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Research article

Non-trivial solutions for a partial discrete Dirichlet nonlinear problem with p-Laplacian

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Abstract: We investigate the non-trivial solutions for a partial discrete Dirichlet nonlinear problem with p-Laplacian by applying Ricceri's variational principle and a two non-zero critical points theorem. In addition, we identify open intervals of the parameter λ under appropriate constraints imposed on the nonlinear term. This allows us to ensure that the nonlinear problem has at least one or two non-trivial solutions.

Keywords: partial discrete nonlinear problem; critical point theory; p-Laplacian; non-trivial solutions
Mathematics Subject Classification: 39A10, 34B15, 35B38

1. Introduction

Let $d, h \in \mathbb{N}^*$ and $\mathbb{N}[1, d] = \{1, 2, 3, ..., d\}$. We study the following partial discrete Dirichlet nonlinear problem:

$$\begin{cases} -\Delta_1 \left(\phi_p \left(\Delta_1 w(l-1, q)\right)\right) - \Delta_2 \left(\phi_p \left(\Delta_2 w(l, q-1)\right)\right) = \lambda g(l, q) f\left(w(l, q)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\,(l, q) \in \mathbb{N}[1, d] \times \mathbb{N}[1, h], \\ w(l, 0) = 0 = w(l, h+1), \quad \forall l \in \mathbb{N}[0, d+1], \\ w(0, q) = 0 = w(d+1, q), \quad \forall q \in \mathbb{N}[0, h+1], \end{cases} \tag{1.1}$$

where $\Delta_1 w(l-1, q) = w(l, q) - w(l-1, q)$ and $\Delta_2 w(l, q-1) = w(l, q) - w(l, q-1)$, the p-Laplacian ϕ_p is defined by $\phi_p(t) = |t|^{p-2}t$, $1 < p < +\infty$, $\lambda > 0$ is a real parameter, $g : \mathbb{N}[1, d] \times \mathbb{N}[1, h] \to]0, +\infty[$ is a positive valued function and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Nonlinear difference equations appear as numerical solutions and as discrete analogues of differential equations which model diverse phenomena in many fields. In the past few years, the study of difference equations has attracted much interest and appeared in a large variety of applications [1]. Guo and Yu [2] were the first to apply the variational method to the difference equations in 2003. Since then, many results on difference equations have been obtained by utilizing variational method and critical point theory [3–20].

The study of partial discrete nonlinear problems with two or more discrete variables is greatly used as mathematical models in a variety of disciplines. Applying critical point theory, some authors have showed the existence findings of non-trivial solutions for the classical partial discrete nonlinear problem with $p = 2$.

In 2010, Galewski and Orpel [5] established at least one non-trivial solution by employing the Mountain Pass Lemma. Molica Bisci and Imbesi [9] in 2014 determined an unbounded sequence of solutions. Heidarkhani and Imbesi [6] in 2015 investigated the existence of three solutions. The same authors in 2019 discussed the existence of non-trivial solutions [7] for a class of partial discrete nonlinear problems with the help of a local minimum theorem. In 2021 [14], we generalized the case of $p = 2$ by adding a weight as follows

$$\begin{cases} -\Delta_1(p(l-1,q)\Delta_1 w(l-1,q)) - \Delta_2(p(l,q-1)\Delta_2 w(l,q-1)) = \lambda f((l,q), w(l,q)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\,(l,q) \in \mathbb{N}\,[1,d] \times \mathbb{N}\,[1,h]\,, \\ w(l,0) = w(l,h+1) = 0, \quad \forall l \in \mathbb{N}[1,d], \\ w(0,q) = w(d+1,q) = 0, \quad \forall q \in \mathbb{N}[1,h], \end{cases}$$

where $p : \mathbb{N}[0,d] \times \mathbb{N}[0,h] \to]0,+\infty[$ fulfills

$$p(0,q) = 0, \quad \forall q \in \mathbb{N}[1,h], \text{ and } \quad p(l,0) = 0, \quad \forall l \in \mathbb{N}[1,d].$$

Moreover, by combining variational methods with the Morse theory, Long [18] investigated a Kirchhoff-type Dirichlet boundary value problem and provided some results on the existence of non-trivial solutions. Specifically, Josef Diblik [19] explored the existence of bounded solutions to discrete equations of fractional order. In addition, Abdelrachid El Amrouss and Omar Hammouti [20] discussed the existence of solutions to a discrete 2n-th order nonlinear problems. As for the case of p-Laplacian, recently in [3, 17], the authors showed the existence of multiple positive solutions by utilizing variational methods.

