathcal{P}(\theta)$, then any $M \geq 1$ such that $A \in \mathcal{P}(M,\theta)$ is called a sectorial bound of A.

Theorem 2.2 (Local existence)**.** *Assume that $u_0 \in H^2_{X,0}(\Omega)$; then, there exists $T > 0$ such that problem (1.1) possesses a unique weak solution u on $[0,T]\times\Omega$, which satisfies*

$$u \in C([0,T]; H^2_{X,0}(\Omega)) \cap C^1((0,T); L^2(\Omega))$$

with $u(0) = u_0$. Moreover, if $T_{\max} := T_{\max}(u_0) = \sup\{T > 0 : u = u(t) \text{ exists on } [0,T]\} < \infty$, then $\lim\limits_{t\to T_{\max}} \|X^2 u(t)\| = \infty$.

Proof. Indeed, according to the results in [44], we know that the operator $A = \Delta_X$ can be seen as an unbounded operator in $X_0 = L^2(\Omega)$ with the domain $X_1 = H^4_X(\Omega) \cap H^2_{X,0}(\Omega)$. Hence, the operator A, which is the realization of the biharmonic operator Δ_X^2 in $L^2(\Omega)$ under Dirichlet boundary conditions $\frac{\partial u}{\partial \nu} = 0$, is a sectorial operator in $L^2(\Omega)$ with angle θ_A and thus the infinitesimal generator of the analytic semigroup e^{At}. The scale of the fractional power space $\{D(A^\alpha)\}_{\alpha\in[0,1)}$ associated with A satisfies the following conditions:

$$\begin{aligned}
&D(A) = H^4_X(\Omega) \cap H^2_{X,0}(\Omega) \text{ and } \|\cdot\|_{H^4\cap H^2_0} \leq C\|A(\cdot)\|,\\
&D(A^\alpha) = H^{4\alpha}_X(\Omega) \cap H^2_{X,0}(\Omega) \text{ for any } \alpha \in \left(\frac{1}{2}, 1\right) \text{ and } \|\cdot\|_{H^{4\alpha}\cap H^2_0} \leq C\|A^\alpha(\cdot)\|,\\
&D(A^{\frac{1}{2}}) = H^2_{X,0}(\Omega) \text{ and } \|\cdot\|_{H^2_0} \leq C\|A^{\frac{1}{2}}(\cdot)\|,
\end{aligned}$$

where C is a generic constant that may be different for different lines. Moreover, the realization $A^\alpha : X_1 = D(A^\alpha) \to X_0 = L^2(\Omega)$ is an isometry. Thus, by using the notations given above, the problem (1.1) is reduced to the following abstract Cauchy problem:

$$\begin{cases} u_t + Au = \Phi(u), \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $\Phi(u) := \nabla_X \cdot \left(\frac{\nabla_X u}{\sqrt{1+|\nabla_X u|^2}}\right) + |u|^{p-2}u$ is a map relative to the pair (X_1, X_0). Indeed, given that $1 \leq p < \frac{N+2}{N-2}$, $\Omega \subset \mathbb{R}^2$ and Proposition 2, we know that

$$D(A^\eta) = H^{4\eta}_X(\Omega) \cap H^2_{X,0}(\Omega) \hookrightarrow L^{2p}(\Omega),$$

where $\eta = \frac{p-1}{4p} + \frac{3}{4} \in (\frac{1}{2}, 1)$. Thus, for any $u, v \in D(A^\eta)$, a simple calculation shows that

$$\begin{aligned} &\|\Phi(u) - \Phi(v)\| \\ = &\left\| \left(\nabla_X \cdot \left(\frac{\nabla_X u}{\sqrt{1 + |\nabla_X u|^2}} \right) + |u|^{p-2}u \right) - \left(\nabla_X \cdot \left(\frac{\nabla_X v}{\sqrt{1 + |\nabla_X v|^2}} \right) + |v|^{p-2}v \right) \right\| \\ \leq &\left\| \left(\nabla_X \cdot \left(\frac{\nabla_X u}{\sqrt{1 + |\nabla_X u|^2}} \right) \right) - \left(\nabla_X \cdot \left(\frac{\nabla_X v}{\sqrt{1 + |\nabla_X v|^2}} \right) \right) \right\| - \left\| |u|^{p-2}u - |v|^{p-2}v \right\|. \end{aligned} \tag{2.2}$$

For the terms on the right-hand side of (2.2), by using a similar argument as that for Lemma 3.1 in [29], we finally obtain

$$\|\Phi(u) - \Phi(v)\| \leq C\left(1 + \|A^{\frac{1}{2}}u\| + \|A^{\frac{1}{2}}v\|\right)^2 (\|A^\eta u\| + \|A^\eta v\|)\|A^{\frac{1}{2}}u - A^{\frac{1}{2}}v\|.$$

Hence, it follows from Lemma 3.1 in [29] that, for each $u_0 \in X_1$, there exists a $T > 0$, which only depends on $\|X^2 u_0\| \leq C$, such that the problem (2.1) has a unique solution $u \in C([0, T]; H^2_{X,0}(\Omega)) \cap C^1((0, T); L^2(\Omega))$, satisfying

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}\Phi(u(s))\mathrm{d}s.$$

Thus, the solution of problem (1.1) has local existence and uniqueness; in addition, it satisfies that

$$u \in C([0, T]; H^2_{X,0}(\Omega)) \cap C^1((0, T); L^2(\Omega)).$$

Not only that, the local existence time T only depends on the initial datum. Hence, by employing a similar idea as shown in [29], as long as $\|X^2 u\|$ remains bounded, we can also conclude that the local solution obtained above can be continued. Thus, if $T_{\max} = T_{\max}(u_0) < \infty$, we conclude the following:

$$\lim_{t \to T_{\max}} \|X^2 u(t)\| = \infty.$$

3. Basic properties of the ground state solution

To begin in this subsection, we display some well-known properties of the ground state solution Q. First, it is established by using the direct variational method for constrained minimization:

$$E(Q) = \inf\{E(v) \mid v \in H^2_{X,0}(\Omega)\backslash\{0\}, I(v) = 0\}, \tag{3.1}$$

where $I(\cdot) : H^2_{X,0}(\Omega) \to \mathbb{R}$ is the Nehari functional, defined by

$$I(u) := \langle E'(u), u \rangle = \|X^2 u\|^2 + \int_\Omega \left(\frac{|Xu|^2}{\sqrt{1 + |Xu|^2}} \right) \mathrm{d}x - \|u\|_p^p, \tag{3.2}$$

which also helps us to characterize the Nehari manifolds $\mathcal{N} = \{u \in H^2_{X,0}(\Omega)\backslash\{0\} \mid I(u) = 0\}$, as well as the corresponding order-preserving manifolds

$$\mathcal{N}_+ = \{u \in H^2_{X,0}(\Omega) \mid I(u) > 0\} \text{ and } \mathcal{N}_- = \{u \in H^2_{X,0}(\Omega) \mid I(u) < 0\},$$

where $E'(u)$ denotes the Fréchet derivative of $E'(\cdot)$ at u, which is a bounded linear operator from $H^2_{X,0}(\Omega)$ into $\mathbb{R}$. Especially, the ground state solutions of the problem (1.1) can be shown as follows:

$$\begin{cases} \Delta_X^2 u - \nabla_X \cdot \left(\frac{\nabla_X u}{\sqrt{1+|\nabla_X u|^2}} \right) = |u|^{p-2}u, & x \in \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \Omega. \end{cases} \tag{3.3}$$

Indeed, the functional $I(u)$ is well defined on $H^2_{X,0}(\Omega)$. A function $u \in H^2_{X,0}(\Omega)$ is called a week solution of (3.3) if

$$(\Delta_X u, \Delta_X v) - \left(\frac{\nabla_X u}{\sqrt{1 + |\nabla_X u|^2}}, \nabla_X v \right) = (|u|^{p-2}u, v), v \in H^2_{X,0}(\Omega).$$

In addition, all weak solutions of problem (3.3) is denoted by the set

$$\Sigma := \{u \mid u \text{ is a weak solution of the problem (3.3)}\},$$

and we know that $\Sigma \backslash \{0\} \subset \mathscr{N}$ immediately. Thus, the mountain pass level can be denoted by

$$d := E(Q) = \inf_{u \in \mathscr{N}} E(u)$$

and the ground state solution of the problem (3.3) can be regarded as $u \in \Sigma$ with $E(u) = d$. As we all know, the existence of ground-state solutions of elliptic equations is an important research topic that has attracted much attention. We refer the readers to [1, 18] and the corresponding research on the ground state solutions of related elliptic equations. In order to obtain a ground state solution of the corresponding Nehari manifold, we mainly use the Lagrange multiplier method. According to this result, the instability of the ground state solution for the problem (3.3) is discussed, where the definition of instability is displayed below.

Definition 3.1. *Considering every ground state solution Q for the problem* (3.3)*, we claim that Q is unstable, if for any $\varepsilon > 0$, one can find a $u_0 \in H^2_{X,0}(\Omega)$ such that*

$$\|u_0 - Q\|_{H^2_{X,0}(\Omega)} < \varepsilon,$$

and the corresponding solution of problem (1.1) *with the initial datum u_0 blows up in finite time.*

Now, let us give the existence of the ground state solution for the problem (3.3) as follows.

Theorem 3.2. *There exists an element $Q \in \mathscr{N}$ that satisfies the following conditions:*

(i) $E(Q) = \inf_{u \in \mathscr{N}} E(u) = d$;

(ii) Q is a ground-state solution to the problem (3.3).

