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Theory article

On sequences of homoclinic solutions for fractional discrete p-Laplacian equations

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Abstract: In this paper, we consider the following discrete fractional p-Laplacian equations:

$$(-\Delta_1)^s_p u(a) + V(a)|u(a)|^{p-2}u(a) = \lambda f(a, u(a)), \text{ in } \mathbb{Z},$$

where λ is the parameter and $f(a, u(a))$ satisfies no symmetry assumption. As a result, a specific positive parameter interval is determined by some requirements for the nonlinear term near zero, and then infinitely many homoclinic solutions are obtained by using a special version of Ricceri's variational principle.

Keywords: discrete fractional p-Laplacian; homoclinic solutions; Ricceri's variational principle
Mathematics Subject Classification: 35R11, 49M25, 35J20

1. Introduction and main result

In this article, we study the existence of infinitely many homoclinic solutions of the following fractional discrete p-Laplacian equations:

$$(-\Delta_1)^s_p u(a) + V(a)|u(a)|^{p-2}u(a) = \lambda f(a, u(a)), \text{ in } \mathbb{Z}, \tag{1.1}$$

where $s \in (0, 1)$ and $p \in (1, \infty)$ are fixed constants, $V(a) \in \mathbb{R}^+$, λ is a positive parameter, $f(a, \cdot)$ is a continuous function for all $a \in \mathbb{Z}$ and $(-\Delta_1)^s_p$ is the fractional discrete p-Laplacian given by

$$(-\Delta_1)^s_p u(a) = 2 \sum_{b\in\mathbb{Z}, b\neq a} |u(a) - u(b)|^{p-2} (u(a) - u(b))K_{s,p}(a - b), \text{ in } \mathbb{Z},$$

where the discrete kernel $K_{s,p}$ has the following property: There exist two constants $0 < c_{s,p} \le C_{s,p}$, such that

$$\begin{cases} \frac{c_{s,p}}{|d|^{1+ps}} \le K_{s,p}(d) \le \frac{C_{s,p}}{|d|^{1+ps}}, \text{ for all } d \in \mathbb{Z} \setminus \{0\}; \\ K_{s,p}(0) = 0. \end{cases} \tag{1.2}$$

The fractional operator has received more attention recent decades because of its many applications in the real world. Many scholars have paid attention to this kind of problem, and have produced a lot of classical works, see for example [1–3]. As a classical fractional operator, the fractional Laplacian has wide applications in various fields such as optimization, population dynamics and so on. The fractional Laplacian on $\mathbb{R}$ can be defined for $0 < s < 1$ and $v \in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)^s v(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x-y|^{N+2s}} dy, \ x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a positive constant and $P.V.$ denotes the Cauchy principle value. In various cases involving differential equations, the Laplace operator is replaced by either the fractional Laplace operator or other more general operators, and hence the existence results have been obtained by employing variational approaches, see for instance [4–7]. These alternative approaches have been found to offer improved descriptions of numerous phenomena observed in the natural world. Correspondingly, it is necessary to give some qualitative results by employing numerical analysis. The nonlocal feature of the fractional Laplacian is one of the important aspects to be considered in numerical methods, which makes it necessary to study the existence of solutions.

Let $\mathbb{Z}_{\mathcal{H}}$ denote a grid of fixed size $\mathcal{H} > 0$ on $\mathbb{R}$, i.e., $\mathbb{Z}_{\mathcal{H}} = \{\mathcal{H}a | a \in \mathbb{Z}\}$. In [8], the definition of the fractional discrete Laplacian on $\mathbb{Z}_{\mathcal{H}}$ is given by

$$(-\Delta_{\mathcal{H}})^s v(a) = \sum_{b \in \mathbb{Z}, b \neq a} (v(a) - v(b)) K_s^{\mathcal{H}}(a-b),$$

where $s \in (0,1)$, $v \in \ell_s = \left\{ v : \mathbb{Z}_{\mathcal{H}} \to \mathbb{R} \middle| \sum_{\omega \in \mathbb{Z}} \frac{|v(\omega)|}{(1+|\omega|)^{1+2s}} < \infty \right\}$ and

$$K_s^{\mathcal{H}}(a) = \begin{cases} \frac{4^s \Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|} \cdot \frac{\Gamma(|a|-s)}{\mathcal{H}^{2s}\Gamma(|a|+1+s)}, & a \in \mathbb{Z}\backslash\{0\}, \\ 0, & a = 0. \end{cases}$$