However, the existence of multiple non-trivial solutions to problem (1.1) have rarely been discussed in terms of the variational principe of Ricceri and a two non-zero critical points theorem. Motivated by the studies in the references above, it is our first attempt to investigate the non-trivial solutions to problem (1.1), subject to certain criteria imposed on the nonlinear term f which is supposed to be sign-changing.

The following is the structure of this paper. We give some basic preliminaries and an illustration of the framework associated to problem (1.1) in Section 2. We give our primary findings and their proofs in Section 3. We provide a few examples to demonstrate our key findings in Section 4. We reach a conclusion in the final segment.

2. Preliminaries

We introduce the corresponding variational framework. For this, we consider the following dh-dimensional Banach space

$$H = \{w : \mathbb{N}[0, d+1] \times \mathbb{N}[0, h+1] \to \mathbb{R} \text{ such that } w(l, 0) = w(l, h+1) = w(0, q) = w(d+1, q) = 0,\ \forall l \in \mathbb{N}[0, d+1],\ \forall q \in \mathbb{N}[0, h+1]\},$$

which is endowed by the norm

$$\|w\| = \left(\sum_{l=1}^{d} \sum_{q=1}^{h} |w(l, q)|^p \right)^{\frac{1}{p}}, \quad w \in H. \tag{2.1}$$

For $w \in H$, we put

$$\|w\|_\infty = \max \{|w(l, q)|,\ (l, q) \in \mathbb{N}[1, d] \times \mathbb{N}[1, h]\}. \tag{2.2}$$

The functionals $\Phi, \Psi : H \to \mathbb{R}$ are defined by

$$\Phi(w) = \sum_{q=1}^{h} \sum_{l=1}^{d+1} \frac{1}{p} |\Delta_1 w(l-1, q)|^p + \sum_{l=1}^{d} \sum_{q=1}^{h+1} \frac{1}{p} |\Delta_2 w(l, q-1)|^p, \tag{2.3}$$

and

$$\Psi(w) = \sum_{q=1}^{h} \sum_{l=1}^{d} g(l, q) F(w(l, q)), \tag{2.4}$$

where $F(w) = \int_0^w f(s)ds$ for every $w \in \mathbb{R}$.

Define the energy functional $E_\lambda : H \to \mathbb{R}$ of problem (1.1) as

$$E_\lambda(w) = \Phi(w) - \lambda \Psi(w). \tag{2.5}$$

Lemma 2.1. *[3] The functionals Φ, Ψ and E_λ are differentiable in sense of Gâteaux and for any w,v $\in H$ we have*

$$\Phi^{'}(w)(v) = -\sum_{q=1}^{h} \sum_{l=1}^{d} \left[\Delta_1 \left(\phi_p \left(\Delta_1 w(l-1, q) \right) \right) + \Delta_2 \left(\phi_p \left(\Delta_2 w(l, q-1) \right) \right) \right] v(l, q) \tag{2.6}$$

$$\Psi^{'}(w)(v) = \sum_{q=1}^{h} \sum_{l=1}^{d} g(l, q) f(w(l, q)) v(l, q), \tag{2.7}$$

$$\begin{aligned} E_\lambda^{'}(w)(v) &= [\Phi^{'}(w) - \lambda \Psi^{'}(w)](v) \\ &= -\sum_{q=1}^{h} \sum_{l=1}^{d} [\Delta_1 \left(\phi_p \left(\Delta_1 w(l-1, q) \right) \right) + \Delta_2 \left(\phi_p \left(\Delta_2 w(l, q-1) \right) \right) \\ &+ \lambda g(l, q) f(w(l, q))] v(l, q). \end{aligned} \tag{2.8}$$

We say that $w^* \in H$ is a critical point of the functional E_λ if $E_\lambda(w^*)(v) = 0$ for all $v \in H$.