Proof. (i) Due to the relationship between $I(u)$ and $E(t)$ in (6.8) and the definition of d, one can deduce that

$$d = \inf_{u \in \mathscr{N}} E(u) = \inf_{u \in \mathscr{N}} \left(\frac{p-2}{2p} \|X^2 u\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xu|^2}{p\sqrt{1 + |Xu|^2}} - 1 \right) \mathrm{d}x \right),$$

which means that there exists a minimizing sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{N} \subset H^2_{X,0}(\Omega)$ satisfying that

$$E(u_n) = \frac{p-2}{2p}\|X^2u_n\|^2 + \int_\Omega \left(\frac{p+(p-1)|Xu_n|^2}{p\sqrt{1+|Xu_n|^2}} - 1 \right) \mathrm{d}x \to d \text{ as } n \to \infty; \tag{3.4}$$

then, the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $H^2_{X,0}(\Omega)$. Given the fact that $H^2_{X,0}(\Omega)$ is reflexive, $H^2_{X,0}(\Omega) \hookrightarrow H^1_{X,0}(\Omega)$ compactly; we derive from the boundedness that there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$, denoted still by $\{u_n\}_{n=1}^{\infty}$, and a function $w \in H^2_{X,0}(\Omega)$ satisfying

$$u_n \to w \text{ weakly in } H^2_{X,0}(\Omega) \text{ as } n \to \infty.$$

Recalling that the embedding $H^2_{X,0}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ satisfies the conditions for compactness with $p > 1$, then, for $n \to \infty$, one can obtain that

$$Xu_n \to Xw \text{ strongly in } L^2(\Omega), \tag{3.5}$$

$$u_n \to w \text{ strongly in } L^{p+1}(\Omega). \tag{3.6}$$

Next, one can deduce that $w \neq 0$. Otherwise, if $w = 0$, then we obtain from (3.5) that

$$u_n \to 0 \text{ strongly in } L^{p+1}(\Omega). \tag{3.7}$$

Recalling that $\{u_n\}_{n=1}^{\infty} \subset \mathcal{N}$, we infer from the definition of $\mathcal{N}$ that

$$\|X^2u_n\|^2 + \int_\Omega \left(\frac{|Xu_n|^2}{\sqrt{1+|Xu_n|^2}} \right) \mathrm{d}x = \|u_n\|_p^p, \tag{3.8}$$

which, combined with (3.7), implies that

$$\|X^2u_n\|^2 + \int_\Omega \left(\frac{|Xu_n|^2}{\sqrt{1+|Xu_n|^2}} \right) \mathrm{d}x \to 0 \text{ as } n \to \infty; \tag{3.9}$$

this contradicts (3.4) due to the fact that $d > 0$; hence, we conclude that $w \neq 0$.

Let $Q = \epsilon w$, where

$$\epsilon = \left(\frac{\|X^2w\|^2 + \int_\Omega \left(\frac{|Xw|^2}{\sqrt{1+|Xw|^2}} \right) \mathrm{d}x}{\|w\|_p^p} \right)^{\frac{1}{p-2}}$$

is a constant. Moreover, by (iii) in Lemma 1.6, we achieve that $I(Q) = 0$. Thus, $Q \in \mathcal{N}$, and from the definition of d, it follows that

$$E(Q) \geq d. \tag{3.10}$$

Given that $u_n \in \mathcal{N}$, one can easily see that

$$E(u_n) = \sup_{\lambda \geq 0} E(\lambda u_n), \ n = 1, 2, \cdots. \tag{3.11}$$

Indeed, we easily obtain from (ii) in Lemma 1.6 that $E(\lambda u_n)$ achieves its maximum value at

$$\lambda = \lambda_n := \left(\frac{\|X^2 u_n\|^2 + \int_\Omega \left(\frac{|Xu_n|^2}{\sqrt{1+|Xu_n|^2}} \right) \mathrm{d}x}{\|u_n\|_p^p} \right)^{\frac{1}{p-2}}.$$

On the other hand, consider $u_n \in \mathcal{N}$, i.e.,

$$\|X^2 u_n\|^2 + \int_\Omega \left(\frac{|Xu_n|^2}{\sqrt{1+|Xu_n|^2}} \right) \mathrm{d}x = \|u_n\|_p^p,$$

which means that $\lambda = 1$ and (3.11) is satisfied. In consideration of the weak lower semicontinuity of the norms, one can know that

$$\|X^2 w\|^2 \leq \liminf_{n\to\infty} \|X^2 u_n\|^2. \tag{3.12}$$

Thus, we derive, by using (3.11), the definition of $E(u)$, (3.12) and (3.5), that

$$\begin{aligned} d &= \lim_{n\to+\infty} E(u_n) \\ &\geq \lim_{n\to+\infty} E(\epsilon u_n) \\ &= \lim_{n\to+\infty} \left(\frac{p-2}{2p} \|X^2 \epsilon u_n\|^2 + \int_\Omega \left(\frac{p + (p-1)|X\epsilon u_n|^2}{p\sqrt{1+|X\epsilon u_n|^2}} - 1 \right) \mathrm{d}x \right) \\ &\geq \liminf_{n\to\infty} \|X^2 \epsilon u_n\|^2 + \lim_{n\to+\infty} \int_\Omega \left(\frac{p + (p-1)|X\epsilon u_n|^2}{p\sqrt{1+|X\epsilon u_n|^2}} - 1 \right) \mathrm{d}x \\ &\geq \|X^2 \epsilon w\|^2 + \int_\Omega \left(\frac{p + (p-1)|X\epsilon w|^2}{p\sqrt{1+|X\epsilon w|^2}} - 1 \right) \mathrm{d}x \\ &= \|X^2 Q\|^2 + \int_\Omega \left(\frac{p + (p-1)|XQ|^2}{p\sqrt{1+|XQ|^2}} - 1 \right) \mathrm{d}x \\ &= E(Q). \end{aligned}$$

Hence, we can see from (3.10) and the above inequality that $E(Q) = d$. Moreover, in consideration of (3.9), we can deduce that

$$\lim_{n\to\infty} \|X^2 u_n\| = \|X^2 w\|, \tag{3.13}$$

which, together with (3.5) and the fact that $H^2_{X,0}(\Omega)$ is a uniformly convex Banach space, shows that

$$u_n \to w \text{ strongly in } H^2_{X,0}(\Omega) \text{ as } n \to \infty.$$

Then, we further have

$$E(w) = \|X^2 w\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xw|^2}{p\sqrt{1+|Xw|^2}} - 1 \right) \mathrm{d}x$$

$$= \lim_{n\to\infty}\left(\|X^2u_n\|^2 + \int_\Omega\left(\frac{p+(p-1)|Xu_n|^2}{p\sqrt{1+|Xu_n|^2}} - 1\right)\mathrm{d}x\right) = d,$$

which means that $E(w) = d = \inf\limits_{u\in\mathcal{N}} E(u)$.

(ii) Now, we show that Q is indeed the ground state solution of the problem (3.3), i.e., $Q \in \Sigma$ and $E(Q) = \inf\limits_{u\in\Sigma} E(u)$. According to the result in (i), one infers that

$$Q \in \mathcal{N} = \left\{u \in H^2_{X,0}(\Omega)\backslash\{0\} \mid \langle E'(u), u\rangle = I(u) = 0\right\},$$

and $E(w) = d = \inf\limits_{u\in\mathcal{N}} E(u)$. Therefore, utilizing the theory of Lagrange multipliers, there exists a constant $\eta \in \mathbb{R}$ such that

$$E'(Q) - \eta I'(Q) = 0 \text{ in } H_X^{-2}(\Omega). \tag{3.14}$$

Thus, we achieve the following chains of equations by using the definition of $\mathcal{N}$:

$$\eta\langle I'(Q), Q\rangle = \langle E'(Q), Q\rangle = I(Q) = 0. \tag{3.15}$$

On the other hand, for every $Q \in H^2_{X,0}(\Omega)$, one can deduce that

$$\begin{aligned}\langle I'(Q), Q\rangle =& \frac{\mathrm{d}}{\mathrm{d}\tau} I(Q+\tau Q)\,|_{\tau=0}\\ =& \underbrace{\|X^2Q\|^2 - \|Q\|_p^p}_{I_1} + \underbrace{\|X^2Q\|^2 + \int_\Omega\left(\frac{|XQ|^2}{\sqrt{1+|XQ|^2}}\right)\mathrm{d}x - \|Q\|_p^p}_{I_2}\\ &- \int_\Omega\left(\frac{|XQ|^4}{(1+|XQ|^2)^{\frac{3}{2}}}\right)\mathrm{d}x - (p-2)\|Q\|_p^p.\end{aligned} \tag{3.16}$$

Since (3.5)-(3.7) holds, by taking $n \to \infty$ in (3.9), we get

$$\|X^2w\|^2 + \int_\Omega\left(\frac{|Xw|^2}{\sqrt{1+|Xw|^2}}\right)\mathrm{d}x \le \|w\|_p^p, \tag{3.17}$$

which confirms that $I_1 < 0$. Furthermore, by the definition of $I(u)$, it follows that $I_2 = I(w) = 0$. Then, (3.16) becomes $\langle I'(Q), Q\rangle < 0$, which, in combination with (3.15), yields $\eta = 0$. Then, we get from (3.15) that $E'(w) = 0$ in $H_X^{-2}(\Omega)$, i.e.,

$$\begin{aligned}\langle E'(Q), v\rangle :=& \frac{\mathrm{d}}{\mathrm{d}\tau} I(Q+\tau v)\,|_{\tau=0}\\ =&(X^2Q, X^2v) + \left(\frac{XQ}{\sqrt{1+|XQ|^2}}, XQ\right) - (|Q|^{p-2}Q, v)\\ =&0,\ v \in H^2_{X,0}(\Omega).\end{aligned} \tag{3.18}$$

Hence, Q is the weak solution of problem (3.3).

4. Global dynamics analysis of solutions for problem (1.1) with subcritical initial energy $E(0) < d$

4.1. Uniform dynamics below the ground state

In this section, we demonstrate that all of the solutions with an initial energy less than the ground state $E(u_0) < E(Q) = d$ are split into the decaying (for $I(u_0) > 0$) and finite time blow-up (for $I(u_0) < 0$). To trace the behavior of the solutions obtained from some initial data, we need to establish the relationship between the global dynamical behavior of solutions and the initial data. First, we discuss the threshold classifications of the initial data for problem (1.1) under the condition of subcritical initial energy (i.e., $E(0) < d$) and give the following theorem on the global existence and finite time blow-up of the solution in Theorem 4.3.

