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Research article

Sign-changing solutions for the Schrödinger-Poisson system with concave-convex nonlinearities in $\mathbb{R}^3$

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Abstract: In this paper, we consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u + \lambda K(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Under the weakly coercive assumption on V and an appropriate condition on K, we investigate the cases when the nonlinearities are of concave-convex type, that is, $1 < q < 2$ and $4 < p < 6$. By constructing a nonempty closed subset of the sign-changing Nehari manifold, we establish the existence of least energy sign-changing solutions provided that $\lambda \in (-\infty, \lambda_*)$, where $\lambda_* > 0$ is a constant.

Keywords: Schrödinger-Poisson system; sign-changing solutions; concave-convex nonlinearities; variational method
Mathematics Subject Classification: 35A15, 35J20, 35J50

1. Introduction

In the past decades, the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.1}$$

has been studied extensively by many authors, where $V : \mathbb{R}^3 \to \mathbb{R}$ and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. This system can be used to describe the interaction of a charged particle with the electrostatic field in quantum mechanics. In this context, the unknown u and ϕ represent the wave functions related to the particle and electric potentials, respectively. Moreover, the local nonlinearity $f(x,u)$ models the interaction among particles. We refer the reader to [6, 20] for more details on its physical background.

It is worth noting that system (1.1) is a nonlocal problem due to the appearance of the term ϕu, where $\phi = \phi_u$ is presented in (1.4) below. This fact states that problem (1.1) is no longer a pointwise identity and brings some essential difficulties. For example, the term $\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x$ in the corresponding energy functional is homogeneous of degree four, then, compared with the local Schrödinger equation, it seems difficult to obtain the boundedness and compactness for any Palais-Smale sequence. In light of the previous observations, the existence of solutions for problem (1.1) have been widely studied and some open problems have been proposed [3, 11, 15, 16, 19, 25, 28, 30, 35].

In what follows, we are particularly interested in the existence of sign-changing solutions (also known as nodal solutions) for problem (1.1). From this perspective, Wang and Zhou [29] were concerned with the existence and energy property of sign-changing solutions for problem (1.1) with $f(x, u) = |u|^{p-2}u$. By introducing appropriate compactness conditions on V, they used methods different from [5] to prove that the so-called sign-changing Nehari manifold is nonempty provided that $4 < p < 6$. Then, combining some analytical techniques and the Brouwer degree theory, the existence of least energy sign-changing solutions was established. After that, the authors in [21] investigated sign-changing solutions of problem (1.1) when $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfied super-cubic and subcritical growth at infinity, superlinear growth at origin, and a well-known Nehari-type monotonicity condition. In particular, they established the energy doubling [31]. Moreover, the authors in [10, 38] obtained the similar existence results if the nonlinearity f satisfied asymptotically cubic and three-linear growth, respectively. On the other hand, when f satisfies three-sublinear growth, the existence and multiplicity of sign-changing solutions can be obtained by invariant sets of descending flow [13, 18]. For more interesting results, such as the Sobolev critical exponent or bounded domains, we refer to [1, 24, 27, 34, 36, 37] and the references therein.

According to the previous statements, we observe that the nonlinearities always satisfy superlinear growth or convexity (i.e. $f(x, u) = |u|^{p-2}u$, $2 < p < 6$) provided that the sign-changing solution of Schrödinger-Poisson systems in the whole space $\mathbb{R}^3$ is considered. Once the nonlinearity is not constrained by the above forms, the methods mentioned previously cannot be directly used. Therefore, in present paper, we focus on a special type of nonlinearities; that is, the concave-convex type, such as $f(x, u) = |u|^{p-2}u + |u|^{q-2}u$ with $4 < p < 6$ and $1 < q < 2$. The concave-convex nonlinearities were introduced in [2], where the authors proved the existence of infinitely many solutions with negative energy for local elliptic problems in bounded domains. After this work, a great attention has been paid to the existence of solutions to elliptic problems with concave-convex nonlinearities. For example, see [7, 8, 17, 32] for local Schrodinger equations, and [9, 14, 22, 23, 26, 33] for Schrodinger-Poisson systems.

Note that only [7, 8, 17, 33] involve the sign-changing solutions. More precisely, Bobkov [7] considered the following Schrödinger equation

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + |u|^{\gamma-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded connected domain with a smooth boundary, $N \geq 1$, $1 < q < 2 < \gamma < 2^*$ and 2^* is the well-known Sobolev critical exponent. They proved the existence of a sign-changing solution on the nonlocal interval $\lambda \in (-\infty, \lambda_0^*)$, where λ_0^* is determined by the variational principle of nonlinear spectral analysis through the fibering method. Moreover, the author in [8] obtained similar existence results and some interesting properties for the nodal solutions of the elliptic equation

$$\begin{cases} -\Delta u = \lambda k(x)|u|^{q-2}u + h(x)|u|^{\gamma-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where $1 < q < 2 < \gamma < 2^*$, $\lambda \in \mathbb{R}$ and the weight functions k, $h \in L^\infty(\Omega)$ satisfy the conditions $\operatorname{ess}\inf_{x\in\Omega} k(x) > 0$ and $\operatorname{ess}\inf_{x\in\Omega} h(x) > 0$. Note that the methods in [7, 8] cannot be applied to the nonlocal elliptic problem (1.1). To this end, based on the setting of bounded domains, Yang and Ou [33] studied the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + \phi u = \lambda |u|^{p-2}u + |u|^{q-2}u & \text{in } \Omega, \\ \quad -\Delta \phi = u^2 & \text{in } \Omega, \\ \quad u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$ and $1 < p < 2$, $4 < q < 6$, λ is a constant. By constrained variational method and quantitative deformation lemma, they obtained that the problem (1.2) has a nodal solution u_λ with positive energy when $\lambda < \lambda^*$, λ^* is a constant. Here, we point out that if the bounded domain is involved, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $1 \le p < 2^*$, which not only avoids the verification of compactness but also ensures the boundedness of the concave term. However, once the whole space is considered, these points cannot be directly determined. Therefore, motivated by the works described above, in this paper we focus on the following Schrödinger-Poisson system in the whole space $\mathbb{R}^3$ with concave-convex nonlinearities

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u + \lambda K(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $1 < q < 2$, $4 < p < 6$, $\lambda > 0$ and V, K satisfy the assumptions:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \ge a > 0$ for each $A > 0$, $meas\{x \in \mathbb{R}^3 : V(x) \le A\} < \infty$, where a is a constant and $meas$ denotes the Lebesgue measure in $\mathbb{R}^3$;

(K) K is positive and $K \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$.

Here the condition (V) is similar to [17]. This condition also ensures the compactness of embedding $H \hookrightarrow L^p(\mathbb{R}^3)$, $2 \le p < 2^*$, where H is the Hilbert space

$$H = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < +\infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \mathrm{d}x \right)^{\frac{1}{2}} \text{ [4, 39, Lemma 3.4]}.$$

Meanwhile, we point out that the authors in [17] considered the local Schrödinger type equation in $\mathbb{R}^N$

$$-\Delta u + V(x)u = \lambda |u|^{q-2}u + \mu u + \nu |u|^{p-2}u,$$

where $1 < q < 2 < p < 2^*$, $N \geq 2$ and λ, μ, ν are parameters. The above equation involves a combination of concave and convex terms. They obtained infinitely many nodal solutions by using the method of invariant sets. However, it seems that this method cannot be applied to problem (1.3). In order to overcome the previous difficulties, we introduce the condition (K), which guarantees a weak continuity result (see Lemma 2.2 below). Moreover, conditions (V) and (K) allow us to construct a suitable nonempty closed subset of sign-changing Nehari manifold similar to [33], and then a least energy sign-changing solution can be obtained.

