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Research article

Existence of normalized solutions for the Schrödinger equation

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Abstract: In this paper, we devote to studying the existence of normalized solutions for the following Schrödinger equation with Sobolev critical nonlinearities.

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

where $N \geqslant 3$, $2 < q < 2 + \frac{4}{N}$, $p = 2^* = \frac{2N}{N-2}$, $a, \mu > 0$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Since the existence result for $2 + \frac{4}{N} < p < 2^*$ has been proved, using an approximation method, that is let $p \to 2^*$, we obtain that there exists a mountain-pass type solution for $p = 2^*$.

Keywords: normalized solutions; Schrödinger equation; Sobolev critical nonlinearities; approximation method; mountain-pass type solution
Mathematics Subject Classification: 35J15, 35J20, 35J91

1. Introduction

In this paper, we consider the existence of solutions for the following Schrödinger equation.

$$i\psi_t + \Delta\psi + \mu|\psi|^{q-2}\psi + |\psi|^{p-2}\psi = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 3$, $2 < q < 2 + \frac{4}{N}$ and $p = 2^* = \frac{2N}{N-2}$. The Schrödinger equation is a famous equation in Physics and there are numerous papers to study it, we refer the readers to [1–4] and references therein.

For (1.1), we are particularly interested in the stationary waves of the form $\psi(x, t) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{R}$. Then u satisfies the equation

$$-\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u \qquad \text{in } \mathbb{R}^N. \tag{1.2}$$

If we fix the L^2-norm of u, that is, let

$$u \in S_a := \{v \in H^1(\mathbb{R}^N) : \|v\|_2^2 = a^2\},$$

where $a > 0$ is a constant. Then the corresponding functional of (1.2) is

$$E_p(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|u\|_q^q - \frac{1}{p}\|u\|_p^p,$$

and λ appears as a Lagrange multiplier. Solutions of (1.2) with prescribed mass are always called normalized solutions. It seems that there is profound physical significance to study normalized solutions. In fact, for the Schrödinger equation, $|\psi(x,t)|^2$ represents the probability density of a single particle appearing in space x at time t. Naturally, there is

$$\int_{\mathbb{R}^N} |\psi(x,t)|^2 dx = 1.$$

Of course, in mathematics, we often consider

$$\int_{\mathbb{R}^N} |\psi(x,t)|^2 dx = a^2.$$

There are a lot of papers to study the normalized solutions of Schrödinger equations and it is impossible for us to provide complete references. We refer the readers to [5–11] and references therein. Moreover, we refer the readers to [12–14] for the normalized solutions of fractional Schrödinger equations and to [15–17] for the normalized solutions of Schrödinger systems.

When we study the normalized solutions, there will be a L^2-critical exponent $2 + \frac{4}{N}$, which comes from the Gagliardo-Nirenberg inequality [18]: for every $2 < p < 2^*$, there exists an optimal constant $C_{N,p}$ depending on N and p such that

$$\|u\|_p \leqslant C_{N,p}\|\nabla u\|_2^{\gamma_p}\|u\|_2^{1-\gamma_p} \qquad \forall u \in H^1(\mathbb{R}^N),$$

where

$$\gamma_p := \frac{N(p-2)}{2p}.$$

By the Gagliardo-Nirenberg inequality, it is not difficult to prove that if the nonlinearities of equation are L^2-subcritical, then the corresponding functional is bounded from below on S_a. For example,

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p$$

is bounded from below on S_a for $2 < p < 2 + \frac{4}{N}$ and global minimizers of $J|_{S_a}$ can be found, see [8, 19]. However, if the nonlinearities are L^2-supercritical, the functional is unbounded from below on S_a and it seems impossible to search for a global minimizer. The first paper to deal with L^2-supercritical is [5]. In [5], Jeanjean found the normalized solutions of mountain-pass type.

Compared with pure L^2-subcritical or L^2-supercritical case, the mixed case is more complicated. In [9], Soave studied (1.2) for $2 < q < 2 + \frac{4}{N} < p < 2^*$ under L^2 constraint. Since q is L^2-subcritical exponent and p is L^2-supercritical exponent, we call $\mu|u|^{q-2} + |u|^{p-2}u$ mixed nonlinearities. The first existence result of normalized solutions in Sobolev critical case was also obtained by Soave [10].