Lemma 4.1 (Invariant manifold with $E(0) < d$)**.** *Suppose that u is the solution obtained in Theorem 2.2; then, for any initial data u_0 satisfying that $E(0) < d$, we have the following:*

(i) if $I(u_0) > 0$, then $u(t) \in \mathscr{N}_+$ for any $t \in [0, T_{\max})$;
(ii) if $I(u_0) < 0$, then $u(t) \in \mathscr{N}_-$ for any $t \in [0, T_{\max})$.

Proof. In consideration of the respective definitions of $E(t)$ and $I(u)$, one can easily deduce that $E(t) \in C([0, T_{\max}))$ and $I(t) \in C([0, T_{\max}))$. By (1.10), we obtain

$$E(u(t)) < d \text{ for any } t \in [0, T_{\max}). \tag{4.1}$$

For (i), given that $I(u_0) > 0$, arguing by contradiction and using the continuity of $I(u(t)) = 0$ with respect to t, then there exists a first $t_0 > 0$ that yields $I(u(t_0)) = 0$ and $u(t_0) \neq 0$, i.e., $u(t_0) \in \mathscr{N}$, while we get from the definition of $d = E(Q)$ that $E(u(t_0)) \geq d$, which contradicts (4.1).

For (ii), we can similarly deduce that $u(t) \in \mathscr{N}_-$ for any $t \in [0, T_{\max})$ by using a contradiction. let us make the opposite assumption that there exists a t_0 that yields $I(u_0) = 0$ and $I(u(t)) < 0$ for all $t \in [0, t_0)$. Thus, one can infer by (iii) of Lemma 1.8 that

$$\|X^2 u(t)\| \geq \operatorname{dist}(0, \mathscr{N} \cup \mathscr{N}_-) \geq S^{-\frac{2p}{p-2}}, 0 \leq t < t_0.$$

Then, by the continuity of $\|X^2 u(t_0)\|$, we get

$$\|X^2 u(t_0)\| \geq S^{-\frac{2p}{p-2}};$$

together with $I(u_0) = 0$, it is easy to show that $u(t_0) \in \mathscr{N}$. Thus, one can deduce by the definition of d that $E(u(t_0)) > d$; obviously, it contradicts (4.1).

Lemma 4.2. *Assume that $u \in H^2_{X,0}(\Omega)$; then, we claim that*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) = -\|u_t(t)\|^2 \textit{ for any } t \in [0, T_{\max}) \tag{4.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 = -2I(u(t)) \textit{ for any } t \in [0, T_{\max}). \tag{4.3}$$

Proof. The identity (4.2) follows by multiplying the equation (1.1) by u_t and integrating by parts on $\Omega \subset \mathbb{R}^2$. Hence, $t \mapsto E(u(t))$ is decreasing. Multiplying the equation (1.1) by u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 &= (u, u_t)\\ &= -\|X^2u(t)\|^2 - \int_\Omega \left(\frac{|Xu(t)|^2}{\sqrt{1+|Xu(t)|^2}}\right)\mathrm{d}x + \|u(t)\|_p^p\\ &= -I(u(t)), t \in [0, T_{\max}).\end{aligned} \tag{4.4}$$

Now, we will focus on our main results on a sufficient criterion for the global existence and finite time blow-up of the solution, starting with the initial datum under subcritical initial energy as follows.

Theorem 4.3. *Assume that u is the solution to the problem* (1.1) *with subcritical initial energy* $E(0) < d$, *where* $u_0 \in H^2_{X,0}(\Omega)$*; then,*

(i) If $I(u_0) > 0$, *then the solution u exists globally, i.e.,* $T_{\max} = \infty$, *and it has the following boundedness property:*

$$\|X^2u(t)\|^2 + \int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau \le \frac{3p-2}{p-2}d,\ t \in [0, \infty). \tag{4.5}$$

Moreover, there exists a positive constant $C := 1 - S^p\left(\frac{2p}{\lambda_1(p-2)}d\right)^{\frac{p-2}{2}}$ *satisfying*

$$\|u(t)\|^2 \le \|u_0\|^2 e^{-\lambda_1 C t},\ t \in [0, \infty).$$

(ii) If $I(u_0) < 0$, *then the solution u undergoes finite time blow-up, i.e.,* $T_{\max} < \infty$. *Moreover, the following holds:*

$$T_{\max} \le \frac{3(p+1)\|u_0\|^2}{2(p-2)^2(d-E(0))}.$$

Proof. As the conclusion is trivial for $u_0 = 0$, we thus only focus on the case that $u_0 \in \mathcal{N}_+$. From Theorem 2.2, let u be the local solution to the problem (1.1) corresponding to the initial data u_0; we first show that $T_{\max} = \infty$. From (1.10) and (6.8), we deduce that

$$E(t) = \frac{p-2}{2p}\|X^2u(t)\|^2 + \int_\Omega \left(\frac{p+(p-1)|Xu(t)|^2}{p\sqrt{1+|Xu(t)|^2}} - 1\right)\mathrm{d}x + \frac{1}{p}I(u(t)) \tag{4.6}$$

and

$$\int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau + E(u(t)) = E(0) \tag{4.7}$$

for all $0 < t < T_{\max}$. Thus, by Lemma 4.1, we obtain that $u(t) \in \mathcal{N}_+$ for any $t \in [0, T_{\max})$, which means that

$$E(u(t)) \le E(0) < d,\ I(u(t)) \ge 0,\ t \in [0, T_{\max}).$$

Thus, the combination of (4.6) and (4.7) shows that

$$\|X^2u(t)\|^2 + \int_0^t \|u_t(\tau)\|^2 \mathrm{d}\tau \le \frac{3p-2}{p-2}d, \ t \in [0, T_{\max}).$$

Hence, with the help of the continuation principle, we obtain that $T_{\max} = \infty$, that is, $u \in L^\infty(0, \infty; H^2_{X,0}(\Omega))$ with $u_t \in L^2(0, \infty; L^2(\Omega))$ means that the weak solution for the problem (1.1) is global in time; it also satisfies the estimate (4.5) based on the initial data u_0.

We next consider the decay estimate. Given that u is a global solution, by (4.3) in Lemma 4.2, we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = -2I(u(t)), \ t \in [0, \infty). \tag{4.8}$$

Since $E(0) < d$ and $I(u_0) > 0$, from Lemma 4.1, one can infer that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T)$. Given the fact that $I(u(t))$ is continuous with respect to t, one can get that $I(u(t)) > 0$ for any $t \in [0, T)$. Since the energy functional $E(u(t))$ is non-increasing via the energy identity (4.2), we obtain

$$\begin{aligned} E(0) \ge & E(u(t)) \\ = & \frac{p-2}{2p}\|X^2u(t)\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xu(t)|^2}{p\sqrt{1 + |Xu(t)|^2}} - 1 \right) \mathrm{d}x + \frac{1}{p} I(u(t)) \\ > & \frac{p-2}{2p}\|X^2u(t)\|^2 \\ > & \frac{\lambda_1(p-2)}{2p}\|u(t)\|^2, \end{aligned} \tag{4.9}$$

where λ_1 is the best constant of the Sobolev embedding $H^2_{X,0}(\Omega) \hookrightarrow L^2(\Omega)$ in (1.13). On the other hand, one can obtain from $I(u(t)) > 0$ that

$$\|u\|_p^p \le S^p\|X^2u\|^p = S^p\left(\|X^2u\|^2\right)^{\frac{p-2}{2}} \|X^2u\|^2. \tag{4.10}$$

Then, from (4.9), we can deduce that

$$S^p\left(\|X^2u\|^2\right)^{\frac{p-2}{2}} \le S^p\left(\frac{2p}{\lambda_1(p-2)}E(0)\right)^{\frac{p-2}{2}} < S^p\left(\frac{2p}{\lambda_1(p-2)}d\right)^{\frac{p-2}{2}} := \eta,$$

which, combined with Lemma 1.8, shows that $\eta < 1$. Hence, taking $\eta^* := 1 - \eta > 0$, we obtain from (4.10) that

$$\|u\|_p^p \le (1-\eta)\|X^2u\|^2,$$

which immediately gives

$$\lambda_1\eta^*\|u(t)\|^2 \le \eta^*\|X^2u\|^2 \le I(u(t)).$$

Therefore, utilizing the definition of $I(u(t))$ and (4.3) in Lemma 4.2, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = -2I(u(t)) \le -2\lambda_1\eta^*\|u(t)\|^2. \tag{4.11}$$

Then, by means of Gronwall's inequality, it follows that

$$\|u(t)\|^2 \le e^{-\lambda_1\eta^*}\|u_0\|^2, t \in [0, \infty).$$

5. Global dynamics analysis of solutions for problem (1.1) with critical initial energy $E(0) = d$

5.1. Summary of the dynamics near the ground state

As shown in Theorem 4.3, we found that the global solution approaches zero asymptotically as $t \to +\infty$. Obviously, it is easy to understand that zero is a trivial solution to the problem (3.3). Therefore, we further want to know the following: Do all global solutions for the problem (1.1) eventually converge to the general nontrivial solution? To address the question, we have the following result.