Remark 2.2. *Every critical point of the functional E_λ is a weak solution for problem (1.1).*

Indeed, let $w^* \in H$ be an arbitrary critical point of functional E_λ. So

$$w^*(l,0) = w^*(l,h+1) = 0, \quad \forall l \in \mathbb{N}[1,d], \quad w^*(0,q) = w^*(d+1,q) = 0, \quad \forall q \in \mathbb{N}[1,h],$$

and

$$E_\lambda(w^*)(v) = 0, \qquad \text{for all } v \in H.$$

Then, from (2.8) we have

$$-\sum_{q=1}^{h}\sum_{l=1}^{d}[\Delta_1\left(\phi_p\left(\Delta_1 w^*(l-1,q)\right)\right) + \Delta_2\left(\phi_p\left(\Delta_2 w^*(l,q-1)\right)\right) + \lambda g(l,q)f(w^*(l,q))]v(l,q) = 0.$$

Since $v \in H$ is arbitrary, therefore

$$-\Delta_1\left(\phi_p\left(\Delta_1 w^*(l-1,q)\right)\right) - \Delta_2\left(\phi_p\left(\Delta_2 w^*(l,q-1)\right)\right) - \lambda g(l,q)f(w^*(l,q)) = 0,$$

for every $(l,q) \in \mathbb{N}[1,d] \times \mathbb{N}[1,h]$. We conclude that every critical point of the functional E_λ is a weak solution for problem (1.1).

Lemma 2.3. *[3] For all $w \in H$, we have*

$$\|w\|_\infty^p \le \frac{p}{4^p}(d+h+2)^{p-1}\Phi(w).$$

Lemma 2.4. *For all $w \in H$, we have*

$$\frac{4^p}{pdh(d+h+2)^{p-1}}\|w\|^p \le \Phi(w) \le \frac{2^{p+1}}{p}\|w\|^p.$$

Proof. Let $w \in H$, from Lemma 2.3 we get

$$|w(l,q)|^p \le \frac{p}{4^p}(d+h+2)^{p-1}\Phi(w)$$

for all $(l,q) \in \mathbb{N}[1,d] \times \mathbb{N}[1,h]$. Then

$$\|w\|^p = \sum_{l=1}^{d}\sum_{q=1}^{h}|w(l,q)|^p \le \frac{dhp}{4^p}(d+h+2)^{p-1}\Phi(w), \quad \forall w \in H,$$

so

$$\frac{4^p}{dhp(d+h+2)^{p-1}}\|w\|^p \le \Phi(w).$$

Furthermore, for all $(l,q) \in \mathbb{N}[1,d] \times \mathbb{N}[1,h]$ we obtain

$$|\Delta_1 w(l-1,q)|^p = |w(l,q) - w(l-1,q)|^p \le \left(|w(l,q)| + |w(l-1,q)|\right)^p.$$

Since the function $t \longmapsto |t|^p$ $(t \ge 0)$ is convex, then we get

$$|\Delta_1 w(l-1,q)|^p \le 2^{p-1}\left(|w(l,q)|^p + |w(l-1,q)|^p\right),$$

so

$$\begin{aligned}\sum_{l=1}^{d+1}\sum_{q=1}^{h}|\Delta_1 w(l-1,q)|^p &\leq 2^{p-1}\left(\sum_{l=1}^{d+1}\sum_{q=1}^{h}|w(l,q)|^p+\sum_{l=1}^{d+1}\sum_{q=1}^{h}|w(l-1,q)|^p\right)\\ &\leq 2\times 2^{p-1}\left(\sum_{l=1}^{d}\sum_{q=1}^{h}|w(l,q)|^p\right)\\ &\leq 2^p\|w\|^p.\end{aligned}$$

Similarly, we prove that

$$\sum_{l=1}^{d}\sum_{q=1}^{h+1}|\Delta_2 w(l,q-1)|^p \leq 2^p\|w\|^p.$$

Thus

$$\Phi(w) \leq \frac{2^{p+1}}{p}\|w\|^p, \quad \forall\, w \in H.$$

□

Now, we recall the basic tools that will be used in the next section. According to the following Ricceri's variational principle described in [[21], Theorem 2.1], we can achieve the first result.