The above discrete kernel $K_s^{\mathcal{H}}$ has the following property: There exist two constants $0 < c_s \le C_s$, such that for all $a \in \mathbb{Z}\backslash\{0\}$ there holds

$$\frac{c_s}{\mathcal{H}^{2s}|a|^{1+2s}} \le K_s^{\mathcal{H}}(a) \le \frac{C_s}{\mathcal{H}^{2s}|a|^{1+2s}}.$$

In [8], Ciaurri *et al.* also proved that if v is bounded then $\lim_{s \to 1^-} (-\Delta_{\mathcal{H}})^s v(a) = -\Delta_{\mathcal{H}} v(a)$, where $\Delta_{\mathcal{H}}$ is there discrete Laplacian on $\mathbb{Z}_{\mathcal{H}}$, i.e.

$$\Delta_{\mathcal{H}} v(a) = \frac{1}{\mathcal{H}^2}(v(a+1) - 2v(a) + v(a-1)).$$

Moreover, under some suitable conditions and $\mathcal{H} \to 0$, the fractional Laplacian can be approximated by the fractional discrete Laplacian.

Subsequently, let us give some existence results on the fractional difference equations. Xiang *et al.* [9] first investigated the fractional discrete Laplacian equations based on variational methods:

$$\begin{cases}(-\Delta_1)^s v(a) + V(a)v(a) = \lambda f(a, v(a)), & \text{for } a \in \mathbb{Z},\\ v(a) \to 0, & \text{as } |a| \to \infty,\end{cases} \tag{1.3}$$

where $f(a, \cdot)$ is a continuous function for all $a \in \mathbb{Z}$, $\lambda > 0$, $V(a) \in \mathbb{R}^+$ and

$$(-\Delta_1)^s v(a) = 2 \sum_{b \in \mathbb{Z}, b \neq a} (v(a) - v(b))K_s(a - b), \text{ in } \mathbb{Z}.$$

Using the mountain pass theorem and Ekeland's variational principle under some suitable conditions, they obtained two homoclinic solutions for problem (1.3). It is evident that when $p = 2$, the discrete fractional p-Laplace operator corresponds to the discrete fractional Laplace operator. After that, Ju *et al.* [10] studied the following fractional discrete p-Laplacian equations

$$\begin{cases}(-\Delta_1)^s_p v(d) + V(d)|v(d)|^{p-2}v(d) = \lambda a(d)|v(d)|^{q-2}v(d) + b(d)|v(d)|^{r-2}v(d), & \text{for } d \in \mathbb{Z},\\ v(d) \to 0, & \text{as } |d| \to \infty,\end{cases} \tag{1.4}$$

where $a \in \ell^{\frac{p}{p-q}}$, $b \in \ell^\infty$, $1 < q < p < r < \infty$, $\lambda > 0$, $V(a) \in \mathbb{R}^+$, $(-\Delta_1)^s_p$ is the fractional discrete p-Laplacian. Under certain conditions, they employed the Nehari manifold method to achieve the existence of at least two homoclinic solutions for problem (1.4). In [11], Ju *et al.* investigated the existence of multiple solutions for the fractional discrete p-Laplacian equations with various nonlinear terms via different Clark's theorems. In a recent study conducted by Ju *et al.* in [12], it was demonstrated that using the fountain theorem and the dual fountain theorem under the same hypotheses, two separate sequences of homoclinic solutions were derived for the fractional discrete Kirchhoff-Schrödinger equations. Based on the findings from [8], it could be deduced that Eq.(1.3) can be reformulated as the renowned discrete version of the Schrödinger equation

$$-\Delta\mu(\xi) + V(\xi)\mu(\xi) = \lambda f(\xi, \mu(\xi)), \text{ in } \mathbb{Z}. \tag{1.5}$$

It is worth mentioning that in [13], Agarwal *et al.* first employed the variational methods to analyze Eq.(1.5). Here, We give some literature on the study of difference equations using the critical point theory, see [14–16].