Let the solution obtained in Theorem 4.3 exist globally; then, the ω-limit set $\omega(u_0)$ is defined by

$$\omega(u_0) = \bigcap_{t>0} \overline{\bigcup_{s>t} \{u(s)\}},$$

where the bar over a set refers to the closure, which is taken in $H^2_{X,0}(\Omega)$. Meanwhile, we know that

$$\omega(u_0) = \{\, u_\infty \in H^2_{X,0}(\Omega) \,|\exists \{t_n\}_{n=1}^{\infty} \subset (0,\infty),\ \lim_{n\to\infty} t_n = \infty \text{ such that } \lim_{n\to\infty} \|X^2(u(t_n) - u_\infty)\| = 0 \,\}.$$

Theorem 5.1. *Suppose that u is a global bounded solution for the problem* (1.1)*, which implies that* $w(u_0) \neq \emptyset$*; then, there exist an increasing sequence* $\{t_n\}_{n=1}^{+\infty}$*, with* $t_n \to +\infty$ *as* $n \to +\infty$*, and a solution* $w \in \Sigma$ *to the problem* (3.3) *satisfying that*

$$u(t_n) \to w \text{ weakly in } H^2_{X,0}(\Omega).$$

Moreover, $\omega(u_0) = \{0\}$ *if* $0 \in \omega(u_0)$.

Proof. We take a monotonically increasing sequence $\{t_n\}_{n=1}^{+\infty}$ satisfying the condition that $t_n \to +\infty$ as $n \to +\infty$, and we set $u_n = u(t_n)$. By the boundedness of the solution, we demonstrate that there exit a subsequence of $\{u_n\}_{n=1}^{+\infty}$, denoted still by $\{u_n\}_{n=1}^{+\infty}$ and a function $w \in H^2_{X,0}(\Omega)$, which satisfies that

$$u_m \to w \text{ weakly in } H^2_{X,0}(\Omega) \text{ and } u_n \to w \text{ a.e. in } \Omega. \tag{5.1}$$

For every $\widehat{T} < +\infty$, one can choose two functions $\phi(x)$ and $\iota(s)$ that satisfy

$$\phi(x) \in H^2_{X,0}(\Omega), \iota(s) \in C_0^2(0,\widehat{T}), \iota(s) \geq 0, \int_0^{\widehat{T}} \iota(s)\mathrm{d}s = 1,$$

and let

$$\rho(x,t) := \begin{cases} \iota(t-t_n)\phi(x), & (x,t) \in \Omega' \times (t_n, +\infty), \\ 0, & (x,t) \in \Omega' \times [0, t_n]. \end{cases}$$

In consideration of the definition of the above functions, one can deduce that

$$\int_0^{t_n+\widehat{T}} \int_\Omega u_t(\tau)\rho \mathrm{d}x\mathrm{d}\tau = \int_{t_n}^{t_n+\widehat{T}} \int_\Omega u_t(\tau)\iota(\tau-\tau_n)\phi \mathrm{d}x\mathrm{d}\tau$$

$$= \int_\Omega \left(u(t_n + \widehat{T})\iota(\widehat{T})\phi - u(t_n)\iota(0)\phi \right) \mathrm{d}x$$
$$- \int_{t_n}^{t_n+\widehat{T}} \int_\Omega u(\tau)(\iota(\tau - \tau_n)\phi)_t \mathrm{d}x\mathrm{d}\tau$$
$$= - \int_{t_n}^{t_n+\widehat{T}} \left(u(\tau)\iota'(\tau - \tau_n)\phi \right) \mathrm{d}x\mathrm{d}\tau.$$

Thus, choosing the test function $v = \rho$ in (1.9), for the problem (1.1), it follows that

$$\int_{t_n}^{t_n+\widehat{T}} \int_\Omega \Big(u_t(\tau)\iota'(\tau - \tau_n)\phi - \iota(\tau - \tau_n)X^2u(\tau)X^2\phi - \iota(\tau - \tau_n)\left(\frac{Xu(\tau)}{\sqrt{1 + |Xu(\tau)|^2}} X\phi \right) + |u(\tau)|^{p-2}u(\tau)\phi \Big) \mathrm{d}x\mathrm{d}\tau = 0. \tag{5.2}$$

Making a transformation $s = t - t_n$ in (5.2), one can derive

$$\int_0^{\widehat{T}} \int_\Omega \Big(u_t(\tau_n + s)\iota'(s)\phi - \iota(s)X^2u(\tau_n + s)X^2\phi - \iota(s)\left(\frac{Xu(\tau_n + s)}{\sqrt{1 + |Xu(\tau_n + s)|^2}} X\phi \right) + |u(\tau_n + s)|^{p-2}u(\tau_n + s)\phi \Big) \mathrm{d}x\mathrm{d}\tau = 0. \tag{5.3}$$

In consideration of $H^2_{X,0}(\Omega) \hookrightarrow L^p(\Omega)$ and (5.1), we know that, for any $s \in [0, \widehat{T}]$, there exist a subsequence of $\{u_n\}_{n=1}^{+\infty}$, denoted still by $\{u_n\}_{n=1}^{+\infty}$, and a function $\widetilde{w} \in L^p(\Omega)$ such that

$$u(t_n + s) \to \widetilde{w} \text{ weakly in } L^p(\Omega) \text{ and } u(t_n) \to w \text{ weakly in } L^p(\Omega). \tag{5.4}$$

We next confirm that $\widetilde{w} = w$ almost everywhere in Ω. Indeed, by employing a contradiction argument, we claim that $E(u(t)) \geq 0$ for any $t \geq 0$ via the assumption that the solution is global in time. In addition, if there exist a $t_0 \in [0, \infty)$ satisfying that $E(u(t_0)) < 0$, then, from the relation in (1.12), we obtain that $I(u(t_0)) < 0$; thus, given the conclusion presented in Theorem 4.3, the solution undergoes the finite time blow-up, which contradicts the assumption that the solution is global.

Given that $0 \leq E(u(t)) \leq E(0)$ for any $t \in [0, \infty)$ and the non-increasing property of $E(u(t))$ with respect to t in (4.2), there exists a constant $c > 0$ satisfying

$$\lim_{t\to\infty} E(u(t)) = c.$$

Then, we infer from (1.10) that

$$\int_0^{+\infty} \|u_t(\tau)\|^2 \mathrm{d}\tau \leq E(0) < +\infty, \tag{5.5}$$

thus, combining this with Hölder's inequality and (5.5), we obtain

$$\int_\Omega |u(t_n + s) - u(t_n)|^2 \mathrm{d}x = \int_\Omega \left(\int_{t_n}^{t_n+s} u_t(\tau)\mathrm{d}\tau \right)^2 \mathrm{d}x$$

$$\leq s\int_{t_n}^{t_n+s}\int_\Omega (u_t(t))^2\,\mathrm{d}x\mathrm{d}\tau$$
$$\leq T\int_{t_n}^{t_n+s}\|u_t(\tau)\|^2\mathrm{d}\tau\to 0$$

as $t_n\to+\infty$, i.e.,

$$u(t_n+s)\to u(t_n)\text{ strongly in }\ L^2(\Omega)\text{ as }t_n\to+\infty.$$

Therefore, $\widetilde{w}=w$ almost everywhere in Ω, which confirms our claim. Taking $n\to+\infty$ in (5.3), one can deduce from the dominated convergence theorem, (5.1), (5.5) and the choice of ι that

$$\int_0^{\widehat{T}}\int_\Omega\Big(w\iota'(s)\phi-\iota(s)X^2wX^2\phi-\iota(s)\left(\frac{Xw}{\sqrt{1+|Xw|^2}}X\phi\right)+|w|^{p-2}w\phi\Big)\mathrm{d}x\mathrm{d}\tau=0. \tag{5.6}$$

Since $\iota(0)=\iota(T)=0$, we deduce that

$$\int_0^{\widehat{T}}\int_\Omega w\iota'(s)\phi\mathrm{d}x\mathrm{d}\tau=\int_\Omega\Big(w\iota(\widehat{T})-w\iota(0)\phi\Big)\,\mathrm{d}x=0.$$

Thus, from (5.6), we obtain

$$\begin{aligned}&\int_\Omega\left(X^2wX^2\phi-\left(\frac{Xw}{\sqrt{1+|Xw|^2}}X\phi\right)-|w|^{p-2}w\phi\right)\mathrm{d}x\\=&\frac{1}{\widehat{T}}\int_0^{\widehat{T}}\int_\Omega\left(X^2wX^2\phi-\left(\frac{Xw}{\sqrt{1+|Xw|^2}}X\phi\right)-|w|^{p-2}w\phi\right)\mathrm{d}x\mathrm{d}\tau=0.\end{aligned}$$

Hence, by

$$\begin{aligned}\langle E(u(t)),\phi\rangle&=(X^2u(t),X^2\phi)+\left(\frac{Xu(t)}{\sqrt{1+|Xu(t)|^2}}X\phi\right)-|u(t)|^{p-2}u(t)\phi\\&=-(u'(t),\phi),\end{aligned} \tag{5.7}$$

we know that w is a weak solution of problem (1.1). Finally, we show that $\omega(u_0)=\{0\}$ if $0\in\omega(u_0)$. Since $0\in\omega(u_0)$, there exists an increasing sequence $\{t_k\}_{k=1}^\infty$, $t_k\to\infty$ as $k\to\infty$ such that $u(t_k)\to 0$ in $H^2_{X,0}(\Omega)$ as $k\to\infty$; together with (4.2) in Lemma 4.2, it follows that $E(0)=0$ and $E(t)$ is decreasing to 0 as t increases to ∞. Therefore, there exists a constant $t_0\geq 0$ such that

$$E(u(t_0))<d^*, \tag{5.8}$$

where $d^*:=\frac{p-2}{2p}S^{-\frac{2p}{p-2}}\leq d$. Because $d^*<d$ and the solution exists globally, we obtain from Theorem 4.3 that $I(u(t_0))\geq 0$. For the case of $I(u(t_0)=0$, $E(u(t_0))=d^*<d=\inf_{w\in\mathcal{N}}E(w)$ implies that $u(t_0)=0$. Thus, it follows that the uniqueness of solutions satisfies that $u(t)\equiv 0$ for all $t\geq t_0$, so $\omega(u_0)=\{0\}$. For the case of $I(u(t_0)>0$, it follows from Theorem 4.3 that $\|X^2u(t)\|$ exponentially decays to 0 for $t\geq t_0$; thus, $\omega(u_0)=\{0\}$.