Before proceeding, we discuss the basic framework for dealing with our problem. The usual norm in the Lebesgue space $L^r(\mathbb{R}^3)$ is denoted by $|u|_r = \left(\int_{\mathbb{R}^3} |u|^r \mathrm{d}x\right)^{\frac{1}{r}}$, $r \in [1, +\infty)$. It is well known that, by the Lax-Milgram theorem, when $u \in H$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u = u^2$, where

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\mathrm{d}y. \tag{1.4}$$

Substituting (1.4) into (1.3), we can rewrite system (1.3) as the following equivalent form

$$-\Delta u + V(x)u + \phi_u u = |u|^{p-2}u + \lambda K(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

Therefore, the energy functional associated with system (1.3) is defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^3} K(x)|u|^q \mathrm{d}x, \ \forall u \in H.$$

The functional $I_\lambda(u)$ is well-defined for every $u \in H$ and belongs to $C^1(H, \mathbb{R})$. Furthermore, for any $v \in H$,

$$\langle I'_\lambda(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + V(x)uv)\mathrm{d}x + \int_{\mathbb{R}^3}\phi_u uv \mathrm{d}x - \int_{\mathbb{R}^3}|u|^{p-2}uv \mathrm{d}x - \lambda\int_{\mathbb{R}^3} K(x)|u|^{q-2}uv \mathrm{d}x.$$

As is well known, the solution of problem (1.5) is the critical point of the functional $I_\lambda(u)$. Moreover, if $u \in H$ is a solution of problem (1.5) and $u^\pm \neq 0$, then u is a sign-changing solution of system (1.3), where

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

Naturally, we introduce the Nehari manifold of I_λ as

$$\mathcal{N}_\lambda = \{u \in H\backslash\{0\} : \langle I'_\lambda(u), u\rangle = 0\},$$

which is related to the behavior of the map $\varphi_u : r \to I_\lambda(ru)$ $(r > 0)$ (see [12] for the introduction of this map). For $u \in H$, we have

$$\varphi_u(r) = \frac{1}{2}r^2\|u\|^2 + \frac{1}{4}r^4\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{1}{p}r^p|u|_p^p - \frac{\lambda}{q}r^q\int_{\mathbb{R}^3} K(x)|u|^q \mathrm{d}x.$$

It is well known that, for any $u \in H\backslash\{0\}$, $\varphi'_u(r) = 0$ if and only if $ru \in \mathcal{N}_\lambda$, which also implies that $\varphi'_u(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. This manifold is always used to find the positive ground state solution. In order to obtain sign-changing solutions of problem (1.3), it is necessary to consider the sign-changing Nehari manifold

$$\mathcal{M}_\lambda = \{u \in H : u^\pm \neq 0, \ \langle I'_\lambda(u), u^\pm\rangle = 0\}.$$

Hoeever, this manifold cannot be directly applied due to appearance of concave term $\lambda k(x)|u|^{q-2}u$. As we will see, inspired by [33], we can construct the set $\mathcal{M}_\lambda^* \subset \mathcal{M}_\lambda$ and prove this set is a nonempty closed set, where $\mathcal{M}_\lambda^*$ is defined by (2.5) below. We then show that the minimization problem $m_\lambda := \inf\limits_{u\in\mathcal{M}_\lambda^*} I_\lambda(u)$ is attained by some $u_\lambda \in \mathcal{M}_\lambda^*$ with positive energy. Finally, the classical deformation lemma [32, Lemma 2.3] states that the u_λ is a weak solution of problem (1.3). Up to now, the main results can be stated as follows.

Theorem 1.1. *Assume that* (V) *and* (K) *hold. Then there exists a constant* $\lambda^* > 0$ (*determined in* (2.13)) *such that for any* $\lambda \in (-\infty, \lambda^*)$*, problem* (1.3) *possesses a least energy sign-changing solution* u_λ *with positive energy.*

Remark 1.2. As mentioned previously, our Theorem 1.1 extends the result of [33] to the whole space $\mathbb{R}^3$. Moreover, for nonlinearities that do not involve concave terms, such as $f(x,u) = |u|^{p-2}u$ $(4 < p < 6)$, with the aid of classic techniques in [29], one can obtain that the sign-changing Nehari manifold is nonempty. However, in our paper, $1 < q < 2$ and $4 < p < 6$ mean that $\lambda K(x)|u|^{q-2}u$ is concave and $|u|^{p-2}u$ is convex, which is different from [29]. At this point, it is difficult to directly prove that the set $\mathcal{M}_\lambda$ is nonempty. To this end, we carefully analyze the behavior of $f_u(s,t) = I_\lambda(su^+ + tu^-)$ and then introduce the set $\mathcal{M}_\lambda^*$. In particular, by determining some important lower bound estimates, we verify that $\mathcal{M}_\lambda^* \neq \emptyset$. On the other hand, we point out that the range of parameter λ can be negative, which is similar to previous results.

The remainder of this paper is organized as follows. In section 2, we present some preliminary lemmas that are crucial for proving our main results. Section 3 is devoted to proving Theorem 1.1.

2. Preliminears

In this section, we present some preliminary lemmas that are crucial for proving our main results. First, we recall some well-known properties of ϕ_u that are a collection of results in [11, 19].

Lemma 2.1. *For any* $u \in H$*, we get*

(i) there exists $C > 0$ *such that* $\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \le C\|u\|^4$*;*
(ii) $\phi_u \ge 0$*, for any* $u \in H$*;*
(iii) $\phi_{tu} = t^2\phi_u$*, for any* $t > 0$ *and* $u \in H$*;*
(iv) if $u_n \rightharpoonup u$ *in* H*, then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$ *and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x.$$

Next, we verify a weak continuity of the concave term. The proof is similar to [32, Lemma 2.13], but we state the proof here for the readers convenience.

Lemma 2.2. *Assume that* $1 < q < 2$ *and* (K) *hold, then the functional*

$$G : H \to \mathbb{R} : \ u \mapsto \int_{\mathbb{R}^3} K(x)|u|^q \mathrm{d}x$$

is weakly continuous.

Proof. Undoubtedly, it is sufficient to prove that if $u_n \rightharpoonup u$ in H, then $\int_{\mathbb{R}^3} K(x)|u_n|^q \mathrm{d}x \to \int_{\mathbb{R}^3} K(x)|u|^q \mathrm{d}x$ as $n \to \infty$. In fact, if $u_n \rightharpoonup u$ in H, going if necessary to a subsequence, we can assume that $u_n \to u$ a.e. on $\mathbb{R}^3$. Since $u_n \rightharpoonup u$ in H, we get that $\{u_n\}$ is bounded in $L^6(\mathbb{R}^3)$ and $\{u_n^q\}$ is bounded in $L^{\frac{6}{q}}(\mathbb{R}^3)$. Therefore, $u_n^q \rightharpoonup u^q$ in $L^{\frac{6}{q}}(\mathbb{R}^3)$. Combining with (K) and the definition of weak convergence, we obtain

$$\int_{\mathbb{R}^3} K(x)|u_n|^q \mathrm{d}x \to \int_{\mathbb{R}^3} K(x)|u|^q \mathrm{d}x \text{ as } n \to \infty.$$

Now, for any $u \in H$ with $u^{\pm} \neq 0$, we introduce the map $f_u : [0,+\infty)\times[0,+\infty) \to \mathbb{R}$ defined by $f_u(s,t) = I_\lambda(su^+ + tu^-)$, i.e.,

$$\begin{aligned} & f_u(s,t) \\ = & \frac{1}{2}s^2\|u^+\|^2 + \frac{1}{4}s^4\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x + \frac{1}{2}s^2t^2\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x - \frac{1}{p}s^p|u^+|_p^p - \frac{\lambda}{q}s^q\int_{\mathbb{R}^3}K(x)|u^+|^q\mathrm{d}x \\ & + \frac{1}{2}t^2\|u^-\|^2 + \frac{1}{4}t^4\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x - \frac{1}{p}t^p|u^-|_p^p - \frac{\lambda}{q}t^q\int_{\mathbb{R}^3}K(x)|u^-|^q\mathrm{d}x. \end{aligned} \tag{2.1}$$

Here we have used the fact that

$$\int_{\mathbb{R}^3}\phi_{u^+}|u^-|^2\mathrm{d}x = \int_{\mathbb{R}^3}\phi_{u^-}|u^+|^2\mathrm{d}x.$$

Moreover, we have that

$$\begin{aligned} \nabla f_u(s,t) &= (\langle I'_\lambda(su^+ + tu^-), u^+\rangle,\ \langle I'_\lambda(su^+ + tu^-), u^-\rangle) \\ &= \left(\frac{1}{s}\langle I'_\lambda(su^+ + tu^-), su^+\rangle,\ \frac{1}{t}\langle I'_\lambda(su^+ + tu^-), tu^-\rangle\right), \end{aligned}$$

which implies that for any $u \in H$ with $u^{\pm} \neq 0$, $su^+ + tu^- \in \mathcal{M}_\lambda$ if and only if the pair (s,t) is a critical point of f_u.