In particular, we observe that both the nonlinear terms in [11, 12] have the following symmetry condition:

($\mathcal{S}$) $f(a, v)$ is odd in v.

Therefore, in this paper, we consider the nonlinear term without condition ($\mathcal{S}$), and study the existence of multiple homoclinic solutions of problem (1.1). For this, let us first recall if the solution v of Eq.(1.1) satisfies $v(d) \to 0$ as $|d| \to \infty$, then v is called a homoclinic solution. Suppose that $V(a)$ and $f(a, u(a))$ in problem (1.1) satisfy the following assumptions:

($\mathcal{V}$) $V \in \ell^1$ and there is a constant $V_0 \in (0, \inf_{a\in\mathbb{Z}} V(a)]$; ($\ell^1$ is defined in next section)
($\mathcal{F}$) $|f(a, u)| \leq C\left(|u|^{p-1} + |u|^{t-1}\right)$ for any $a \in \mathbb{Z}$ and $u \in \mathbb{R}$, where $p < t < \infty$ and $C > 0$ is a constant.

Set

$$A := \liminf_{\tau\to 0^+} \frac{\max_{|\zeta|\le\tau} \sum_{a\in\mathbb{Z}} F(a,\zeta(a))}{\tau^p}, B := \limsup_{\tau\to 0^+} \frac{\sum_{a\in\mathbb{Z}} F(a,\tau(a))}{\tau^p}, \theta := \frac{C_s V_0}{C_b \|V\|_1},$$

where $F(a,u) = \int_0^u f(a,\eta)d\eta$ and C_s, C_b will appear in next section. Here we give the main conclusion of our paper as follows.

Theorem 1.1. *Suppose that* ($\mathcal{V}$) *and* ($\mathcal{F}$) *are satisfied. Furthermore, the following inequality holds:* $A < \theta B$. *Then, for every* $\lambda \in \left(\frac{C_b\|V\|_1}{pB}, \frac{C_s V_0}{pA}\right)$, *problem* (1.1) *possesses infinitely many nontrivial homoclinic solutions. In addition, their critical values and their* ℓ^∞*-norms tend to zero.*

The rest of this article is arranged as follows: In Section 2, we introduce some definitions and give some preliminary results. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give an example to demonstrate the main result.

Here we illustrate some notations used in this paper:

- C, $C_{s,\omega}$, C_s, C_b and C_∞ are diverse positive constants.
- $\hookrightarrow$ denotes the embedding.
- $\to$ denotes the strong convergence.

2. Preliminaries

First we give some basic definitions.

Let $1 \le \omega \le \infty$, we give the definition of the space $(\ell^\omega, \|\cdot\|_\omega)$ as follows:

$$\ell^\omega := \begin{cases} \left\{\mu : \mathbb{Z} \to \mathbb{R} \,\middle|\, \sum_{a\in\mathbb{Z}} |\mu(a)|^\omega < \infty\right\}, & \text{if } 1 \le \omega < \infty; \\ \left\{\mu : \mathbb{Z} \to \mathbb{R} \,\middle|\, \sup_{a\in\mathbb{Z}} |\mu(a)| < \infty\right\}, & \text{if } \omega = \infty; \end{cases}$$

$$\|\mu\|_\omega := \begin{cases} \left(\sum_{a\in\mathbb{Z}} |\mu(a)|^\omega\right)^{1/\omega}, & \text{if } 1 \le \omega < \infty; \\ \sup_{a\in\mathbb{Z}} |\mu(a)|, & \text{if } \omega = \infty. \end{cases}$$

Through the corresponding conclusions in [17], we know that ℓ^ω is a Banach space. Moreover, $\ell^{\omega_1} \hookrightarrow \ell^{\omega_2}$ and $\|\mu\|_{\omega_2} \le \|\mu\|_{\omega_1}$ if $1 \le \omega_1 \le \omega_2 \le \infty$.