Theorem 2.5. *Assume that $\Phi, \Psi : X \to \mathbb{R}$ are two Gâteaux differentiable functionals, where X is a reflexive real Banach space. Furthermore, Φ is strongly continuous, sequentially weakly lower semicontinuous as well as coercive in X, whereas Ψ is sequentially weakly upper semicontinuous in X. Let $J_\lambda = \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and define*

$$\varphi(r) = \inf_{w\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Psi(v)-\Psi(w)}{r-\Phi(w)}.$$

Then, for any $r > \inf_X \Phi$ and $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the functional J_λ restricted to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point of J_λ in X.

Our second technique is based on a two non-zero critical points theorem proved in [[22],Theorem 2.1] and [23].

Theorem 2.6. *Assume that $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals fulfilling $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$, where X is a real finite dimensional Banach space. Suppose that there exist $r \in \mathbb{R}$ and $\tilde{w} \in X$, with $0 < \Phi(\tilde{w}) < r$, such that*

$$\frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Psi(v)}{r} < \frac{\Psi(\tilde{w})}{\Phi(\tilde{w})},$$

and for each

$$\lambda \in \Lambda = \left] \frac{\Phi(\tilde{w})}{\Psi(\tilde{w})}, \frac{r}{\sup\limits_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)} \right[,$$

the functional $J_\lambda = \Phi - \lambda\Psi$ *satisfies the (PS)-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda$, *the functional* J_λ *has at least two non-zero critical points* $w_{\lambda,1}$, $w_{\lambda,2}$ *such that* $J_\lambda(w_{\lambda,1}) < 0 < J_\lambda(w_{\lambda,2})$.

3. Main results

Let

$$G = \sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q), \quad \alpha = \frac{2^{2p-1}}{(d+h)(d+h+2)^{p-1}}, \quad g_0 = \min_{\mathbb{N}[1,d]\times\mathbb{N}[1,h]} g(l,q).$$

Theorem 3.1. *Assume that*

$$\limsup_{w\to 0} \frac{F(w)}{w^p} = +\infty.$$

Then there is a $\lambda_0 > 0$ *such that, for any* $\lambda \in]0, \lambda_0[$, *problem (1.1) has at least one non-trivial solution in* H.

Proof. Lemma 2.1 implies Φ, Ψ and E_λ are the functionals in $C^1(H, \mathbb{R})$. The functional Φ is of class C^1 on the finite dimensional space H, so is sequentially weakly lower semicontinuous. Also, the functional Ψ is of class C^1 on the finite dimensional space H, so is sequentially weakly upper semicontinuous. Moreover, according to Lemma 2.4, we infer that Φ is coercive.

Put $r = \dfrac{d+h+2}{p} > 0$ and $c = \dfrac{d+h+2}{4}$. For any $w \in H$ such that $\Phi(w) < r$, from Lemma 2.3, we get $\|w\|_\infty \le c$. Consequently

$$\sup_{\Phi(w)<r} \Psi(w) = \sup_{w\in\Phi^{-1}(]-\infty,r[)} \left(\sum_{l=1}^{d}\sum_{q=1}^{h} g(l,q)F(w(l,q)) \right) \le G \max_{|t|\le c} F(t). \tag{3.1}$$

Let

$$\varphi(r) = \inf_{\Phi(w)<r} \frac{\sup\limits_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(w)}{r - \Phi(w)}.$$

From (3.1), we get

$$\varphi(r) \le \frac{\sup\limits_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)}{r} \le \frac{1}{r} G \max_{|t|\le c} F(t),$$

so

$$\frac{1}{\varphi(r)} \geqslant \frac{r}{G \max\limits_{|t|\le c} F(t)}.$$

Put

$$\lambda_0 = \frac{d+h+2}{pG\max\limits_{|t|\le c} F(t)}.$$

Therefore, owing to Theorem 2.5, for all $\lambda \in]0, \lambda_0[\subset \left]0, \frac{1}{\varphi(r)}\right[$ problem (1.1) has at least one solution $w_\lambda \in \Phi^{-1}(]-\infty, r[)$.