Lemma 2.3. *Assume that $1 < q < 2$, $4 < p < 6$ and the assumption (K) hold, then there exists a constant $\lambda_1 > 0$ such that for any $u \in H$ with $u^{\pm} \neq 0$, there holds that*

(i) if $\lambda \in (0,\lambda_1)$, then for any fixed $t \geq 0$, $f_u(s,t)$ has exactly two critical points, $0 < s_1(t) < s_2(t)$; $s_1(t)$ is the minimum point and $s_2(t)$ is the maximum point; moreover, if $\lambda \leq 0$, then for any fixed $t \geq 0$, $f_u(s,t)$ has exactly one critical point, $s_3(t) > 0$, and it is the maximum point;

(ii) if $\lambda \in (0,\lambda_1)$, then for any fixed $s \geq 0$, $f_u(s,t)$ has exactly two critical points, $0 < t_1(s) < t_2(s)$; $t_1(s)$ is the minimum point and $t_2(s)$ is the maximum point; moreover, if $\lambda \leq 0$, then for any fixed $s \geq 0$, $f_u(s,t)$ has exactly one critical point, $t_3(s) > 0$, and it is the maximum point.

Proof. We define $f_\mu(s,t) : [0,+\infty)\times[0,+\infty) \to \mathbb{R}$ by

$$\begin{aligned} f_\mu(s,t) = & \frac{1}{2}\|su^+ + tu^-\|^2 + \frac{1}{4}\mu\int_{\mathbb{R}^3}\phi_{su^+ + tu^-}(su^+ + tu^-)^2\mathrm{d}x - \frac{1}{p}|su^+ + tu^-|_p^p \\ & - \frac{\lambda}{q}\int_{\mathbb{R}^3}K(x)|su^+ + tu^-|^q\mathrm{d}x, \end{aligned}$$

where μ is a nonnegative parameter.

(i) For any fixed $t \geq 0$, a direct calculation gives

$$\begin{aligned}\frac{\partial f_\mu}{\partial s}(s,t) =& s\|u^+\|^2 + \mu s^3 \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + \mu s t^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x - s^{p-1}|u^+|_p^p \\ &- \lambda s^{q-1} \int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x \\ =& s^{q-1}\Big(s^{2-q}\|u^+\|^2 + \mu s^{4-q} \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + \mu s^{2-q} t^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x \\ &- s^{p-q}|u^+|_p^p - \lambda \int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x\Big).\end{aligned}$$

Then, if $s > 0$, $\frac{\partial f_\mu}{\partial s}(s,t) = 0$ is equivalent to

$$\begin{aligned}\beta_\mu(s) =& s^{2-q}\|u^+\|^2 + \mu s^{4-q} \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + \mu s^{2-q} t^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x - s^{p-q}|u^+|_p^p \\ &- \lambda \int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x = 0.\end{aligned}$$

For $\beta_\mu(s)$, we can obtain that

$$\begin{aligned}\beta'_\mu(s) =& s^{1-q}\Big((2-q)\|u^+\|^2 + \mu(4-q)s^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + \mu(2-q)t^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x \\ &- (p-q)s^{p-2}|u^+|_p^p\Big).\end{aligned}$$

Clearly, for any fixed $t \geq 0$, $1 < q < 2$ and $4 < p < 6$, we can infer that β_μ has exactly one critical point $s_\mu > 0$, where s_μ is related to t for any $\mu \geq 0$. Moreover, β_μ is strictly increasing in $(0, s_\mu)$ and strictly decreasing in $(s_\mu, +\infty)$.

Noting that if $\mu = 0$, we have

$$\beta_0(s) = s^{2-q}\|u^+\|^2 - s^{p-q}|u^+|_p^p - \lambda \int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x$$

and

$$\beta'_0(s) = s^{1-q}((2-q)\|u^+\|^2 - (p-q)s^{p-2}|u^+|_p^p).$$

Hence,

$$s_0 = \left(\frac{(2-q)\|u^+\|^2}{(p-q)|u^+|_p^p}\right)^{\frac{1}{p-2}}$$

and

$$\beta_0(s_0) = \frac{p-2}{p-q}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} \frac{\|u^+\|^{\frac{2(p-q)}{p-2}}}{|u^+|_p^{\frac{p(2-q)}{p-2}}} - \lambda \int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x.$$

Let

$$\alpha_\mu(s) = s^{2-q}\|u^+\|^2 + \mu s^{4-q} \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + \mu s^{2-q} t^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x - s^{p-q}|u^+|_p^p$$

and

$$\lambda_1^+ = \inf_{u\in H, u^\pm\neq 0} \frac{\alpha_0(s_0)}{\int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x},$$

then it follows from Sobolev embedding that

$$\lambda_1^+ \geq \frac{p-2}{p-q}\Big(\frac{2-q}{p-q}\Big)^{\frac{2-q}{p-2}} \frac{1}{S_6^q S_p^{\frac{p(2-q)}{p-2}} |K|_{\frac{6}{6-q}}} > 0. \tag{2.2}$$

Therefore, if $\lambda \in (0, \lambda_1^+)$, we can deduce $\beta_\mu(s_0) > \beta_0(s_0) > 0$ for any $u \in H$ with $u^\pm \neq 0$. For any fixed $t \geq 0$, if $\lambda \in (0, \lambda_1^+)$, there exist unique $s_1(t)$ and $s_2(t)$ with $0 < s_1(t) < s_2(t)$, such that $\beta_\mu(s) = 0$ and $\beta_\mu'(s_1(t)) > 0$, $\beta_\mu'(s_2(t)) < 0$. On the other hand, when $\lambda \leq 0$, there is a unique $s_3(t) > 0$ such that $\beta_\mu(s) = 0$ and $\beta_\mu'(s_3(t)) < 0$.

Finally, considering that

$$\begin{aligned}
\frac{\partial^2 f_\mu}{\partial s^2}(s,t) &= \|u^+\|^2 + 3\mu s^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + \mu t^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x \\
&\quad - (p-1)s^{p-2}|u^+|_p^p - \lambda(q-1)s^{q-2} \int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x \\
&= s^{q-2}\Big(s^{2-q}\|u^+\|^2 + 3\mu s^{4-q} \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + \mu s^{2-q} t^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x \\
&\quad - (p-1)s^{p-q}|u^+|_p^p - \lambda(q-1) \int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x\Big),
\end{aligned}$$

we can find that

$$\frac{\partial^2 f_\mu}{\partial s^2}(s,t) = (q-1)s^{q-2}\beta_\mu(s) + s^{q-1}\beta_\mu'(s).$$

Note that $\frac{\partial f_\mu}{\partial s}(s,t) = 0$ is equivalent to $\beta_\mu(s) = 0$, then we have

$$\frac{\partial^2 f_\mu}{\partial s^2}(s,t) = s^{q-1}\beta_\mu'(s) \ \text{ if } \ \frac{\partial f_\mu}{\partial s}(s,t) = 0.$$

Thus, the facts that $\beta_\mu'(s_1(t)) > 0$, $\beta_\mu'(s_2(t)) < 0$ and $\beta_\mu'(s_3(t)) < 0$ signify that

$$\frac{\partial^2 f_\mu}{\partial s^2}(s_1(t),t) > 0, \ \frac{\partial^2 f_\mu}{\partial s^2}(s_2(t),t) < 0 \text{ and } \frac{\partial^2 f_\mu}{\partial s^2}(s_3(t),t) < 0.$$

In particular, when $\mu = 1$, the results still hold.