Next, we give the variational framework and some lemmas of this paper.

The space $\left(Q, \|\cdot\|_Q\right)$ is defined by

$$Q = \left\{\sigma : \mathbb{Z} \to \mathbb{R} \,\middle|\, \sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}} |\sigma(a)-\sigma(b)|^p K_{s,p}(a-b) + \sum_{d\in\mathbb{Z}} V(d)|\sigma(d)|^p < \infty\right\};$$

$$\begin{aligned} \|\sigma\|_Q^p &= [\sigma]_{s,p}^p + \sum_{d\in\mathbb{Z}} V(d)|\sigma(d)|^p \\ &= \sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}} |\sigma(a)-\sigma(b)|^p K_{s,p}(a-b) + \sum_{d\in\mathbb{Z}} V(d)|\sigma(d)|^p. \end{aligned}$$

Lemma 2.1. (see [10, Lemma 2.1]) *If* $\xi \in \ell^\omega$, *then* $[\xi]_{s,\omega} \le C_{s,\omega}\|\xi\|_\omega < \infty$.

Lemma 2.2. (see [11, Lemma 2.2]) *Under the hypothesis* $(\mathcal{V})$, $\left(Q, \|\cdot\|_Q\right)$ *is a reflexive Banach space, and*

$$\|\sigma\| := \left(\sum_{a\in\mathbb{Z}} V(a)|\sigma(a)|^p\right)^{1/p}.$$

is an equivalent norm of Q.

Through Lemma 2.2, we obtain that there exist $0 \le C_s \le C_b$ such that

$$C_s \|\mu\|^p \le \|\mu\|_Q^p \le C_b \|\mu\|^p. \tag{2.1}$$

Lemma 2.3. *Under the hypothesis* $(\mathcal{V})$, $Q \hookrightarrow \ell^r$ *is continuous for all* $p \le r \le \infty$.

Proof. Using the above conclusions and $(\mathcal{V})$, we can deduce that

$$\|\sigma\|_r \le \|\sigma\|_p = \left(\sum_{a\in\mathbb{Z}} |\sigma(a)|^p\right)^{\frac{1}{p}} \le V_0^{-\frac{1}{p}} \left(\sum_{a\in\mathbb{Z}} V(a)\,|\sigma(a)|^p\right)^{\frac{1}{p}}, \ \forall\, \sigma \in Q.$$

As desired.

Lemma 2.4. (see [10, Lemma 2.4]) *If* $\mathcal{W} \subset Q$ *is a compact subset, then for* $\forall\ \iota > 0$, $\exists\ a_0 \in \mathbb{N}$ *such that*

$$\left[\sum_{|a|>a_0} V(a)|\xi(a)|^p\right]^{1/p} < \iota, \text{ for each } \xi \in \mathcal{W}.$$

For all $u \in Q$, we define

$$K(u) = D(u) - \lambda E(u)$$

where

$$D(u) = \frac{1}{p}\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}} |u(a) - u(b)|^p\, K_{s,p}(a-b) + \frac{1}{p}\sum_{d\in\mathbb{Z}} V(d)|u(d)|^p = \frac{1}{p}\,\|u\|_Q^p$$

and

$$E(u) = \sum_{d\in\mathbb{Z}} F(d, u(d)).$$

Clearly

$$\inf_Q D(\mu) = \inf_Q \frac{1}{p}\,\|\mu\|_Q^p = D(0) = 0. \tag{2.2}$$

Lemma 2.5. (see [10, Lemma 2.5]) *Under the hypothesis* $(\mathcal{V})$, *then* $D(\sigma) \in C^1(Q, \mathbb{R})$ *with*

$$\begin{aligned}\langle D'(\sigma), \xi\rangle = &\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}} |\sigma(a) - \sigma(b)|^{p-2}\,(\sigma(a) - \sigma(b))\,(\xi(a) - \xi(b))\,K_{s,p}(a-b)\\ &+ \sum_{d\in\mathbb{Z}} V(d)|\sigma(d)|^{p-2}\sigma(d)\xi(d),\end{aligned}$$

for all $\sigma, \xi \in Q$.