Since the L^2 constraint, there are some difficulties to observe the structure of $E_p|_{S_a}$. A possible method is to consider the function

$$\Psi_u^p(s) := E_p(s \star u) = \frac{1}{2}e^{2s}\|\nabla u\|_2^2 - \frac{\mu}{q}e^{q\gamma_q s}\|u\|_q^q - \frac{1}{p}e^{p\gamma_p s}\|u\|_p^p,$$

where

$$s \star u := e^{\frac{Ns}{2}} u(e^s \cdot).$$

It is not difficult to prove that $s \star u \in S_a$ for all $s \in \mathbb{R}$ if $u \in S_a$ and hence we can study the structure of Ψ_u^p to speculate the structure of $E_p|_{S_a}$.

If u is a critical point of $E_p|_{S_a}$, then 0 may be a critical point of Ψ_u^p. If 0 is a critical point of Ψ_u^p, then $(\Psi_u^p)'(0) = 0$, that is

$$\|\nabla u\|_2^2 = \mu\gamma_q\|u\|_q^q + \gamma_p\|u\|_p^p. \tag{1.3}$$

In fact, by Pohozaev identity, u satisfies (1.3) as long as u is a critical point of E_p. Now, we can define a manifold

$$\mathcal{P}_{a,p} := \{u \in S_a : P_p(u) = 0\},$$

where

$$P_p(u) := \|\nabla u\|_2^2 - \mu\gamma_q\|u\|_q^q - \gamma_p\|u\|_p^p.$$

It is clear that all critical points of $E_p|_{S_a}$ belong to $\mathcal{P}_{a,p}$ and $s \star u \in \mathcal{P}_{a,p}$ if and only if $(\Psi_u^p(s))' = 0$. We divide $\mathcal{P}_{a,p}$ into three parts.

$$\mathcal{P}_{a,p}^+ = \{u \in \mathcal{P}_{a,p} : (\Psi_u^p)''(0) > 0\} = \{u \in \mathcal{P}_{a,p} : 2\|\nabla u\|_2^2 > \mu q\gamma_q^2\|u\|_q^q + p\gamma_p^2\|u\|_p^p\},$$

$$\mathcal{P}_{a,p}^0 = \{u \in \mathcal{P}_{a,p} : (\Psi_u^p)''(0) = 0\} = \{u \in \mathcal{P}_{a,p} : 2\|\nabla u\|_2^2 = \mu q\gamma_q^2\|u\|_q^q + p\gamma_p^2\|u\|_p^p\},$$

and

$$\mathcal{P}_{a,p}^- = \{u \in \mathcal{P}_{a,p} : (\Psi_u^p)''(0) < 0\} = \{u \in \mathcal{P}_{a,p} : 2\|\nabla u\|_2^2 < \mu q\gamma_q^2\|u\|_q^q + p\gamma_p^2\|u\|_p^p\}.$$

Define

$$m(a,p) := \inf_{u \in \mathcal{P}_{a,p}} E_p(u) \qquad \text{and} \qquad m^\pm(a,p) := \inf_{u \in \mathcal{P}_{a,p}^\pm} E_p(u).$$

For $2 < q < 2 + \frac{4}{N} < p \leqslant 2^*$, since $q\gamma_q < 2$ and $p\gamma_p > 2$, the function Ψ_u^p may have two critical points on $\mathbb{R}$, one is local minimum point and the other is global maximum point. Moreover, if we assume s_u is the local minimum and t_u is the global maximum. Then, it is not difficulty to check that $s_u \star u \in \mathcal{P}_{a,p}^+$ and $t_u \star u \in \mathcal{P}_{a,p}^-$ (see [9, Lemma 5.3] and [10, Lemma 4.2] for more details). Therefore, it is natural to speculate that E_p has two critical points on S_a under appropriate assumptions, one is a local minimizer on S_a and is also a minimizer on $\mathcal{P}_{a,p}^+$, the other is a mountain-pass type critical point and is also a minimizer on $\mathcal{P}_{a,p}^-$.

In fact, the local minimizer and mountain-pass type solution of $E_p|_{S_a}$ for $2 < q < 2 + \frac{4}{N} < p < 2^*$ have been found by Soave, see [9, Theorem 1.3]. For $2 < q < 2 + \frac{4}{N} < p = 2^*$, Soave obtained the local minimum, but due to $H_{rad}^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, there are some difficulties to obtain the mountain-pass type solution (see Theorem 1.1 and Remark 1.1 in [10]). Therefore, it is natural to ask the following question:

(Q) Does $E_{2^*}|_{S_a}$ has a second critical point of mountain pass type when $2 < q < 2 + \frac{4}{N}$?