(*ii*) For any fixed $s \geq 0$, let

$$\begin{aligned}
\lambda_1^- &= \inf_{u\in H, u^\pm\neq 0} \frac{\alpha_0(s_0)}{\int_{\mathbb{R}^3} K(x)|u^-|^q \mathrm{d}x} \\
&= \inf_{u\in H, u^\pm\neq 0} \frac{p-2}{p-q}\Big(\frac{2-q}{p-q}\Big)^{\frac{2-q}{p-2}} \frac{\|u^-\|^{\frac{2(p-q)}{p-2}}}{|u^-|_p^{\frac{p(2-q)}{p-2}} \int_{\mathbb{R}^3} K(x)|u^-|^q \mathrm{d}x}.
\end{aligned} \tag{2.3}$$

Analogously with the proof (*i*), we can derive the conclusion.

At last, it is easy to see that $\lambda_1^+ = \lambda_1^-$. Indeed, $u^\pm \neq 0$ indicates that $(-u)^\pm \neq 0$, which yields that $\lambda_1^+ = \lambda_1^-$. Let $\lambda_1 = \lambda_1^+ = \lambda_1^-$, from (*i*) and (*ii*), then the proof is completed.

Let

$$\lambda_2 = \inf_{u\in H\backslash\{0\}} \frac{p-2}{p-q}\Big(\frac{2-q}{p-q}\Big)^{\frac{2-q}{p-2}} \frac{\|u\|^{\frac{2(p-q)}{p-2}}}{|u|_p^{\frac{p(2-q)}{p-2}} \int_{\mathbb{R}^3} K(x)|u|^q \mathrm{d}x}.$$

Undoubtedly,

$$\lambda_1 \geq \lambda_2 \geq \frac{p-2}{p-q}\Big(\frac{2-q}{p-q}\Big)^{\frac{2-q}{p-2}} \frac{1}{S_6^q S_p^{\frac{p(2-q)}{p-2}} |K|_{\frac{6}{6-q}}} > 0. \tag{2.4}$$

Here, the notation S_p represents the embedding constant of $H \hookrightarrow L^p(\mathbb{R}^3)$, which has a value depending on $p \in [2, 6]$. According to Lemma 2.3, the following corollary is a direct result.

Corollary 2.4. *Assume that* $1 < q < 2$, $4 < p < 6$, *the assumption* (K) *and* $0 < \lambda < \lambda_2$ *hold, then for any* $u \in H\backslash\{0\}$, $\varphi_u(r)$ *has exactly two critical points,* $0 < r_1(u) < r_2(u)$ *and* $\varphi_u''(r_1(u)) > 0$, $\varphi_u''(r_2(u)) < 0$. *On the other hand, when* $\lambda \leq 0$, $\varphi_u(r)$ *has exactly one critical point,* $r_3(u) > 0$, *and* $\varphi_u''(r_3(u)) < 0$.

Lemma 2.5. *Assume that* $1 < q < 2$, $4 < p < 6$, *the assumption* (K) *and* $\lambda < \lambda_1$ *hold, then for any* $u \in \mathcal{M}_\lambda$, $(\partial^2 f_u/\partial s^2)(1,1) \neq 0$ *and* $(\partial^2 f_u/\partial t^2)(1,1) \neq 0$. *Moreover,* $I_\lambda(u) \to +\infty$ *as* $\|u\| \to +\infty$, *i.e., the functional* I_λ *is coercive and bounded from below on* $\mathcal{N}_\lambda$.

Proof. If $0 < \lambda < \lambda_1$, from Lemma 2.3, it follows that $f_u(s,1)$ has exactly two critical points $s_1(1)$, $s_2(1)$ and $\frac{\partial^2 f_u}{\partial s^2}(s_1(1),1) > 0$, $\frac{\partial^2 f_u}{\partial s^2}(s_2(1),1) < 0$. Since $u \in \mathcal{M}_\lambda$, we have $\frac{\partial f_u}{\partial s}(1,1) = 0$, which means that $s_1(1) = 1$ or $s_2(1) = 1$. Hence, $\frac{\partial^2 f_u}{\partial s^2}(1,1) \neq 0$. Analogously, we can conclude that $\frac{\partial^2 f_u}{\partial t^2}(1,1) \neq 0$.

If $\lambda \leq 0$, it follows from Lemma 2.3 that $f_u(s,1)$ has exactly one critical point $s_3(1)$ and $\frac{\partial^2 f_u}{\partial s^2}(s_3(1),1) < 0$. Combining with $u \in \mathcal{M}_\lambda$, we have $\frac{\partial f_u}{\partial s}(1,1) = 0$, which shows that $s_3(1) = 1$. Therefore, we get $\frac{\partial^2 f_u}{\partial s^2}(1,1) \neq 0$. Similarly, we can deduce that $t_3(1) = 1$, $\frac{\partial^2 f_u}{\partial t^2}(1,t_3(1)) < 0$ and the claim is clearly true.

Note that $u \in \mathcal{M}_\lambda \subset \mathcal{N}_\lambda$, then the Sobolev embedding indicates that

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I_\lambda'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \Big(\frac{1}{4} - \frac{1}{p}\Big)\int_{\mathbb{R}^3} |u|^p \mathrm{d}x + \lambda\Big(\frac{1}{4} - \frac{1}{q}\Big)\int_{\mathbb{R}^3} K(x)|u|^q \mathrm{d}x \\ &\geq \frac{1}{4}\|u\|^2 + \lambda\Big(\frac{1}{4} - \frac{1}{q}\Big)|K|_{\frac{6}{6-q}} |u|_6^q \\ &\geq \frac{1}{4}\|u\|^2 + \lambda_1\Big(\frac{1}{4} - \frac{1}{q}\Big)|K|_{\frac{6}{6-q}} S_6^q \|u\|^q. \end{aligned}$$

Hence, combining $1 < q < 2$ and $4 < p < 6$, we derive $I_\lambda(u) \to +\infty$ as $\|u\| \to +\infty$. That is, the functional $I_\lambda(u)$ is coercive and bounded from below on $\mathcal{N}_\lambda$. The proof is completed.

Similarly, we obtain the following result.

Corollary 2.6. *Assume that* $1 < q < 2$, $4 < p < 6$, *the assumption* (K) *and* $\lambda < \lambda_2$ *hold, then for any* $u \in \mathcal{N}_\lambda$, $\varphi_u''(1) \neq 0$.

In what follows, we construct the following sets

$$\mathcal{M}_\lambda^- = \left\{u \in \mathcal{M}_\lambda : \frac{\partial^2 f_u}{\partial s^2}(1,1) < 0,\ \frac{\partial^2 f_u}{\partial t^2}(1,1) < 0\right\}$$

and

$$\mathcal{M}_\lambda^* = \left\{u \in \mathcal{M}_\lambda : \frac{\partial^2 f_u}{\partial s^2}(1,1) < 0,\ \frac{\partial^2 f_u}{\partial t^2}(1,1) < 0,\ \varphi_u''(1) < 0\right\}. \tag{2.5}$$

Unquestionably, $\mathcal{M}_\lambda^* \subset \mathcal{M}_\lambda^-$. According to the properties of f_u mentioned above, we can verify that the set $\mathcal{M}_\lambda^*$ is nonempty and $\mathcal{M}_\lambda^* = \mathcal{M}_\lambda^-$ (see Lemma 2.9). To this end, we first get the following fact.