Next, we claim that w_λ is non-zero. Indeed, let $A > 0$ large enough, since $\limsup\limits_{w\to 0} \frac{F(w)}{w^p} = +\infty$, then there exists $\rho > 0$ such that

$$F(w) \ge \frac{2(d+h)}{pG}(A+1)\,|w|^p$$

for all $|w| < \rho$. Moreover, for a fixed sequence $\{w_k\} \subset \mathbb{R}$ such that $\lim\limits_{k\to\infty} w_k = 0$, one has

$$\limsup_{k\to\infty} \frac{F(w_k)}{|w_k|^p} = +\infty.$$

Let $v_k = w_k v$ for all $k \in \mathbb{N}$, where $v \in H$ such that $\forall (l,q) \in \mathbb{N}[1,d] \times \mathbb{N}[1,h] : v(l,q) = 1$.

It is clear that $v_k \in H$ for all $k \in \mathbb{N}$, and $\|v_k\| = |w_k|\|v\| \to 0$ as $k \to \infty$. Therefore, for k that is large enough, we obtain

$$\|v_k\| \le \frac{1}{2}\left(\frac{d+h+2}{2}\right)^{\frac{1}{p}},$$

so by Lemma 2.4, we infer that $\Phi(v_k) < r$ and $v_k \in \Phi^{-1}(]-\infty, r[)$ for all $k \in \mathbb{N}$.

Furthermore, for $k \in \mathbb{N}$ sufficiently large, we deduce that

$$\begin{aligned}\frac{\Psi(v_k)}{\Phi(v_k)} &= \frac{p\sum\limits_{q=1}^{h}\sum\limits_{l=1}^{d} g(l,q)F(v_k(l,q))}{2(d+h)|w_k|^p} \\ &\ge \frac{2(d+h)(A+1)|w_k|^p}{pG} \times \frac{pG}{2(d+h)|w_k|^p} \\ &= A+1 > A,\end{aligned}$$

then we have shown that $\limsup\limits_{k\to+\infty} \frac{\Psi(v_k)}{\Phi(v_k)} = +\infty$.

Consequently, we obtain that

$$E_\lambda(v_k) = \Phi(v_k) - \lambda\Psi(v_k) < 0.$$

Since $v_k \in \Phi^{-1}(]-\infty, r[)$ and w_λ is a global minimun of E in $\Phi^{-1}(]-\infty, r[)$, then

$$E_\lambda(w_\lambda) \le E_\lambda(v_k) < 0 = E_\lambda(0_H),$$

which yields that w_λ is non-zero. □

Second, on a basic of a two non-zero critical points theorem, we get the result as follows.

Theorem 3.2. *Suppose that*

(H_1) *There exist $a, b > 0$ with $b < a\alpha^{\frac{1}{p}}$,*

(H_2) *There exist constants $\mu > 0$ and $\theta > p$ such that*

$$0 < \theta F(w) \le wf(w), \quad \forall |w| \ge \mu,$$

(H_3) $\dfrac{\max_{|w|\le a} F(w)}{(4a)^p} < \dfrac{F(b)}{2(d+h)(d+h+2)^{p-1}b^p}.$

Then, for all $\lambda \in \left] \dfrac{2(d+h)b^p}{pGF(b)}, \dfrac{(4a)^p}{pG(d+h+2)^{p-1}\max_{|w|\le a} F(w)} \right[$, *there exist at least two non-trivial solutions of problem (1.1).*

Proof. From Lemma 2.1, the functionals Φ and Ψ given by (2.3)-(2.4) are differentiable in sense of Gâteaux. Clearly, H is a finite dimensional Banach space and

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

First, according to condition (H_2) there exists $C > 0$ such that

$$F(w) \ge C|w|^\theta, \quad \forall |w| \ge \mu.$$

For $s > 1$ large enough and $w \in H\backslash\{0\}$, we have

$$\begin{aligned} \Psi(sw) &\ge C \sum_{q=1}^{h} \sum_{l=1}^{d} g(l,q)|sw(l,q)|^\theta \\ &\ge s^\theta C g_0 \sum_{q=1}^{h} \sum_{l=1}^{d} |w(l,q)|^\theta. \end{aligned}$$

Therefore, from (2.5) and Lemma 2.4, for all $k \in \mathbb{N}$ we get

$$E_\lambda(sw) \le \left(\frac{2^{p+1}}{p} \|w\|^p \right) s^p - \left(\lambda C g_0 \sum_{q=1}^{h} \sum_{l=1}^{d} |w(l,q)|^\theta \right) s^\theta. \tag{3.2}$$

Since $\theta > p$, one has $E_\lambda(sw) \to -\infty$ as $s \to +\infty$. Then the functional E_λ is unbounded from below.