Theorem 5.2. *Assume that Q is a ground state solution for the problem* (3.3)*, and that u is the solution to the problem* (1.1)*; then, the following claims hold:*

(i) For every $\epsilon > 0$, there exists the initial data satisfying the property

$$\|X^2u_0 - X^2Q\| < \epsilon,$$

leading to the solution u blows up in finite time.

(ii) If the initial data satisfies

$$\|X^2u_0\|^2 + \int_\Omega \left(\frac{|Xu_0|^2}{\sqrt{1+|Xu_0|^2}} \right) \mathrm{d}x = \|Q\|_p^p,$$

then the solution u exists globally. Meanwhile, for any $t \in [0, \infty)$, the boundedness property is obtained as shown in (4.5)*.*

Proof. (i) For any constant $\eta > 1$, we let

$$u_0 := \eta Q; \tag{5.9}$$

then, for any $\varepsilon > 0$, one can choose a suitable $\eta > 1$, which satisfies

$$\|X^2(u_0 - Q)\| = (\eta - 1)^2 \|X^2Q\| < \varepsilon.$$

By (5.9), (iii) in Lemma 1.6 and $Q \in \mathcal{N}$, we know that

$$I(u_0) = I(\eta Q) < I(Q) = 0.$$

Furthermore, we obtain from (ii) in Lemma 1.6 and (i) in Theorem 3.2 that

$$E(u_0) = E(\eta u_0) < E(Q) = d.$$

Hence, for such a initial datum u_0, the assumption in (ii) of Theorem 4.3 is satisfied; thus, the corresponding solution we obtained here blows up in finite time.

(ii) To begin, let us recall the assumption on the initial data, i.e.,

$$\|X^2u_0\|^2 + \int_\Omega \left(\frac{|Xu_0|^2}{\sqrt{1+|Xu_0|^2}} \right) \mathrm{d}x < \frac{p-2}{p}\|Q\|_p^p, \tag{5.10}$$

where Q is a ground state solution for the problem (3.3). Given the fact that $Q \in \mathcal{N}$, i.e., $I(Q) = 0$, one can derive

$$\|X^2Q\|^2 + \int_\Omega \left(\frac{|XQ|^2}{\sqrt{1+|XQ|^2}} \right) \mathrm{d}x = \|Q\|_p^p;$$

thus, it follows from the definition of $E(t)$ that

$$d = E(Q) = \frac{p-2}{2p}\|Q\|_p^p. \tag{5.11}$$

We conclude that $I(u_0) > 0$ via contradiction. Regarding the contradiction, by (iii) in Lemma 1.6, we deduce that there exists a constant $\mu \in (0, 1]$ satisfying that $I(\mu u_0) = 0$, i.e., $\mu u_0 \in \mathscr{N}$; thus, from the definition of d, we get that $E(\mu u_0) \geq d$. Furthermore, for the fixed μ, we obtain from the definition of $E(t)$, (5.10) and (5.11) that

$$E(\mu u_0) =\frac{\mu^2}{2}\left(\|X^2u_0\|^2 + \int_\Omega\left(\frac{|Xu_0|^2}{\sqrt{1+|Xu_0|^2}}\right)\mathrm{d}x\right) - \frac{p-2}{p}\|u_0\|_p^p \tag{5.12}$$

$$<\frac{\mu^2}{2}\left(\|X^2u_0\|^2 + \int_\Omega\left(\frac{|Xu_0|^2}{\sqrt{1+|Xu_0|^2}}\right)\mathrm{d}x\right) \tag{5.13}$$

$$\leq\frac{1}{2}\left(\|X^2u_0\|^2 + \int_\Omega\left(\frac{|Xu_0|^2}{\sqrt{1+|Xu_0|^2}}\right)\mathrm{d}x\right) \tag{5.14}$$

$$<\frac{p-2}{2p}\|Q\|_p^p = d, \tag{5.15}$$

which yields a contradiction. Next, we prove that $E(u_0) < d$, it is easy to obtain via the chains of inequality in (5.12), i.e.,

$$\begin{aligned} E(0) =&\frac{1}{2}\left(\|X^2u_0\|^2 + \int_\Omega\left(\frac{|Xu_0|^2}{\sqrt{1+|Xu_0|^2}}\right)\mathrm{d}x\right) - \frac{p-2}{p}\|u_0\|_p^p \\ <&\frac{1}{2}\left(\|X^2u_0\|^2 + \int_\Omega\left(\frac{|Xu_0|^2}{\sqrt{1+|Xu_0|^2}}\right)\mathrm{d}x\right) \\ <&\frac{p-2}{2p}\|Q\|_p^p = d. \end{aligned}$$

Therefore, in consideration of (i) of Theorem 4.3, we conclude that the solution exists globally and the estimate (4.5) holds.

5.2. *Global existence and decay of solutions for problem* (1.1) *with* $E(0) = d$

Theorem 5.3. *Suppose that* $u_0 \in H^2_{X,0}(\Omega)$ *satisfies that* $E(0) = d$ *and* $I(u_0) \geq 0$*; then, the solution* u *obtained in Theorem 2.2 exists globally, i.e.,* $T_{\max} = \infty$*. Moreover, the following holds:*

(i) If $I(u_0) = 0$*, then* $u_0 \in H^2_{X,0}(\Omega)$ *is a ground state solution of problem* (3.3) *and* $u(t) \equiv u_0$ *for all* $t \geq 0$*.*

(ii) If $I(u_0) > 0$*, then the* ω*-limit set* $\omega(u_0) = \{0\}$*. Moreover, there exists a constant* $\xi \gg 1$ *such that the exponential decay estimates in (ii) of Theorem 4.3 holds for* $t \geq \xi$ *by replacing* u_0 *with* $u(\xi)$*.*

Proof. We divide the proof into two cases.

Case 1: $I(u_0) = 0$ with $E(0) = d$. Since $E(0) = d > 0$, it obvious that $u_0 \not\equiv 0$. By the proof of Theorem 3.2, $u_0 \in H^2_{X,0}(\Omega)$ is a ground state solution of problem (3.3); then, by the uniqueness of the solution, we have that $u(t) \equiv u_0$ for all $t \geq 0$.

Case 2: $I(u_0) > 0$ with $E(0) = d$. Since $u \in C([0, T); H^2_{X,0}(\Omega))$, we get that $I(u(t)) \in C([0, T))$ which, together with $I(u_0) > 0$, implies that there exists a constant $t_0 > 0$ small enough such that $I(u(t)) > 0$ for any $t \in [0, t_0]$. Then, by $(u, u_t) = -I(u(t))$, we get that $\|u_t(t)\| > 0$ for any $t \in [0, t_0]$. Thus, we obtain

$$E(u(t_0)) =E(0) - \int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau$$

$$=d - \int_0^t \|u_t(\tau)\|^2 d\tau < d.$$

Then, we conclude that $I(u(t_0)) > 0$, and Theorem 4.3 tells us that u exists globally. By Theorem 3.2, there exist a $\widetilde{u}$ in $\omega(u_0) \cap \Sigma$ and an increasing sequence $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to} \|X^2 u(t_k) - X^2\widetilde{u}\| = 0$. Then, it follows that $E(u(t)) \in C([0,\infty))$, $E(u(t))$ is non-increasing with respect to t and $E(u(t_0)) < d$ that

$$E(\widetilde{u}) = \lim_{k\to\infty} E(u(t_k)) \le E(u(t_0)) < d.$$

Since $\widetilde{u} \in \Sigma$, we get that $I(\widetilde{u}) = 0$. Then, one can infer from the definition of d that $\widetilde{u} = 0$, i.e., $0 \in \omega(u_0)$. Hence, we infer from Theorem 3.2 that $\omega(u_0) = \{0\}$.

We next show the exponential decay of the solution for $t \gg 1$. Indeed, since $\omega(u_0) = \{0\}$, there exists $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k = \infty$ and $u(t_k) \to 0$ in $H^2_{X,0}(\Omega)$ as $k \to \infty$. Since $E(0) = 0$ and $E(v)$ is continuous with $v \in H^2_{X,0}(\Omega)$, there exists a constant $\epsilon \gg 1$ such that $E(u(\epsilon)) < d^*$, where $d^* = \frac{p-2}{2p} S^{-\frac{2p}{p-2}} \le d$. Given the sign of $I(u(\epsilon))$, it follows that

(a) if $I(u(\epsilon)) < 0$, because $d^* < d$, we know from Theorem 4.3 that u blows up in finite time, which contradicts the fact that u exists globally;
(b) if $I(u(\epsilon)) = 0$, it infers that $u(\epsilon) = 0$. Otherwise, if $u(\epsilon) \ne 0$, then, by the definition of d, we get that $E(u(\epsilon)) > d$, which contradicts that $E(u_0) < d^* \le d$. Then, by the uniqueness of the solution, we obtain that $u(t) \equiv 0$ for all $t \ge \epsilon$, and the conclusion holds;
(c) if $I(u(\epsilon)) < 0$, in consideration of $E(u(\epsilon)) < d^*$, the result follows from Theorem 4.3 immediately.

5.3. Finite time blow-up of solutions for problem (1.1) *with* $E(0) = d$

Theorem 5.4. *Suppose that $u_0 \in H^2_{X,0}(\Omega)$ satisfies that $E(0) = d$ and $I(u_0) \ge 0$; then, the solution u obtained in Theorem 2.2 blows up in finite time, i.e., $T_{\max} < \infty$.*

Proof. Let u be any local solution of problem (1.1) obtained by applying Theorem 2.2 with $E(0) = d$, $u_0 \in \mathcal{N}_-$. From the continuities of $E(t)$ and $I(u(t))$ with respect to t, one can deduce that there exists a sufficiently small $t_1 > 0$, with $t_1 < T_{\max}$ satisfying that $E(t_1) > 0$ and $I(u(t_1)) < 0$ for any $t \in [0, t_1]$. We then have that $(u(t), u_t(t)) = -2I(u(t)) > 0$ and $\|u_t(t)\| > 0$ for $t \in [0, t_1]$. From this and the continuity of $\int_0^t \|u_t(\tau)\|^2 d\tau$, we can choose a t_1 such that

$$0 < E(u(t_1)) = d - \int_0^{t_1} \|u_t(\tau)\|^2 d\tau = d_1 < d. \tag{5.16}$$

Multiplying equation (1.1) by u_t in first and then integrating over (t_1, t) gives

$$E(u(t)) + \int_{t_1}^{t} \|u_t(\tau)\|^2 d\tau = E(u(t_1)). \tag{5.17}$$

Taking $v(t) = u(t + t_1)$ as the new initial data, similar to the situation in which $E(0) < d$ in Lemma 4.1, it still holds that $v(t) \in \mathcal{N}_-$ for all $t \in [t_1, T_{\max})$; the remainder of the proof is similar to the proof in Theorem 4.3; hence, it is omitted.