Lemma 2.7. *Assume that* $1 < q < 2$, $4 < p < 6$ *and the assumption* (K) *hold, there exists* $\sigma > 0$, *which is independent of u and* λ, *such that*

$$\|u^\pm\| > \sigma > 0$$

for any $u \in \mathcal{M}_\lambda^-$.

Proof. For any $u \in \mathcal{M}_\lambda^-$, from $\frac{\partial^2 f_u}{\partial s^2}(1,1) < 0$, $\frac{\partial^2 f_u}{\partial t^2}(1,1) < 0$ and Sobolev embedding, it follows that

$$\begin{aligned}(2-q)\|u^\pm\|^2 &< (2-q)\|u^\pm\|^2 + (4-q)\int_{\mathbb{R}^3}\phi_{u^\pm}(u^\pm)^2\mathrm{d}x + (2-q)\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x\\ &< (p-q)|u^\pm|_p^p \le (p-q)S_p^p\|u^\pm\|^p,\end{aligned}$$

which implies

$$\|u^\pm\| > \Big(\frac{2-q}{(p-q)S_p^p}\Big)^{\frac{1}{p-2}} := \sigma > 0. \tag{2.6}$$

Hence, the proof is finished.

In order to prove that $\mathcal{M}_\lambda^* \neq \emptyset$, let us define

$$\lambda_3 = \inf_{u\in\mathcal{M}_\lambda^-}\left\{\frac{(p-2)\|u^+\|^2 + (p-4)\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x}{(p-q)\int_{\mathbb{R}^3}K(x)|u^+|^q\mathrm{d}x}, \frac{(p-2)\|u^-\|^2 + (p-4)\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x}{(p-q)\int_{\mathbb{R}^3}K(x)|u^-|^q\mathrm{d}x}\right\}.$$

From Sobolev embedding and Lemma 2.7, it follows that

$$\frac{(p-2)\|u^\pm\|^2 + (p-4)\int_{\mathbb{R}^3}\phi_{u^\pm}(u^\pm)^2\mathrm{d}x}{(p-q)\int_{\mathbb{R}^3}K(x)|u^\pm|^q\mathrm{d}x} \ge \frac{(p-2)\|u^\pm\|^{2-q}}{(p-q)S_6^q|K|_{\frac{6}{6-q}}} > \frac{(p-2)\sigma^{2-q}}{(p-q)S_6^q|K|_{\frac{6}{6-q}}} > 0,$$

where σ is given by (2.6). Therefore, we have

$$\lambda_3 \ge \frac{(p-2)\sigma^{2-q}}{(p-q)S_6^q|K|_{\frac{6}{6-q}}} > 0. \tag{2.7}$$

Furthermore, we compute that

$$\begin{aligned}\frac{\partial^2 f_u}{\partial s^2}(s,t) =& \|u^+\|^2 + 3s^2\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x + t^2\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x\\ &- (p-1)s^{p-2}|u^+|_p^p - \lambda(q-1)s^{q-2}\int_{\mathbb{R}^3}K(x)|u^+|^q\mathrm{d}x,\end{aligned}$$

$$\frac{\partial^2 f_u}{\partial t^2}(s,t) = \|u^-\|^2 + 3t^2\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x + s^2\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x$$
$$-(p-1)t^{p-2}|u^-|_p^p - \lambda(q-1)t^{q-2}\int_{\mathbb{R}^3}K(x)|u^-|^q\mathrm{d}x$$

and

$$\frac{\partial^2 f_u}{\partial s\partial t}(s,t) = 2st\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x, \quad \frac{\partial^2 f_u}{\partial t\partial s}(s,t) = 2st\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x.$$

For any $u \in \mathcal{M}_\lambda$, we obtain

$$\begin{aligned}
&\frac{\partial^2 f_u}{\partial s^2}(1,1)\\
&= \|u^+\|^2 + 3\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x + \int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x - (p-1)|u^+|_p^p - \lambda(q-1)\int_{\mathbb{R}^3}K(x)|u^+|^q\mathrm{d}x\\
&= (2-q)\|u^+\|^2 + (4-q)\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x + (2-q)\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x - (p-q)|u^+|_p^p\\
&= (2-p)\|u^+\|^2 + (4-p)\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x + (2-p)\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x\\
&\quad - \lambda(q-p)\int_{\mathbb{R}^3}K(x)|u^+|^q\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2 f_u}{\partial t^2}(1,1)\\
&= \|u^-\|^2 + 3\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x + \int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x - (p-1)|u^-|_p^p\\
&\quad - \lambda(q-1)\int_{\mathbb{R}^3}K(x)|u^-|^q\mathrm{d}x\\
&= (2-q)\|u^-\|^2 + (4-q)\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x + (2-q)\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x - (p-q)|u^-|_p^p\\
&= (2-p)\|u^-\|^2 + (4-p)\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x + (2-p)\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x\\
&\quad - \lambda(q-p)\int_{\mathbb{R}^3}K(x)|u^-|^q\mathrm{d}x.
\end{aligned}$$

Lemma 2.8. *Assume that $1 < q < 2$, $4 < p < 6$, the assumption (K) and $\lambda < \lambda_1$ hold, then for any $u \in H$ with $u^\pm \neq 0$, there exists a unique pair $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^-$. Moreover, if $\lambda < \lambda_3$, then $I_\lambda(s_u u^+ + t_u u^-) = \max_{s,t>0} I_\lambda(su^+ + tu^-)$.*

Proof. First, we only prove the case of $0 < \lambda < \lambda_1$ since the proof of $\lambda \leq 0$ is very similar. Let $u \in H$ with $u^\pm \neq 0$, from the proof of Lemma 2.3, then we have that $\frac{\partial f_u}{\partial s}(s,t)$ satisfies the conditions:

(*i*) $\frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$ for all $s \geq 0$;
(*ii*) $\frac{\partial f_u}{\partial t}(s,t)$ is continuous and has continuous partial derivatives in $[0,+\infty)\times[0,+\infty)$;

(*iii*) $\frac{\partial^2 f_u}{\partial t^2}(s, t_2(s)) < 0$ for all $s \geq 0$.

Hence, we can obtain that if $0 < \lambda < \lambda_1$, $\frac{\partial f_u}{\partial t}(s,t) = 0$ determines an implicit function $t_2(s)$ with continuous derivative on $[0, +\infty)$ by using the implicit function theorem. Analogously, if $0 < \lambda < \lambda_1$, $\frac{\partial f_u}{\partial s}(s,t) = 0$ determines an implicit function $s_2(t)$ with continuous derivative on $[0, +\infty)$.