Lemma 2.6. (see [12, Lemma 2.6]) *Under the hypotheses* $(\mathcal{V})$ *and* $(\mathcal{F})$*, then* $E(\sigma) \in C^1(Q, \mathbb{R})$ *with*

$$\langle E'(\sigma), \xi \rangle = \sum_{d \in \mathbb{Z}} f(d, \sigma(d))\xi(d)$$

for all $\sigma, \xi \in Q$.

Combining Lemma 2.5 and Lemma 2.6, we know that $K(\sigma) \in C^1(Q, \mathbb{R})$.

Lemma 2.7. *Under the hypotheses* $(\mathcal{V})$ *and* $(\mathcal{F})$*, then for* $\forall\ \lambda > 0$*, every critical point of* K *is a homoclinic solution of problem* (1.1).

Proof. Assume σ be a critical point of K, we get for $\forall\ \xi \in Q$

$$\begin{aligned}&\sum_{a \in \mathbb{Z}} \sum_{b \in \mathbb{Z}} |\sigma(a) - \sigma(b)|^{p-2} (\sigma(a) - \sigma(b)) (\xi(a) - \xi(b)) K_{s,p}(a - b) + \sum_{a \in \mathbb{Z}} V(a)|\sigma(a)|^{p-2}\sigma(a)\xi(a) \\ =&\lambda \sum_{a \in \mathbb{Z}} f(a, \sigma(a))\xi(a).\end{aligned} \tag{2.3}$$

For each $a \in \mathbb{Z}$, we define $e_d \in Q$ as follows:

$$e_d(a) := \begin{cases} 0, & \text{if } a \neq d; \\ 1, & \text{if } a = d. \end{cases}$$

Taking $\xi = e_d$ in (2.3), we have

$$2 \sum_{b \in \mathbb{Z}, b \neq a} |u(a) - u(b)|^{p-2} (u(a) - u(b))K_{s,p}(a - b) + V(a)|u(a)|^{p-2}u(a) = \lambda f(a, \sigma(a)).$$

So σ is a solution of problem (1.1). Moreover, by Lemma 2.3 and $\sigma \in Q$, we know $\sigma(a) \to 0$ as $|a| \to \infty$. Thus, σ is a homoclinic solution of problem (1.1).

3. Proof of main result

In this section, we shall use the following Thoerem 3.1 to prove our main result. In fact, this theorem is a special version of Ricceri's variational principle [18, Lemma 2.5].

Theorem 3.1. (see [19, Lemma 2.1]) *Let* Q *be a reflexive Banach space,* $K(\mu) := D(\mu) + \lambda E(\mu)$ *for each* $\mu \in Q$*, where* $D, E \in C^1(Q, \mathbb{R})$*,* D *is coercive, and* λ *is a real positive parameter. For every* $\gamma > \inf_Q D(\mu)$*, let*

$$\eta(\gamma) := \inf_{\mu \in D^{-1}((-\infty, \gamma))} \frac{\left(\sup_{\nu \in D^{-1}((-\infty,\gamma))} E(\nu)\right) - E(\mu)}{\gamma - D(\mu)},$$

and

$$\rho := \liminf_{\gamma \to (\inf_Q D(\mu))^+} \eta(\gamma)$$

If $\rho < +\infty$*, then for every* $\lambda \in (0, \frac{1}{\rho})$*, the following conclusions holds only one:*

(*a*) *there exists a global minimum of* D *which is a local minimum of* K.