6. Global dynamics analysis of solutions for problem (1.1) with supercritical initial energy $E(0) > d$

6.1. A sharp-like threshold condition for problem (1.1) with supercritical initial energy $E(0) > d$

Theorem 6.1 (Sharp-like threshold condition). *Assume that* $u_0 \in H^2_{X,0}(\Omega)$. *For any* $E(0) < \kappa$ *with* $\kappa \in (d, +\infty)$*, the following conclusions hold:*

(i) If $u_0 \in \mathcal{N}_+$ *satisfies that* $\|u_0\|^2 \leq \lambda_\kappa$*, then the solution* u *for the problem* (1.1) *exists globally in time and* $u(t) \to 0$ *as* $t \to +\infty$;

(ii) If $u_0 \in \mathcal{N}_-$ *satisfies that* $\|u_0\|^2 \geq \Lambda_\kappa$*, then the solution* u *for the problem* (1.1) *blows up in finite time.*

Proof. (i) Suppose that $u_0 \in \mathcal{N}_+$ with $E(0) < \alpha$ and $\|u_0\|^2 \leq \lambda_\alpha$. Recalling that the non-increasing property of λ_α, we know that $\|u_0\|^2 \leq \lambda_\alpha \leq \lambda_{E(0)}$. Then, we need to show that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_{\max}(u_0))$ provided that $u_0 \in \mathcal{N}_+$. Arguing by contradiction, given the continuity of $I(u(t))$ with respect to t, one can assume that there exists the first time $t_1 \in (0, T_{\max}(u_0))$ that yields $u(t) \in \mathcal{N}_+$ for any $t \in [0, t_1)$ and $u(t_1) \in \mathcal{N}$. Combining $u(t) \in \mathcal{N}_+$ for any $t \in [0, t_1)$, $\|u_0\|^2 \leq \lambda_\alpha \leq \lambda_{E(0)}$ and (4.3) in Lemma 4.2, we obtain

$$\|u(t_1)\|^2 < \|u_0\|^2 \leq \lambda_{E(0)} \tag{6.1}$$

and

$$E(u(t_1)) < E(u_0). \tag{6.2}$$

Therefore, from $u(t_1) \in \mathcal{N}$ and (6.2), one can deduce that $u(t_1) \in \mathcal{N}_{E(0)}$. Combining the definition of $\lambda_{E(0)}$ and $u(t_1) \in \mathcal{N}_{E(0)}$, one can conclude that $\|u(t_1)\|^2 \geq \lambda_{E(0)}$, which contradicts (6.1); then, we obtain $u(t) \in \mathcal{N}_+$ for any $t \in [0, T_{\max}(u_0))$. Moreover, taking (6.2) together with (ii) of Lemma 1.8, $\|Xu\|^2$ is bounded in $H^2_{X,0}(\Omega)$, i.e., $\|Xu\|^2 < \frac{2pE(0)}{p-2}$, which tells us that the weak solution is global in time; this means that $T_{\max}(u_0) = \infty$.

In order to prove the asymptotic behavior of the solution as $t \to +\infty$, we need several of the results proved above. For any $u_0 \in H^2_{X,0}(\Omega)$, there exists a maximal solution for the problem (1.1) on $[0, T)$, denoted by $S(t)u_0 \equiv u(t, u_0)$. Suppose that $\sup_{t>0} \|S(t)u_0\|_{H^1} \leq M$ for some $M > 0$. Thus, $\{S(t)u_0\}_{t>0}$ is a relatively compact set in $H^2_{X,0}(\Omega)$, which is a consequence of the fact that the operator $e^{i\theta}\Delta$ has a compact resolvent; therefore, we denote the ω-limit set of $u_0 \in \mathcal{N}_+$ satisfying that $E(0) < \alpha$ and $\|u_0\|^2 \leq \lambda_\alpha$.

Because we know that $u(t) \in \mathcal{N}_+$ for any $t \geq 0$, considering this together with (i) in Lemma 1.8, one deduces from (4.2) in Lemma 4.2 that $E(u(t)) > 0$ is bounded below and decreasing in t, which means that there exists a constant $c \geq 0$ satisfying that $\lim_{t\to+\infty} E(u(t)) = c$. Obviously, for every $u_0^* \in \omega(u_0)$, we get that $E(u^*(t)) = c$ for sufficiently large $t \geq 0$, where $u^*(t)$ denotes the semiflow for problem (1.1) with the initial datum u_0^*. At the same time, we obtain that $u^*(t) = u_0^*$ for any $t \geq 0$, which, together with (4.3) in Lemma 4.2, shows that

$$-I(u^*(t)) = 0, t \in [0, \infty). \tag{6.3}$$

Thus, formula (6.3) implies that

$$\omega(u_0) \in \mathcal{N} \cup \{0\}. \tag{6.4}$$

However, if $u_0^* \in \omega(u_0)$, given $u_0^* \in \mathscr{N}_+$ satisfying that $E(u_0^*) < \alpha$ and $\|u_0^*\|^2 \le \lambda_\alpha$, then we know that $u_0^* \notin \mathscr{N} \cap \{E(u) < \alpha\}$ and

$$\omega(u_0) \cap \mathscr{N} = \emptyset. \tag{6.5}$$

Hence, from (6.4) and (6.5), it is easy to get that $\omega(u_0) = \{0\}$, which implies that the solution exists globally and approaches 0 as $t \to +\infty$, as long as $u_0 \in \mathscr{N}_+$ satisfies that $E(0) < \alpha$ and $\|u_0\|^2 \le \lambda_\alpha$.

(ii) If $u_0 \in \mathscr{N}_-$ satisfies that $E(0) < \alpha$ and $\|u_0\|^2 \ge \Lambda_\alpha$, by employing a similar discussion in the proof of (i), it yields that $u(t) \in \mathscr{N}_-$ for any $t \in [0, T_{\max}(u_0))$. To obtain the finite time blow-up, we assume that $T_{\max}(u_0) = +\infty$. Given that (4.2) in Lemma 4.2 describes the non-increasing property of $E(u(t))$ in t, one of the two cases below may exist:

(a) There is a constant C satisfying that $\lim\limits_{t\to+\infty} E(u(t)) = C$;

(b) $\lim\limits_{t\to+\infty} E(u(t)) = -\infty$.

In what follows, we will explain that neither of these cases hold by applying the contradiction to $T_{\max}(u_0) = +\infty$, which infers that $T_{\max}(u_0) < \infty$.

Suppose that the case (a) happens; by (4.2) in Lemma 4.2 and $\lim\limits_{t\to+\infty} E(u(t)) = C$, one can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) = -\|u_t(t)\|^2 \to 0 \text{ as } t \to +\infty,$$

which means that the solution u is trending to the stationary solution to the problem (1.1) as $t \to +\infty$, that is, $u(t) \in \mathscr{N}$ or $u(t) = 0$ as $t \to +\infty$. Alternatively, $u(t) \in \mathscr{N}_-$ for every $t > 0$ means that $u(t) \notin \mathscr{N}$ for all $t > 0$, which yields that $u(t) = 0$ as $t \to +\infty$. From the other side, $u(t) \in \mathscr{N}_-$, together with (iii) in Lemma 4.2, implies that $u(t) \neq 0$ as $t \to +\infty$, which contradicts that $u(t) = 0$ as $t \to +\infty$. Thus, the case (a) cannot occur.

For the case (b), repeating what we did for the case (a), we assume that case (b) occurs; since there is the continuity of $E(t)$ in t, one can infer that there exists a first time $t_1 < T_{\max}(u_0)$ satisfying that $E(t_1) < 0$. We have proved that $u(t) \in \mathscr{N}_-$ for all $t \in [0, T_{\max}(u_0))$; thus, $u(t_1) \in \mathscr{N}_-$. Choosing $u(t_1)$ as a new initial datum, Theorem 4.3 tells us that the corresponding solution $U(t) = u(t_1 + t)$ blows up in finite time, which conflicts with our hypothesis that the solution u is global. Therefore, the case (b) also cannot occur.

Totally, together with (a) and (b), one can deduce that $T_{\max}(u_0) < \infty$, that is, the solution to the problem (1.1) blows up in finite time provided that $u_0 \in \mathscr{N}_-$ satisfies that $E(0) < \alpha$ and $\|u_0\|^2 \ge \Lambda_\alpha$.

Theorem 6.2. *Let $u_0 \in H^2_{X,0}(\Omega)$ satisfy that $E(0) > d$ and*

$$\frac{2p}{\lambda_1(p-2)}|\Omega|^{\frac{p-2}{2}} E(0) \le \|u_0\|^p, \tag{6.6}$$

where $|\Omega|$ is the volume of Ω; then, the solution obtained in Theorem 2.2 blows up in finite time, i.e., $T_{\max} < \infty$.