On the other hand, for every $s \geq 0$, from $\frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$ and $\frac{\partial f_u}{\partial t}(s,t) \leq 0$ for sufficiently large $t > 0$, we can show that

$$t_2(s) < s \text{ for large enough } s. \tag{2.8}$$

Otherwise, if $t_2(s) \geq s$, where s is large enough, it follows from the definition of $\frac{\partial f_u}{\partial t}(s,t)$ that $\frac{\partial f_u}{\partial t}(s, t_2(s)) < 0$, which contradicts $\frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$. Similarly, we get

$$s_2(t) < t \text{ for sufficiently large } t. \tag{2.9}$$

Therefore, by (2.8), (2.9), $t_2(0) > 0, s_2(0) > 0$, the continuity of $t_2(s)$ and $s_2(t)$, we conclude that the curves of $t_2(s)$ and $s_2(t)$ must intersect at some point $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$. That is, $\frac{\partial f_u}{\partial t}(s_u, t_u) = \frac{\partial f_u}{\partial s}(s_u, t_u) = 0$. Additionally, noting that

$$t_2'(s) = -\left(\frac{\partial^2 f_u}{\partial t \partial s} \Big/ \frac{\partial^2 f_u}{\partial t^2}\right)(s, t_2(s)) > 0$$

for any $s > 0$, we obtain that the function $t_2(s)$ is strictly increasing in $(0, +\infty)$. Similarly, the function $s_2(t)$ is strictly increasing in $(0, +\infty)$. Consequently, there is a unique pair $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\frac{\partial f_u}{\partial s}(s_u, t_u) = \frac{\partial f_u}{\partial t}(s_u, t_u) = 0$$

and

$$\frac{\partial^2 f_u}{\partial s^2}(s_u, t_u) < 0, \ \frac{\partial^2 f_u}{\partial t^2}(s_u, t_u) < 0;$$

that is, $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^-$.

Next, we prove that (s_u, t_u) is the unique maximum point of $f_u(s,t)$ on $[0, +\infty) \times [0, +\infty)$. In fact, if $u \in \mathcal{M}_\lambda^-$, we only show that $(s_u, t_u) = (1, 1)$ is the pair of numbers such that $I_\lambda(s_u u^+ + t_u u^-) = \max\limits_{s,t>0} I_\lambda(su^+ + tu^-)$. Define

$$H(u) = \left(\frac{\partial^2 f_u}{\partial s^2}\frac{\partial^2 f_u}{\partial t^2} - \frac{\partial^2 f_u}{\partial t \partial s}\frac{\partial^2 f_u}{\partial s \partial t}\right)\Big|_{(1,1)}.$$

If we verify that $H(u) > 0$, then $(1, 1)$ is a local maximum point of $f_u(s,t)$. Combining uniqueness of (s_u, t_u), we have $(1, 1)$ as a global maximum point of $f_u(s,t)$. Let $u \in \mathcal{M}_\lambda^-$, then

$$\begin{aligned}
&H(u)\\
&= \left(\frac{\partial^2 f_u}{\partial s^2}\frac{\partial^2 f_u}{\partial t^2} - \frac{\partial^2 f_u}{\partial t \partial s}\frac{\partial^2 f_u}{\partial s \partial t}\right)\Big|_{(1,1)}\\
&= \Big((2-p)\|u^+\|^2 + (4-p)\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + (2-p)\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x - \lambda(q-p)\int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x\Big)\\
&\times\Big((2-p)\|u^-\|^2 + (4-p)\int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2 \mathrm{d}x + (2-p)\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x - \lambda(q-p)\int_{\mathbb{R}^3} K(x)|u^-|^q \mathrm{d}x\Big)\\
&- 4\left(\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x\right)^2.
\end{aligned}$$

From $\frac{\partial^2 f_u}{\partial s^2}(1,1) < 0$, $\frac{\partial^2 f_u}{\partial t^2}(1,1) < 0$, if $\lambda < \lambda_3$, we derive

$$\begin{aligned}
&-\frac{\partial^2 f_u}{\partial s^2}(1,1) - 2\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x \\
&= (p-2)\|u^+\|^2 + (p-4)\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x + (p-4)\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x \\
&\quad - \lambda(p-q)\int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x \\
&> (p-2)\|u^+\|^2 + (p-4)\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 \mathrm{d}x - \lambda_3(p-q)\int_{\mathbb{R}^3} K(x)|u^+|^q \mathrm{d}x \\
&> 0
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{\partial^2 f_u}{\partial t^2}(1,1) - 2\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x \\
&> (p-2)\|u^-\|^2 + (p-4)\int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2 \mathrm{d}x - \lambda_3(p-q)\int_{\mathbb{R}^3} K(x)|u^-|^q \mathrm{d}x \\
&> 0,
\end{aligned}$$

which show that $H(u) > 0$.

If $u \notin \mathcal{M}_\lambda^-$, then there exists a unique pair (s'_u, t'_u) of positive numbers such that $s'_u u^+ + t'_u u^- \in \mathcal{M}_\lambda^-$. Let $v = s'_u u^+ + t'_u u^-$, i.e., $v \in \mathcal{M}_\lambda^-$. Repeat the above steps and we will get $H(v) > 0$. Hence, the proof is completed.

Lemma 2.9. *If* $1 < q < 2$, $4 < p < 6$, *the assumption* (K) *and* $\lambda < \min\{\lambda_1, \lambda_2, \lambda_3\}$ *hold, then* $\mathcal{M}_\lambda^* \neq \emptyset$. *Moreover, we get* $\mathcal{M}_\lambda^* = \mathcal{M}_\lambda^-$.

Proof. By the definitions of $\mathcal{M}_\lambda^-$ and $\mathcal{M}_\lambda^*$, $\mathcal{M}_\lambda^* \subset \mathcal{M}_\lambda^-$ is obvious. Hence, we only need to prove that if $\lambda < \min\{\lambda_1, \lambda_2, \lambda_3\}$, then $\mathcal{M}_\lambda^- \subset \mathcal{M}_\lambda^*$. That is, for any $u \in \mathcal{M}_\lambda^-$, φ_u reaches its maximum at point $r = 1$. It follows from $\lambda < \lambda_1$ and Lemma 2.8 that $\mathcal{M}_\lambda^- \neq \emptyset$, and from Lemma 2.8, for any $u \in \mathcal{M}_\lambda^-$, we obtain $H(u) > 0$ when $\lambda < \lambda_3$. Combining $f_u(r, r) = \varphi_u(r)$, it follows that $r = 1$ is a maximum of φ_u. Therefore, $\mathcal{M}_\lambda^- \subset \mathcal{M}_\lambda^*$. This completes the proof of Lemma 2.9.

Corollary 2.10. *If* $1 < q < 2$, $4 < p < 6$, *the assumption* (K) *and* $\lambda < \min\{\lambda_1, \lambda_2, \lambda_3\}$ *hold, for* $u \in H$ *and* $u^\pm \neq 0$, *then there exists a unique pair* $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ *such that* $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda^*$ *and* $I_\lambda(s_u u^+ + t_u u^-) = \max_{s,t>0} I_\lambda(su^+ + tu^-)$.

Lemma 2.11. *If* $1 < q < 2$, $4 < p < 6$, *the assumption* (K) *and* $\lambda < \lambda_4$ *hold, for all* $u \in H\backslash\{0\}$, *then there exists* $r_u > 0$ *such that* $\varphi_u(r_u) > 0$, *where* $\lambda_4 > 0$.

Proof. Fixed $u \in H\backslash\{0\}$, let

$$E_u(r) = \frac{1}{2}r^2\|u\|^2 - \frac{1}{p}r^p|u|_p^p$$

for any $r \geq 0$, then we have

$$\begin{aligned}
\varphi_u(r) &= \frac{1}{2}r^2\|u\|^2 + \frac{1}{4}r^4\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{1}{p}r^p|u|_p^p - \frac{\lambda}{q}r^q\int_{\mathbb{R}^3}K(x)|u|^q\mathrm{d}x \\
&\geq \frac{1}{2}r^2\|u\|^2 - \frac{1}{p}r^p|u|_p^p - \frac{\lambda}{q}r^q\int_{\mathbb{R}^3}K(x)|u|^q\mathrm{d}x \\
&= E_u(r) - \frac{\lambda}{q}r^q\int_{\mathbb{R}^3}K(x)|u|^q\mathrm{d}x.
\end{aligned} \tag{2.10}$$