(*b*) *there exists a sequence* $\{\mu_m\}$ *of pairwise distinct critical points (local minima) of* K*, with* $\lim_{m\to\infty} D(\mu_m) = \inf_Q D(\mu)$*, which converges to a global minimum of* D*.*

Remark 3.1. *Obviously,* $\rho \geq 0$*. In addition, when* $\rho = 0$*, we think that* $\frac{1}{\rho} = +\infty$*.*

Proof of Theorem 1.1. Let us recall

$$A = \liminf_{\tau\to 0^+} \frac{\max_{|\zeta|\leq\tau} \sum_{a\in\mathbb{Z}} F(a,\zeta(a))}{\tau^p}, B = \limsup_{\tau\to 0^+} \frac{\sum_{a\in\mathbb{Z}} F(a,\tau(a))}{\tau^p}, \theta = \frac{C_s V_0}{C_b \|V\|_1},$$

where $F(a,u) = \int_0^u f(a,\omega)d\omega$. Fix $\lambda \in \left(\frac{C_b\|V\|_1}{pB}, \frac{C_s V_0}{pA}\right)$ and set K, D, E as in Section 2. By Lemma 2.2, Lemma 2.5 and Lemma 2.6, we know Q be a reflexive Banach space and $D, E \in C^1(Q,\mathbb{R})$. Because of

$$D(\mu) = \frac{1}{p}\sum_{a\in\mathbb{Z}}\sum_{b\in\mathbb{Z}} |\mu(a)-\mu(b)|^p K_{s,p}(a-b) + \frac{1}{p}\sum_{d\in\mathbb{Z}} V(d)|\mu(d)|^p = \frac{1}{p}\|\mu\|_Q^p \to +\infty$$

as $\|\mu\|_Q \to +\infty$, i.e. D is coercive. Now, we show that $\rho < +\infty$. For this purpose, let $\{\delta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \delta_n = 0$ and

$$\lim_{n\to\infty} \frac{\max_{|\zeta|\leq\delta_n} \sum_{a\in\mathbb{Z}} F(a,\zeta(a))}{\delta_n^p} = A.$$

Put

$$\gamma_n := \frac{C_s V_0}{p}\delta_n^p,$$

for all $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} \gamma_n = 0$. For $n > 0$ is big enough, by Lemma 2.2 and (2.1), we can derive that

$$D^{-1}((-\infty,\gamma_n)) \subset \{\nu \in Q : |\nu(d)| \leq \delta_n,\ d \in \mathbb{Z}\}. \tag{3.1}$$

Since $D(0) = E(0) = 0$, for each n large enough, by (3.1), we get

$$\begin{aligned}
\eta(\gamma_n) &= \inf_{\mu\in D^{-1}((-\infty,\gamma_n))} \frac{\left(\sup_{\nu\in D^{-1}((-\infty,\gamma_n))} \sum_{a\in\mathbb{Z}} F(a,\nu(a))\right) - E(\mu)}{\gamma_n - D(\mu)} \\
&\leq \frac{\left(\sup_{\nu\in D^{-1}((-\infty,\gamma_n))} \sum_{a\in\mathbb{Z}} F(a,\nu(a))\right) - E(0)}{\gamma_n - D(0)} \\
&= \frac{\sup_{\nu\in D^{-1}((-\infty,\gamma_n))} \sum_{a\in\mathbb{Z}} F(a,\nu(a))}{\gamma_n} \\
&\leq \frac{\max_{|w|\leq\delta_n} \sum_{a\in\mathbb{Z}} F(a,w(a))}{\gamma_n} \\
&= \frac{p\max_{|w|\leq\delta_n} \sum_{a\in\mathbb{Z}} F(a,w(a))}{C_s V_0 \delta_n^p}.
\end{aligned}$$

Therefore, by (2.2), we acquire that

$$\begin{aligned}
\rho = \liminf_{\gamma\to(\inf_Q D(\mu))^+} \eta(\gamma) = \liminf_{\gamma_n\to 0^+} \eta(\gamma_n) \leq \lim_{n\to\infty} \eta(\gamma_n) \\
\leq \lim_{n\to\infty} \frac{p\max_{|w|\leq\delta_n} \sum_{a\in\mathbb{Z}} F(a,w(a))}{C_s V_0 \delta_n^p} = \frac{pA}{C_s V_0}.
\end{aligned} \tag{3.2}$$

From (3.2), we get

$$\lambda \in \left(\frac{C_b\,\|V\|_1}{pB}, \frac{C_s V_0}{pA}\right) \subset \left(0, \frac{1}{\rho}\right).$$