Next, we show that the functional E_λ satisfies the (PS) condition. Arguing by contradiction, for this, suppose that there exists an unbounded sequence $\{w_k\} \subset H$ such that $\{E_\lambda(w_k)\}$ is bounded and $E'_\lambda(w_k) \to 0$ as $k \to +\infty$. Then, there exists a positive constant A such that

$$E_\lambda(w_k) \le A \quad \text{and} \quad \|w_k\| \ge -\frac{1}{\theta} \left(E'_\lambda(w_k), w_k \right),$$

for $k \in \mathbb{N}$ large enough,

$$A + \|w_k\| \geq E_\lambda(w_k) - \frac{1}{\theta}\left(E'_\lambda(w_k), w_k\right). \tag{3.3}$$

Moreover, we have

$$\begin{aligned}
&E_\lambda(w_k) - \frac{1}{\theta}\left(E'_\lambda(w_k), w_k\right)\\
&= \Phi(w_k) - \lambda\Psi(w_k) - \frac{1}{\theta}\left(p\Phi(w_k) - \lambda\sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q)f(w_k(l,q))w_k(l,q)\right)\\
&= \left(1 - \frac{p}{\theta}\right)\Phi(w_k) - \frac{\lambda}{\theta}\sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q)\left(\theta F(w_k(l,q)) - f(w_k(l,q))w_k(l,q)\right),
\end{aligned}$$

and from assumption (H_2), one has

$$\begin{aligned}
\sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q)\left(\theta F(w_k(l,q)) - f(w_k(l,q))w_k(l,q)\right) &\leq \sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q)\max_{|w|\leq\mu}|\theta F(w) - wf(w)|\\
&\leq G\max_{|w|\leq\mu}|\theta F(w) - wf(w)|.
\end{aligned}$$

Then, from Lemma 2.4, we deduce that

$$A + \|w_k\| \geq \frac{4^p}{pdh(d+h+2)^{p-1}}\left(1 - \frac{p}{\theta}\right)\|w_k\|^p - \frac{\lambda}{\theta}G\max_{|w|\leq\mu}|\theta F(w) - wf(w)|.$$

However, this is absurd since $p > 1$ and $\theta > p$. Therefore, the sequence $\{w_k\}$ is bounded in H which is a finite dimensional space, then $\{w_k\}$ has a convergent subsequence. This shows that E_λ fulfills the (PS) condition.

On the other hand, put

$$r = \frac{(4a)^p}{p(d+h+2)^{p-1}}.$$

For all $w \in H$, with $\Phi(w) < r$, from Lemma 2.3 we get $\|w\|_\infty \leq a$, and we have

$$\Psi(w) = \sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q)F(w(l,q)) \leq G\max_{|t|\leq a}F(t).$$

Then

$$\frac{\sup\limits_{\Phi(w)<r}\Psi(w)}{r} \leq \frac{p(d+h+2)^{p-1}}{4^p} \times \frac{G\max\limits_{|t|\leq a}F(t)}{a^p}.$$

Choose $\tilde{w}$ defined by $\tilde{w}(l,q) = b$ for all $(l,q) \in \mathbb{N}[1,d]\times\mathbb{N}[1,h]$, $w(l,0) = 0 = w(l,h{+}1)$, $l \in \mathbb{N}[0,d{+}1]$ and $w(0,q) = 0 = w(d+1,q)$, $q \in \mathbb{N}[0,h+1]$. It is obvious that $\tilde{w} \in H$ and from assumption (H_1), one has

$$\Phi(\tilde{w}) = \frac{2(d+h)b^p}{p} < r.$$

Moreover, we have

$$\frac{\Psi(\tilde{w})}{\Phi(\tilde{w})} = \frac{\sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q)F(\tilde{w}(l,q))}{\Phi(\tilde{w})} = \frac{p}{2(d+h)b^p}\sum_{q=1}^{h}\sum_{l=1}^{d} g(l,q)F(b) = \frac{pGF(b)}{2(d+h)b^p}.$$

According to (H_3), we deduce that

$$\frac{\sup_{\Phi(w)<r} \Psi(w)}{r} < \frac{\Psi(\tilde{w})}{\Phi(\tilde{w})}.$$