Considering $E_u(r)$, we obtain that there is a unique $r_1(u) = \left(\frac{\|u\|^2}{|u|_p^p}\right)^{\frac{1}{p-2}} > 0$ such that $E_u(r)$ achieves its maximum at $r_1(u)$ and the maximum value is $E_u(r_1(u)) = \frac{p-2}{2p}\left(\frac{\|u\|}{|u|_p}\right)^{\frac{2p}{p-2}}$. Moreover, from Sobolev embedding and (2.10), it is clear to calculate that

$$\begin{aligned}
\varphi_u(r_1(u)) &\geq E_u(r_1(u)) - \frac{\lambda}{q}(r_1(u))^q\int_{\mathbb{R}^3}K(x)|u|^q\mathrm{d}x \\
&\geq E_u(r_1(u)) - \frac{\lambda}{q}(r_1(u))^q|K|_{\frac{6}{6-q}}S_6^q\|u\|^q \\
&= E_u(r_1(u)) - \frac{\lambda}{q}S_6^q|K|_{\frac{6}{6-q}}\Big(\frac{2p}{p-2}\Big)^{\frac{q}{2}}(E_u(r_1(u)))^{\frac{q}{2}} \\
&= (E_u(r_1(u)))^{\frac{q}{2}}\Big((E_u(r_1(u)))^{\frac{2-q}{2}} - \frac{\lambda}{q}S_6^q|K|_{\frac{6}{6-q}}\Big(\frac{2p}{p-2}\Big)^{\frac{q}{2}}\Big).
\end{aligned} \tag{2.11}$$

Consequently, by taking

$$\begin{aligned}
\lambda_4 &= \frac{(p-2)q}{2p|K|_{\frac{6}{6-q}}S_6^q}\inf_{u\in H\backslash\{0\}}\left(\frac{\|u\|}{|u|_p}\right)^{\frac{p(2-q)}{p-2}} \\
&\geq \frac{(p-2)q}{2p|K|_{\frac{6}{6-q}}S_6^q S_p^{\frac{p(2-q)}{p-2}}} > 0,
\end{aligned} \tag{2.12}$$

we conclude that if $\lambda < \lambda_4$, it holds

$$\begin{aligned}
\frac{\lambda}{q}S_6^q|K|_{\frac{6}{6-q}}\left(\frac{2p}{p-2}\right)^{\frac{q}{2}} &< \frac{\lambda_4}{q}S_6^q|K|_{\frac{6}{6-q}}\left(\frac{2p}{p-2}\right)^{\frac{q}{2}} \\
&\leq \frac{1}{q}S_6^q|K|_{\frac{6}{6-q}}\left(\frac{2p}{p-2}\right)^{\frac{q}{2}}\frac{(p-2)q}{2pS_6^q|K|_{\frac{6}{6-q}}}\left(\frac{\|u\|}{|u|_p}\right)^{\frac{p(2-q)}{p-2}} \\
&= (E_u(r_1(u)))^{\frac{2-q}{2}}
\end{aligned}$$

for any $u \in H\backslash\{0\}$. This together with (2.11) yields that $\varphi_u(r_1(u)) > 0$ for any $\lambda < \lambda_4$.

Let

$$\lambda^* = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \tag{2.13}$$

then it follows from (2.2), (2.4), (2.7) and (2.12) that $\lambda^* > 0$. Now, we consider the properties of the set $\mathcal{M}_\lambda^*$.

Lemma 2.12. *If* $1 < q < 2$, $4 < p < 6$, $\lambda < \lambda^*$ *and the assumptions* (V) *and* (K) *hold, then* $\mathcal{M}_\lambda^*$ *is a closed set.*

Proof. Letting $\{u_n\} \subset \mathcal{M}_\lambda^*$ satisfy $u_n \to u_0$ as $n \to \infty$ in H, we now prove that $u_0 \in \mathcal{M}_\lambda^*$. From $\{u_n\} \subset \mathcal{M}_\lambda^*$, we obtain

$$\langle I_\lambda'(u_0), u_0^\pm \rangle = \lim_{n\to\infty} \langle I_\lambda'(u_n), u_n^\pm \rangle = 0, \tag{2.14}$$

$$\frac{\partial^2 f_{u_0}}{\partial s^2}(1,1) = \lim_{n\to\infty} \frac{\partial^2 f_{u_n}}{\partial s^2}(1,1) \le 0, \tag{2.15}$$

$$\frac{\partial^2 f_{u_0}}{\partial t^2}(1,1) = \lim_{n\to\infty} \frac{\partial^2 f_{u_n}}{\partial t^2}(1,1) \le 0, \tag{2.16}$$

$$\varphi_{u_0}''(1) = \lim_{n\to\infty} \varphi_{u_n}''(1) \le 0. \tag{2.17}$$

From Lemma 2.7, it follows that $\|u_n^\pm\| > \sigma > 0$ for any $u_n \in \mathcal{M}_\lambda^-$ and hence $\|u_0^\pm\| = \lim_{n\to\infty} \|u_n^\pm\| > \sigma > 0$, which indicates $u_0^\pm \neq 0$. Using this and (2.14), we obtain $u_0 \in \mathcal{M}_\lambda$ and $r = 1$ is a critical point of φ_{u_0}. Consequently, by (2.15)-(2.17), Lemma 2.5 and Corollary 2.6, we derive that

$$\frac{\partial^2 f_{u_0}}{\partial s^2}(1,1) < 0,\ \frac{\partial^2 f_{u_0}}{\partial t^2}(1,1) < 0,\ \varphi_{u_0}''(1) < 0.$$

Hence, $u_0 \in \mathcal{M}_\lambda^*$ and $\mathcal{M}_\lambda^*$ is a closed set.

Lemma 2.13. *If* $1 < q < 2$, $4 < p < 6$, $\lambda < \lambda^*$ *and assumptions* (V) *and* (K) *hold, then the infimum* $m_\lambda := \inf_{u\in\mathcal{M}_\lambda^*} I_\lambda(u)$ *can be achieved by some* $u_\lambda \in \mathcal{M}_\lambda^*$ *and* $m_\lambda > 0$.

Proof. According to Lemma 2.5, $m_\lambda > -\infty$ when $\lambda < \lambda^*$. Let $\{u_n\} \subset \mathcal{M}_\lambda^*$ be a minimizing sequence for the functional I_λ, namely $I_\lambda(u_n) \to m_\lambda$ as $n \to \infty$. Since the functional I_λ is coercive on $\mathcal{M}_\lambda^*$, then $\{u_n\}$ is bounded in H. Going if necessary to a subsequence, we may assume that

$$u_n \rightharpoonup u_\lambda \text{ in } H,\ u_n \to u_\lambda \text{ in } L^p(\mathbb{R}^3).$$

Now, we first claim that $u_\lambda^\pm \neq 0$. In fact, from Lemma 2.2, Lemma 2.7 and the convergence of $\{u_n\}$ in $L^p(\mathbb{R}^3)$, for any $\lambda < \lambda^*$, we conclude that

$$\begin{aligned}
|u_\lambda^\pm|_p^p + \lambda \int_{\mathbb{R}^3} K(x)|u_\lambda^\pm|^q \mathrm{d}x &= \lim_{n\to\infty} \left(|u_n^\pm|_p^p + \lambda \int_{\mathbb{R}^3} K(x)|u_n^\pm|^q \mathrm{d}x\right) \\
&= \lim_{n\to\infty} \left(\|u_n^\pm\|^2 + \int_{\mathbb{R}^3} \phi_{u_n^\pm}(u_n^\pm)^2 \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u_n^+}(u_n^-)^2 \mathrm{d}x\right) \\
&\ge \lim_{n\to\infty} \|u_n^\pm\|^2 > \sigma^2 > 0.
\end{aligned}$$

This means that $u_\lambda^\pm \neq 0$ for any $\lambda < \lambda^*$.