Next, we verify that 0 is not a local minimum of K. First, suppose that $B = +\infty$. Choosing M such that $M > \frac{C_b\|V\|_1}{p\lambda}$ and let $\{h_n\}$ be a sequence of positive numbers, with $\lim_{n\to\infty} h_n = 0$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$\sum_{a\in\mathbb{Z}} F(a, h_n) > M h_n^p. \tag{3.3}$$

Therefore, let $\{l_n\}$ be a sequence in Q defined by

$$l_n(a) := h_n, \text{ for all } a \in \mathbb{Z}.$$

It is easy to infer that $\|l_n(a)\|_Q \to 0$ as $n \to \infty$. By $(\mathcal{V})$, (2.1) and (3.3), we obtain

$$\begin{aligned}
K(l_n) &= D(l_n) - \lambda E(l_n)\\
&= \frac{1}{p}\,\|l_n\|_Q^p - \lambda \sum_{a\in\mathbb{Z}} F\,(a, l_n)\\
&= \frac{1}{p}\,\|l_n\|_Q^p - \lambda \sum_{a\in\mathbb{Z}} F\,(a, h_n)\\
&\leq \frac{C_b}{p}\,\|l_n\|^p - \lambda \sum_{a\in\mathbb{Z}} F\,(a, h_n)\\
&= \frac{C_b}{p}\sum_{a\in\mathbb{Z}} V(a)\,|l_n(a)|^p - \lambda \sum_{a\in\mathbb{Z}} F\,(a, h_n)\\
&= \frac{C_b}{p}\left(\sum_{a\in\mathbb{Z}} V(a)\right)|h_n|^p - \lambda \sum_{a\in\mathbb{Z}} F\,(a, h_n)\\
&< \frac{C_b}{p}\,\|V\|_1\, h_n^p - \lambda M h_n^p\\
&= \left(\frac{C_b}{p}\,\|V\|_1 - \lambda M\right) h_n^p.
\end{aligned}$$

So, $K(l_n) < 0 = K(0)$ for each $n \geq n_1$ big enough. Next, suppose that $B < +\infty$. Since $\lambda > \frac{C_b\|V\|_1}{pB}$, there exists $\varepsilon > 0$ such that $\varepsilon < B - \frac{C_b\|V\|_1}{p\lambda}$. Hence, also choosing $\{h_n\}$ be a sequence of positive numbers, with $\lim_{n\to\infty} h_n = 0$, there is $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$

$$\sum_{a\in\mathbb{Z}} F(a, h_n) > (B - \varepsilon)\, h_n^p. \tag{3.4}$$

Arguing as before and by choosing $\{l_n\}$ in Q as above, we get

$$\begin{aligned}
K(l_n) &= D(l_n) - \lambda E(l_n)\\
&= \frac{1}{p}\,\|l_n\|_Q^p - \lambda \sum_{a\in\mathbb{Z}} F\,(a, l_n)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}\|l_n\|_Q^p - \lambda \sum_{a\in\mathbb{Z}} F(a,h_n) \\
&\leq \frac{C_b}{p}\|l_n\|^p - \lambda \sum_{a\in\mathbb{Z}} F(a,h_n) \\
&= \frac{C_b}{p}\sum_{a\in\mathbb{Z}} V(a)\,|l_n(a)|^p - \lambda \sum_{a\in\mathbb{Z}} F(a,h_n) \\
&= \frac{C_b}{p}\left(\sum_{a\in\mathbb{Z}} V(a)\right)|h_n|^p - \lambda \sum_{a\in\mathbb{Z}} F(a,h_n) \\
&< \frac{C_b}{p}\|V\|_1\, h_n^p - \lambda(B-\varepsilon)h_n^p \\
&= \left(\frac{C_b}{p}\|V\|_1 - \lambda(B-\varepsilon)\right)h_n^p.
\end{aligned}
$$