Proof. Let u be the solution obtained in Theorem 2.2. Given (6.6) and Hölder's inequality, we have

$$\frac{2p}{\lambda_1(p-2)}|\Omega|^{\frac{p-2}{2}} E(0) \le \|u_0\|^p \le \|u_0\|_p^p |\Omega|^{\frac{p-2}{2}}.$$

On the other hand, the relationship between $E(t)$ and $I(u)$ tells us that

$$\begin{aligned} E(0) >& \frac{p-2}{2p}\|u_0\|_p^p + \frac{1}{p}I(u_0) \\ >& E(0) + \frac{1}{p}I(u_0), \end{aligned}$$

which shows that $I(u_0) < 0$. Recalling the assumption in Theorem 6.1, if we want to demonstrate that the solution u blows up in finite time, we need to show that the initial data satisfies that $\|u_0\|^2 \geq \Lambda_{E(0)}$. Obviously, for any $v \in \mathcal{N} \cap \{E(v) < E(u_0)\}$, i.e., $I(v) = 0$ with $v \neq 0$, and $E(v) < E(u_0)$, we obtain from Hölder's inequality that

$$\begin{aligned} \|v\|^p \leq& |\Omega|^{\frac{p-2}{2}} \|v\|_p^p \\ =& |\Omega|^{\frac{p-2}{2}} \left(\|X^2 v\|^2 + \int_\Omega \left(\frac{|Xv|^2}{\sqrt{1+|Xv|^2}} \right) \mathrm{d}x \right) \\ =& |\Omega|^{\frac{p-2}{2}} \frac{2p}{p-2} \left(\left(\frac{1}{2} - \frac{1}{p} \right) \|X^2 v\|^2 + \left(\frac{1}{2} - \frac{1}{p} \right) \int_\Omega \left(\frac{|Xv|^2}{\sqrt{1+|Xv|^2}} \right) \mathrm{d}x \right) \\ \leq& |\Omega|^{\frac{p-2}{2}} \frac{2p}{p-2} \left(\frac{p-2}{2p} \|X^2 v\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xv|^2}{p\sqrt{1+|Xv|^2}} - 1 \right) \mathrm{d}x \right) \\ =& |\Omega|^{\frac{p-2}{2}} \frac{2p}{p-2} \left(\frac{p-2}{2p} \|X^2 v\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xv|^2}{p\sqrt{1+|Xv|^2}} - 1 \right) \mathrm{d}x + \frac{1}{p} I(v) \right) \\ <& |\Omega|^{\frac{p-2}{2}} \frac{2p}{p-2} E(0), \end{aligned}$$

which, together with the definition of $\Lambda_{E(0)}$, shows that

$$\Lambda_{E(0)}^p \leq |\Omega|^{\frac{p-2}{2}} \frac{2p}{p-2} E(0).$$

Hence, by (6.6), we obtain that $\|u_0\|^2 \geq \Lambda_{E(0)}$; we finally conclude the finite time blow-up results by applying (ii) in Theorem 6.1 immediately.

6.2. *A sufficient condition for the finite time blow-up solution for problem* (1.1) *with arbitrary positive initial energy* $E(0) > 0$

To begin, we prove the invariant lemma below, which provides key information for the proof of the finite time blow-up result for arbitrary positive initial energy.

Lemma 6.3. *Let* $u_0 \in H^2_{X,0}(\Omega)$ *satisfies that*

$$\frac{2p}{\lambda_1(p-2)} E(0) \leq \|u_0\|^2; \tag{6.7}$$

then, $u(t) \in \mathcal{N}_-$ *for any* $t \geq 0$, *where* λ_1 *is the optimal constant in* (1.13).

Proof. Given the definition of $E(t)$ and $I(u)$, it follows that

$$E(t) = \frac{p-2}{2p}\|X^2u(t)\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xu(t)|^2}{p\sqrt{1+|Xu(t)|^2}} - 1 \right) dx + \frac{1}{p} I(u(t)); \tag{6.8}$$

then, combining this with the assumption (6.7), we have

$$\begin{aligned} E(0) =& \frac{p-2}{2p}\|X^2u_0\|^2 + \int_\Omega \left(\frac{p + (p-1)|Xu_0|^2}{p\sqrt{1+|Xu_0|^2}} - 1 \right) dx + \frac{1}{p} I(u_0) \\ \geq& \frac{p-2}{2p}\|X^2u_0\|^2 + \frac{1}{p} I(u_0) \\ \geq& \frac{\lambda_1(p-2)}{2p}\|u_0\|^2 + \frac{1}{p} I(u_0), \end{aligned}$$

which means that $I(u_0) < 0$. To obtain $u(t) \in \mathcal{N}_-$ for any $t \in [0, T)$, in consideration of the continuity of $I(u(t))$ with respect to t, one can suppose that there exists a first time $\iota \in (0, T)$ satisfying that $u(t) \in \mathcal{N}_-$ for all $t \in (0, \iota)$ and $u(\iota) \in \mathcal{N}$; thus, from (4.3), it follows that

$$\frac{d}{dt}\|u(t)\|^2 = -2I(u(t)) > 0 \text{ for all } t \in [0, \iota), \tag{6.9}$$

which shows that

$$\|u_0\|^2 < \|u(\iota)\|^2. \tag{6.10}$$

From (4.2) in Lemma 4.2, it is easy to infer that

$$E(u(\iota)) \leq E(u_0). \tag{6.11}$$

Given the relationship between $E(t)$ and $I(u)$ in (1.12), that $u(\iota) \in \mathcal{N}$ and the weighted the Sobolev embedding inequality, it follows that

$$\begin{aligned} E(u(\iota)) >& \frac{p-2}{2p}\|X^2u(t)\|^2 + \frac{1}{p} I(u(t)) \\ \geq& \frac{\lambda_1(p-2)}{2p}\|u(t)\|^2. \end{aligned}$$

Combining (6.11) and (6.7), we get

$$\frac{\lambda_1(p-2)}{2p}\|u(\iota)\|^2 \leq E(0) < \frac{\lambda_1(p-2)}{2p}\|u_0\|^2,$$

that is

$$\|u_0\|^2 > \|u(\iota)\|^2,$$

which contradicts (6.10). Therefore, this lemma has been proved.

Next, the blow-up of the solution for problem (1.1) is given below. In addition, the estimate of the upper bound of the blow-up time is also discussed.

Theorem 6.4 (Finite time blow-up and lifespan estimate)**.** *Assume that* $u_0 \in H^2_{X,0}(\Omega)$ *and* $E(0) > 0$ *satisfies* (6.7)*; then, the local solution u for the problem (1.1) blows up in finite time. Moreover, there exists a time* $T_{\max}$ *satisfying*

$$T_{\max} \leq \frac{2c}{(\alpha-1)\|u_0\|^4}$$

such that

$$\lim_{t\to T_{\max}} \int_0^t \|u(\tau)\|^2 \mathrm{d}\tau = +\infty,$$

where the constant c satisfies

$$c > \frac{1}{4}\rho^{-2}\|u_0\|^4 \text{ and } 0 < \rho < \frac{1}{2\alpha\|u_0\|^2}\left(\xi\|u_0\|^2 - 4\alpha E(0)\right).$$

Proof. Arguing by contradiction, we begin to assume that the maximum existence time of the solution satisfies that $T_{\max} = +\infty$. From the definition of $E(t)$, $I(u)$ and (4.3), one can obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 &= -2I(u(t)) \\
&= -2\left(\|X^2u(t)\|^2 + \int_\Omega \left(\frac{|Xu(t)|^2}{\sqrt{1+|Xu(t)|^2}}\right)\mathrm{d}x - \|u(t)\|_p^p\right) \\
&= -4\left(\frac{1}{2}\|X^2u(t)\|^2 + \int_\Omega \left(\sqrt{1+|Xv|^2}-1\right)\mathrm{d}x - \frac{1}{p}\|u(t)\|_p^p\right) \\
&\quad + 2\int_\Omega \left(\frac{p+(p-1)|Xu(t)|^2}{p\sqrt{1+|Xu(t)|^2}} - 1\right)\mathrm{d}x + \left(2-\frac{4}{p}\right)\|u(t)\|_p^p \\
&= -4E(t) + 2\int_\Omega \left(\frac{p+(p-1)|Xu(t)|^2}{p\sqrt{1+|Xu(t)|^2}} - 1\right)\mathrm{d}x + \frac{2(p-2)}{p}\|u(t)\|_p^p \\
&\geq -4E(t) + \frac{2(p-2)}{p}\|u(t)\|_p^p.
\end{aligned} \tag{6.12}$$

Considering the sign of $E(t)$, we divide the proof into the following two cases.
Case I. $E(t) \geq 0$ for every $t > 0$. From (6.7), we choose α satisfying that

$$1 < \alpha < \frac{\xi\|u_0\|^2}{4E(0)}, \tag{6.13}$$

where $\xi := \frac{\lambda_1(p-2)}{2p}$. Substituting (4.2) into (6.12), and in consideration of $E(u(t)) \geq 0$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 &\geq 4(\alpha-1)E(u(t)) - 4\alpha E(u(t)) + \frac{2(p-2)}{p}\|u(t)\|_p^p \\
&\geq -4\alpha E(u(t)) + \frac{2(p-2)}{p}\|u(t)\|_p^p \\
&\geq -4\alpha E(0) + 4\alpha\int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau + \frac{2(p-2)}{p}\|u(t)\|_p^p.
\end{aligned} \tag{6.14}$$

From Lemma 6.3 (i.e., $I(u) < 0$) and the weighted Sobolev embedding inequality, we have

$$\|u\|_p^p \geq \|X^2 u\|^2 + \int_\Omega \left(\frac{|Xu(t)|^2}{\sqrt{1 + |Xu(t)|^2}} \right) \mathrm{d}x \geq \|X^2 u\|^2 \geq \lambda_1 \|u\|^2,$$

which causes (6.14) to become

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 \geq -4\alpha E(0) + 4\alpha \int_0^t \|u_t(\tau)\|^2 \mathrm{d}\tau + \frac{2\lambda_1(p-2)}{p}\|u(t)\|^2. \tag{6.15}$$