Next, we proof that $u_n \to u_\lambda$ in H. Arguing by contradiction, suppose that

$$\|u_\lambda^+\| < \liminf_{n\to\infty} \|u_n^+\| \text{ or } \|u_\lambda^-\| < \liminf_{n\to\infty} \|u_n^-\|.$$

From Corollary 2.10, there exists a unique pair $(s_{u_\lambda}, t_{u_\lambda})$ such that $\tilde{u}_\lambda = s_{u_\lambda}u_\lambda^+ + t_{u_\lambda}u_\lambda^- \in \mathcal{M}_\lambda^*$ and $I_\lambda(u_n^+ + u_n^-) = \max_{s,t>0} I_\lambda(s_{u_\lambda}u_n^+ + t_{u_\lambda}u_n^-)$. Consequently,

$$m_\lambda \le I_\lambda(\tilde{u}_\lambda) < \liminf_{n\to\infty} I_\lambda(s_{u_\lambda}u_n^+ + t_{u_\lambda}u_n^-) \le \liminf_{n\to\infty} I_\lambda(u_n^+ + u_n^-) = m_\lambda.$$

That is, we get a contradiction. Therefore, $u_n \to u_\lambda$ in H and m_λ is achieved by u_λ. Combining the fact that $\mathcal{M}_\lambda^*$ is closed, so $u_\lambda \in \mathcal{M}_\lambda^*$.

Finally, it follows from $u_\lambda \in \mathcal{M}_\lambda^*$ that $\varphi_{u_\lambda}(r)$ reached its global maximum at $r = 1$. By this and Lemma 2.11, we can deduce that $\varphi_{u_\lambda}(1) > 0$, i.e., $m_\lambda > 0$. This finishes the proof of Lemma 2.13.

3. Existence of sign-changing solutions

The main aim of this section is to prove our results. Thanks to Lemma 2.13, it suffices to check that the minimizer u_λ for m_λ is a sign-changing of problem (1.3).

Proof of Theorem 1.1. Since $u_\lambda \in \mathcal{M}_\lambda^*$, according to Corollary 2.10, we obtain that

$$I_\lambda(su_\lambda^+ + tu_\lambda^-) < I_\lambda(u_\lambda^+ + u_\lambda^-) = m_\lambda, \text{ for } (s,t) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash(1,1).$$

Moreover, we get $I_\lambda(u_\lambda) > 0$, $\varphi''_{u_\lambda}(1,1) < 0$, $(\partial^2 f_{u_\lambda}/\partial s^2)(1,1) < 0$ and $(\partial^2 f_{u_\lambda}/\partial t^2)(1,1) < 0$.

Let $D = (1-\delta, 1+\delta) \times (1-\delta, 1+\delta)$ and $h : D \to H$ by $h(s,t) = su_\lambda^+ + tu_\lambda^-$ for any $(s,t) \in D$. Then there is a constant $0 < \delta < 1$ such that

$$0 < m := \max_{\partial D} I_\lambda(h(s,t)) < m_\lambda, \quad \max_{(s,t)\in D} \frac{\partial^2 f_{h(s,t)}}{\partial s^2}(1,1) < 0, \tag{3.1}$$

$$\max_{(s,t)\in D} \frac{\partial^2 f_{h(s,t)}}{\partial t^2}(1,1) < 0, \quad \max_{(s,t)\in D} \varphi''_{h(s,t)}(1) < 0. \tag{3.2}$$

By the quantitative deformation lemma, we prove that $I'_\lambda(u_\lambda) = 0$. Suppose by contradiction that $I'_\lambda(u_\lambda) \neq 0$, then there exist $\lambda_1 > 0$ and $\xi > 0$ such that

$$\|I'_\lambda(v)\| \ge \lambda_1 \text{ for all } v \in H, \ \|v - u_\lambda\| \le 3\xi.$$

Let $\varepsilon = \min\{\frac{m_\lambda - m}{3}, \frac{\lambda_1 \xi}{8}\}$ and $s_\xi = \{u \in H : \ \|u - u_\lambda\| \le \xi\}$, then the deformation lemma (see [32], Lemma 2.3) shows that there is a deformation $\eta \in C([0,1] \times H, H)$ such that

(*i*) $\eta(d,u) = u$ if $u \notin I_\lambda^{-}([m_\lambda - 2\varepsilon, m_\lambda + 2\varepsilon]) \cap s_{2\xi}$, $d \in [0,1]$;
(*ii*) $I_\lambda(\eta(d,u)) \le I_\lambda(u)$ for all $u \in H$, $d \in [0,1]$;
(*iii*) $I_\lambda(\eta(d,u)) < m_\lambda$, $\forall u \in I_\lambda^{m_\lambda} \cap s_\xi$, $\forall d \in (0,1]$.

First, we need to prove that

$$\max_{(s,t)\in D} I_\lambda(\eta(d, h(s,t))) < m_\lambda \text{ for all } d \in (0,1]. \tag{3.3}$$

In fact, for any $d \in (0,1]$, it follows from Corollary 2.10 and (*ii*) that

$$\max_{\{(s,t)\in D:\ h(s,t)\notin s_\xi\}} I_\lambda(\eta(d, h(s,t))) \le \max_{\{(s,t)\in D:\ h(s,t)\notin s_\xi\}} I_\lambda(h(s,t)) < m_\lambda.$$

Moreover, Corollary 2.10 and (*iii*) imply that

$$\max_{\{(s,t)\in D:\ h(s,t)\in s_{\xi}\}} I_\lambda(\eta(d,h(s,t))) < m_\lambda \text{ for all } d\in(0,1].$$

Hence, (3.3) holds. From the continuity of η and (3.1)-(3.2), there exists a constant $d_0\in(0,1]$ such that

$$\begin{aligned}
&\max_{(s,t)\in D}\frac{\partial^2 f_{\eta(d_0,h(s,t))}}{\partial s^2}(1,1)<0,\\
&\max_{(s,t)\in D}\frac{\partial^2 f_{\eta(d_0,h(s,t))}}{\partial t^2}(1,1)<0,\\
&\max_{(s,t)\in D}\varphi''_{\eta(d_0,h(s,t))}(1)<0.
\end{aligned}\tag{3.4}$$

In the following, we prove that $\eta(d_0,h(D))\cap\mathcal{M}_\lambda^*\neq\emptyset$, which contradicts the definition of m_λ. In fact, let $g(s,t)=\eta(d_0,h(s,t))$ and

$$\psi_1(s,t)=\left(\langle I'_\lambda(h(s,t)),u_\lambda^+\rangle,\langle I'_\lambda(h(s,t)),u_\lambda^-\rangle\right),$$

$$\psi_2(s,t)=\left(\frac{1}{s}\langle I'_\lambda(g(s,t)),g^+(s,t)\rangle,\frac{1}{t}\langle I'_\lambda(g(s,t)),g^-(s,t)\rangle\right).$$

Then Corollary 2.10 and the degree theory yield $\deg(\psi_1,D,0)=1$. On the other hand, we know $\varepsilon<\frac{m_\lambda-m}{3}$, $m<m_\lambda-2\varepsilon$. Hence, from (*i*) we have $\eta(d,h(s,t))=h(s,t)$ for $d\in(0,1]$, $(s,t)\in\partial D$, and it follows that

$$\psi_1(s,t)=\psi_2(s,t) \text{ for any } (s,t)\in\partial D.$$

Combining the homotopy invariance property of the degree, we get $\deg(\psi_2,D,0)=\deg(\psi_1,D,0)=1$. That is, there exists $(s_0,t_0)\in D$ such that $\psi_2(s_0,t_0)=0$. Therefore, using (3.4) and $\psi_2(s_0,t_0)=0$, we have $\eta(d_0,h(s_0,t_0))\in\mathcal{M}_\lambda^*$, i.e., $\eta(d_0,h(D))\cap\mathcal{M}_\lambda^*\neq\emptyset$. From this, u_λ is a critical point of I_λ, i.e., $I'_\lambda(u_\lambda)=0$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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