So, $K(l_n) < 0 = K(0)$ for each $n \geq n_2$ big enough. In general, 0 is not a local minimum of K. By Theorem 3.1, (a) is not valid, then we have a sequence $\{\mu_n\} \subset Q$ of critical points of K such that

$$\lim_{n\to\infty} D(\mu_n) = \lim_{n\to\infty} \frac{1}{p}\|\mu_n\|_Q = \inf_Q D(\mu) = 0$$

and

$$\lim_{n\to\infty} K(\mu_n) = \inf_Q D(\mu) = 0.$$

By Lemma 2.3, we gain

$$\|\mu_n\|_\infty \leq C_\infty \|\mu_n\|_Q \to 0$$

as $n \to \infty$. By Lemma 2.7, the problem (1.1) admits infinitely many nontrivial homoclinic solutions. In addition, their critical values and their ℓ^∞-norms tend to zero. This completes the proof.

4. Example

Here, we give an example of a nonlinear term which can apply Theorem 1.1.

Example 4.1. We define

$$\Psi(n) := \frac{1}{3^{3^{3n}}}, \text{ for } n \in \mathbb{N}_+;$$

$$\Phi(n) := \frac{1}{3^{3^{3n-1}}}, \text{ for } n \in \mathbb{N}_+;$$

$$\chi(n) := \frac{1}{3^{(p+1)3^{3n-3}}}, \text{ for } n \in \mathbb{N}_+.$$

Obviously, we know that $\Phi(n+1) < \Psi(n) < \Phi(n)$ for all $n \in \mathbb{N}_+$ and $\lim_{n\to\infty}\Psi(n) = \lim_{n\to\infty}\Phi(n) = 0$. Set

$$f(a,u) = 0, \ \forall\, a \in \mathbb{Z} \setminus \mathbb{N}_+.$$

And for each $a \in \mathbb{N}_+$, let $f(a,\cdot)$ is a nonnegative continuous function such that

$$f(a,u) = 0,\ \forall\, u \in \mathbb{R} \setminus (\Psi(a), \Phi(a)) \text{ and } \int_{\Psi(a)}^{\Phi(a)} f(a,\eta)d\eta = \chi(a).$$

There are many nonlinear terms that satisfy the above conditions. Here we give one of them as an example.

$$f(a,u) = \sum_{n\in\mathbb{N}_+} \frac{\chi(n)}{2}\left(\left(u + \frac{\Phi(n)+\Psi(n)}{2}\right)\frac{2}{\Phi^2(n)-\Psi^2(n)}\right) e_{\{n\}\times[\Psi(n),\Phi(n)]}(a,u)$$

where $e_{M\times N}$ is the indicator function on $M \times N$. Then

$$\begin{aligned}
A &= \liminf_{\tau\to 0^+} \frac{\max_{|\zeta|\le\tau}\sum_{a\in\mathbb{Z}} F(a,\zeta(a))}{\tau^p} \\
&\le \lim_{n\to\infty} \frac{\max_{|\zeta|\le\Psi(n)}\sum_{a\in\mathbb{Z}} F(a,\zeta(a))}{\Psi^p(n)} \\
&= \lim_{n\to\infty} \frac{\sum_{a=n+1}^{\infty} F(a,\zeta(a))}{\Psi^p(n)} \\
&= \lim_{n\to\infty} \frac{\sum_{a=n+1}^{\infty} \chi(a)}{\Psi^p(n)} \\
&\le \lim_{n\to\infty} \frac{3\chi(n+1)}{\Psi^p(n)} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
B &= \limsup_{\tau\to 0^+} \frac{\sum_{a\in\mathbb{Z}} F(a,\zeta(a))}{\tau^p} \\
&\ge \lim_{n\to\infty} \frac{\sum_{a\in\mathbb{Z}} F(a,\zeta(a))}{\Phi^p(n)} \\
&\ge \lim_{n\to\infty} \frac{F(n,\zeta(n))}{\Phi^p(n)} \\
&\ge \lim_{n\to\infty} \frac{\chi(n)}{\Phi^p(n)} \\
&= +\infty.
\end{aligned}$$

Now it is easy to see that all the assumptions of Theorem 1.1 are satisfied, hence the corresponding conclusion can be delivered by Theorem 1.1.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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