Hence all hypotheses of Theorem 2.6 are fulfilled, thus the functional E_λ has at least two non-trivial critical points for all $\lambda \in \left]\frac{2(d+h)b^p}{pGF(b)}, \frac{(4a)^p}{pG(d+h+2)^{p-1}\max_{|w|\le a} F(w)}\right[$. □

4. Examples

Example 4.1. *We consider problem (1.1) with $d = h = 10$, $p = \sqrt{10}$ and the functions of the second term are given by*

$$g(l,q) = \frac{1}{lq}, \qquad \forall (l,q) \in \mathbb{N}[1,d] \times \mathbb{N}[1,h],$$

$$f(w) = (3-w)w^2e^{-w}, \qquad \forall w \in \mathbb{R}.$$

By simple computations, we get that

$$F(w) = w^3e^{-w}, \quad \forall w \in \mathbb{R}, \quad \max_{|w|\le \frac{11}{2}} F(w) = \frac{27}{e^3},$$

$$\lim_{|w|\to 0} \frac{F(w)}{w^{\sqrt{10}}} = +\infty, \quad G = \sum_{q=1}^{h}\sum_{l=1}^{d} \frac{1}{lq} \approx 8.58.$$

Therefore, according to Theorem 3.1, for any $\lambda \in]0, 0.60[$ the above problem has at least one non-trivial solution.

Example 4.2. *We give an example of function f, which satisfies the assumption (H_2) of Theorem 3.2. In fact, for $p = \frac{10}{3}$, take*

$$f(w) = \begin{cases} 1 + \frac{11}{3}w^{\frac{8}{3}} & , \quad w \ge 0, \\ 1 - \frac{11}{3}(-w)^{\frac{8}{3}} & , \quad w < 0, \end{cases}$$

we get

$$F(w) = \int_0^w f(t)dt = w + |w|^{\frac{11}{3}}, \quad \forall w \in \mathbb{R}.$$

Put $\mu = \left(\frac{46}{3}\right)^{\frac{3}{8}}$, $\theta = \frac{7}{2}$, and follow the analysis below.

1. *For all* $w \in \mathbb{R}$ *such that* $|w| \geq \left(\frac{46}{3}\right)^{\frac{3}{8}}$, *we get*

$$F(w) \geq |w|\left(-1 + |w|^{\frac{8}{3}}\right) \geq \mu\left(-1 + \mu^{\frac{8}{3}}\right) = \frac{43}{3}\left(\frac{46}{3}\right)^{\frac{3}{8}} > 0.$$

2. *For* $w < 0$,

$$wf(w) - \theta F(w) = w + \frac{11}{3}(-w)^{\frac{11}{3}} - \frac{7}{2}\left(w + |w|^{\frac{11}{3}}\right) = \frac{1}{6}|w|^{\frac{11}{3}} - \frac{5}{2}w > 0.$$

3. *For* $w \geq \left(\frac{46}{3}\right)^{\frac{3}{8}}$,

$$\begin{aligned} wf(w) - \theta F(w) &= w + \frac{11}{3}w^{\frac{11}{3}} - \frac{7}{2}\left(w + |w|^{\frac{11}{3}}\right) = \frac{1}{6}w^{\frac{11}{3}} - \frac{5}{2}w \\ &= \frac{1}{6}w\left(w^{\frac{8}{3}} - 15\right) \geq \frac{1}{18}\left(\frac{46}{3}\right)^{\frac{3}{8}} > 0. \end{aligned}$$

Then assumption (H_2) *holds as well.*

5. Conclusion

In this work, we study the existence and multiplicity of non-trivial solutions for a discrete nonlinear problem in a dh-dimensional Banach space. The approach allows us to prove that the energy functional has at least one or two non-trivial critical points that are solutions of the associated problem. In order to demonstrate how the findings might be applied to real-world situations, two examples are presented in which a variety of presumptions are shown to be accurate.

Moreover, we have already discussed problem (1.1) in the case where g is a positive function. As for the case where the function g changes sign, it has been left as an open question for future research. Besides, we can tackle the existence of solution for problem (1.1), where $(l, q) \in \mathbb{Z} \times \mathbb{Z}$ and $\lim_{|l|+|q| \longrightarrow +\infty} w(l, q) = 0$.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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