Hence, we obtain the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 - 4\xi\|u(t)\|^2 \geq -4\alpha E(u_0),$$

which yields

$$\|u(t)\|^2 \geq \|u_0\|^2 e^{4\xi t} + 4\alpha E(0)\left(1 - e^{4\xi t}\right). \tag{6.16}$$

Next, we denote $\mathcal{F}(t) := \int_0^t \|u(\tau)\|^2 \mathrm{d}\tau$. Because the solution has already been assumed to be global, $\mathcal{F}(t)$ is bounded for any $t \in [0, \infty)$. Hence, we derive

$$\mathcal{F}'(t) = \|u(t)\|^2 \text{ and } \mathcal{F}''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2. \tag{6.17}$$

Taking a constant ρ satisfies

$$0 < \rho < \frac{1}{2\alpha\|u_0\|^2}\left(\xi\|u_0\|^2 - 4\alpha E(0)\right); \tag{6.18}$$

then, we derive

$$\xi\|u_0\|^2 - 4\alpha E(0) \geq 2\alpha\rho\|u_0\|^2. \tag{6.19}$$

Substituting (6.16) into (6.15), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 > 4\alpha \int_0^t \|u_t(\tau)\|^2 \mathrm{d}\tau + \left(\|u_0\|^2 - 4\alpha E(0)\right) e^{\xi t}. \tag{6.20}$$

Thus, given (6.19) and that $e^t \geq 1$, (6.20) becomes

$$\begin{aligned} \mathcal{F}''(t) &> 4\alpha \int_0^t \|u_t(\tau)\|^2 \mathrm{d}\tau + \left(\|u_0\|^2 - 4\alpha E(0)\right) e^{\xi t} \\ &> 4\alpha \int_0^t \|u_t(\tau)\|^2 \mathrm{d}\tau + 2\alpha\rho\|u_0\|^2. \end{aligned} \tag{6.21}$$

Take

$$\mathcal{G}(t) := \left(\int_0^t \|u(\tau)\|^2 \mathrm{d}\tau\right)^2 + \rho^{-1}\|u_0\|^2 \int_0^t \|u(\tau)\|^2 \mathrm{d}\tau + c, \tag{6.22}$$

where

$$c > \frac{1}{4}\rho^{-2}\|u_0\|^4. \tag{6.23}$$

According to the definition of $\mathcal{G}(t)$, it is easy to derive

$$\mathcal{G}'(t) = \left(2\int_0^t \|u(\tau)\|^2 \mathrm{d}\tau + \rho^{-1}\|u_0\|^2\right)\|u(t)\|^2 \tag{6.24}$$

and

$$\mathcal{G}''(t) = \left(2\mathcal{F}(t) + \rho^{-1}\|u_0\|^2\right)\mathcal{F}''(t) + 2(\mathcal{F}'(t))^2. \tag{6.25}$$

To obtain the following results, we define

$$\delta := 4c - \rho^{-2}\|u_0\|^4 > 0,$$

where the positivity can be confirmed by (6.23); then, (6.24) gives

$$\begin{aligned}(\mathcal{G}'(t))^2 &= \left(4\mathcal{F}^2(t) + 4\varepsilon^{-1}\|u_0\|^2\mathcal{F}(t) + \varepsilon^{-2}\|u_0\|^4\right)(\mathcal{F}'(t))^2 \\ &= \left(4\mathcal{F}^2(t) + 4\varepsilon^{-1}\|u_0\|^2\mathcal{F}(t) + 4c - \delta\right)(\mathcal{F}'(t))^2 \\ &= (4\mathcal{F}(t) - \delta)(\mathcal{F}'(t))^2,\end{aligned} \tag{6.26}$$

which implies that

$$4\mathcal{G}(t)(\mathcal{F}'(t))^2 = (\mathcal{G}'(t))^2 + \delta(\mathcal{F}'(t))^2. \tag{6.27}$$

Taking advantage of the inner product in $L^2(\Omega)$, we have the following identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = (u(t), u_t(t)).$$

Integrating the above formula over $[0, t]$ yields

$$\|u(t)\|^2 = \|u_0\|^2 + 2\int_0^t (u(\tau), u_t(\tau))\mathrm{d}\tau. \tag{6.28}$$

Combining (6.17) with (6.28), and by applying Hölder's and Cauchy's inequalities, it follows that

$$\begin{aligned}\|u\|^4 =& \left(\|u_0\|^2 + 2\int_0^t (u, u_t)\mathrm{d}\tau\right)^2 \\ \leq& \left(\|u_0\|^2 + 2\left(\int_0^t \|u(\tau)\|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\right)^2 \\ =& \|u_0\|^4 + 4\|u_0\|^2\left(\int_0^t \|u(\tau)\|^2\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau\right)^{\frac{1}{2}} \\ &+ 4\int_0^t \|u(\tau)\|^2\mathrm{d}\tau\int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau \\ \leq& \|u_0\|^4 + 2\varepsilon\|u_0\|^2\int_0^t \|u(\tau)\|^2\mathrm{d}\tau + 2\varepsilon^{-1}\|u_0\|^2\int_0^t \|u_t(\tau)\|^2\mathrm{d}\tau\end{aligned} \tag{6.29}$$

$$
\begin{aligned}
&+4\int_0^t \|u(\tau)\|^2 \mathrm{d}\tau \int_0^t \|u_t(\tau)\|^2 \mathrm{d}\tau \\
:=&\mathcal{K}(t).
\end{aligned}
$$

From (6.25), (6.27) and (6.21), we obtain

$$
\begin{aligned}
2\mathcal{G}(t)\mathcal{G}''(t) =&2\left(\left(2\mathcal{F}(t)+\rho^{-1}\|u_0\|^2\right)\mathcal{F}''(t)+2(\mathcal{F}'(t))^2\right)\mathcal{F}(t)\\
=&2\left(2\mathcal{F}(t)+\rho^{-1}\|u_0\|^2\right)\mathcal{F}''(t)\mathcal{G}(t)+4(\mathcal{F}'(t))^2\mathcal{G}(t)\\
=&2\left(2\mathcal{F}(t)+\rho^{-1}\|u_0\|^2\right)\mathcal{F}''(t)\mathcal{G}(t)+(\mathcal{G}'(t))^2+\delta(\mathcal{F}'(t))^2\\
>&4\alpha\mathcal{G}(t)\left(2\mathcal{F}(t)+\rho^{-1}\|u_0\|^2\right)\left(2\cos^2\theta\int_0^t\|u_t(\tau)\|^2\mathrm{d}\tau+\rho\|u_0\|^2\right)\\
&+(\mathcal{G}'(t))^2+\delta(\mathcal{F}'(t))^2\\
=&4\alpha\mathcal{G}(t)\mathcal{K}(t)+(\mathcal{G}'(t))^2+\delta(\mathcal{F}'(t))^2.
\end{aligned} \tag{6.30}
$$

Combining (6.30), (6.26) and (6.29), it follows that

$$
\begin{aligned}
2\mathcal{G}(t)\mathcal{G}''(t)-(1+\alpha)(\mathcal{G}'(t))^2 >&4\alpha\mathcal{G}(t)\mathcal{K}(t)+\delta(\mathcal{F}'(t))^2-\alpha(\mathcal{G}'(t))^2\\
=&4\alpha\mathcal{G}(t)\mathcal{K}(t)-\alpha\left(4\mathcal{G}(t)-\delta\right)(\mathcal{F}'(t))^2+\delta(\mathcal{F}'(t))^2\\
=&4\alpha\mathcal{G}(t)\mathcal{K}(t)-4\alpha\mathcal{G}(t)(\mathcal{F}'(t))^2+\delta(1+\alpha)(\mathcal{F}'(t))^2\\
\geq&4\alpha\mathcal{G}(t)\mathcal{K}(t)-4\alpha\mathcal{G}(t)\mathcal{K}(t)\\
=&0,
\end{aligned}
$$

i.e.,

$$
\mathcal{G}(t)\mathcal{G}''(t)-\frac{1+\alpha}{2}(\mathcal{G}'(t))^2>0,\quad t\in[0,+\infty),
$$

which yields

$$
\left(\mathcal{G}^{-\beta}(t)\right)''=-\frac{\beta}{\mathcal{G}^{\beta+2}(t)}\left(\mathcal{G}''(t)\mathcal{G}(t)-(\beta+1)(\mathcal{G}'(t))^2\right)<0,\ t\in[0,+\infty),
$$

where $\beta=\frac{\alpha-1}{2}>0$. Given that $\mathcal{G}(0)=c>\frac{1}{4}\varepsilon^{-2}\|u_0\|^4>0$ and $\mathcal{G}'(0)=\varepsilon^{-1}\|u_0\|^4>0$, by Lemma 1.12, it follows that

$$
0<T\leq\frac{2\mathcal{G}(0)}{(\alpha-1)\mathcal{G}'(0)} \tag{6.31}
$$

satisfies that $\lim_{t\to T}\mathcal{G}^{-\beta}(t)=0$, i.e., $\lim_{t\to T}\mathcal{G}(t)=+\infty$, which obviously contradicts with $T_{\max}=+\infty$. And, in consideration of the continuity property of both $\mathcal{G}(t)$ and $\mathcal{F}(t)$ in t, one deduces that $\mathcal{F}(t)\to+\infty$ as t approaches T, while the assumption is that $T_{\max}=+\infty$.

Case II. There exists some $\tilde{t}$ such that $E(\tilde{t})<0$.

Because $E(0)>0$, we can suppose from the continuity property of $E(t)$ in t that there exists a first time $t_0>0$ satisfying that $E(u(t_0))=0$ and $E(u(\hat{t}))<0$ for some $\hat{t}>t_0$. Choosing $u(\hat{t})$ as a new initial datum, together with Lemma 4.1, it follows that $u(t)\in\mathcal{N}_-$ for any $t>\hat{t}$. Following the argument in Theorem 4.3, we similarly obtain that the solution for the problem (1.1) undergoes a the finite time blow-up.

Based on the above two cases, one can deduce that the solution u blows up in finite time.

Use of AI tools declaration

The author declares that he has not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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