In [6], Jeanjean and Le proved $E_{2^*}|_{S_a}$ has a mountain-pass type solution and the solution is also a minimizer on $\mathcal{P}_{a,2^*}^-$ when $N \geqslant 4$. They constructed a minimax structure and proved a strict inequality $m^-(a,2^*) < m^+(a,2^*) + \frac{1}{N}S^{\frac{N}{2}}$ to obtain the compactness of a Palais-smale(PS) sequence. The proof of [6] is complicated especially the proof of the strict inequality, see Propositions 1.10, 1.11 and 1.12 for more details. After that, Wei and Wu [11] gave a simpler proof of $m^-(a,2^*) < m^+(a,2^*) + \frac{1}{N}S^{\frac{N}{2}}$ and proved that the answer is also positive for (Q) when $N = 3$. Different from [6], We and Wu didn't construct the minimax structure, but directly proved the convergence of the minimizing sequence for $m^-(a,2^*)$, see Lemma 3.1 and Proposition 3.1 of [11] for more details.

Our main goal is giving a new proof of (Q) and the method we call the Sobolev subcritical approximation method. The idea of the Sobolev subcritical approximation method is: by [9, Theorem1.3 (ii)], we know $E_p|_{S_a}$ has a mountain-pass type solution u_p when $2 + \frac{4}{N} < p < 2^*$. Let $p \to 2^*$, it is not difficult to prove that $u_p \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then, we prove that u is the solution of (1.2), $u_p \to u$ in $H^1(\mathbb{R}^N)$, u is a critical point of $E_{2^*}|_{S_a}$ and is the minimum of E_{2^*} on $\mathcal{P}^-_{a,2^*}$. Proving strong convergence is a crucial step in our proof, we also need use the strict inequality $m^-(a,2^*) < m^+(a,2^*) + \frac{1}{N}S^{\frac{N}{2}}$.

Let

$$C' = \left(\frac{2^* S^{\frac{2^*}{2}}(2-\gamma_q q)}{2(2^*-\gamma_q q)}\right)^{\frac{2-\gamma_q q}{2^*-2}} \frac{q(2^*-2)}{2C^q_{N,q}(2^*-\gamma_q q)} \tag{1.4}$$

and

$$C'' = \frac{22^*}{N\gamma_q C^q_{N,q}(2^*-\gamma_q q)}\left(\frac{\gamma_q q S^{\frac{N}{2}}}{2-\gamma_q q}\right)^{\frac{2-\gamma_q q}{2}}.$$

Define $\alpha(N,q) := \min\{C', C''\}$. Our main result can be stated as follows.

Theorem 1.1. *Let $N \geqslant 3$, $2 < q < 2 + \frac{4}{N}$, $p = 2^*$ and $a, \mu > 0$. Moreover, let us suppose that $\mu a^{q(1-\gamma_q)} < \alpha(N,q)$. Then $E_{2^*}|_{S_a}$ has a critical point of mountain-pass type which is positive, radially symmetric and solves* (1.2) for some $\lambda < 0$.

Remark 1.1. The definition of $\alpha(N,q)$ comes from [10, (1.6)] to ensure that Ψ^p_u has two critical points.

Remark 1.2. The Sobolev subcritical approximation method has been used by [20, Remark 1.3] and [7]. In [7], Li considered the normalized solutions of (1.2) with $2 + \frac{4}{N} < q < p = 2^*$ and proved (1.2) has a normalized ground state for every $\mu > 0$, see [7, Theorem 1.4]. Li solve an open problem

(Q') Does $E_{2^*}|_{S_a}$ have a ground state if $\mu > 0$ and $\mu a^{(1-\gamma_q)q}$ large?

which is raised by Soave [10, Page 7]. For $2 < q < 2 + \frac{4}{N} < p = 2^*$, if we follow the step of Li, the last inequality is invalid since $q\gamma_q < 2$ (see [7, Page 13]) and we can not prove $u \in S_a$. In fact, we refer some ideas of [10,11] to obtain strong convergence in $H^1(\mathbb{R}^N)$.

2. Preliminaries

In this section, we collect some results which will be used in the rest of the paper. First, let us recall the Sobolev inequality.

Lemma 2.1. *For every $N \geqslant 3$, there is an optimal constant $S > 0$ depending only on N such that*

$$S\|u\|^2_{2^*} \leqslant \|\nabla u\|^2_2 \qquad \forall u \in D^{1,2}(\mathbb{R}^N),$$

where $D^{1,2}(\mathbb{R}^N)$ denotes the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm $\|u\|_{D^{1,2}} := \|\nabla u\|_2$.

It is well known [21] that S is achieved by

$$U_{\varepsilon,y}(x) = [N(N-2)]^{\frac{N-2}{4}}\Big(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2}\Big)^{\frac{N-2}{2}} \qquad \text{for } \forall\varepsilon > 0 \text{ and } y \in \mathbb{R}^N,$$

and $U_{\varepsilon,y}$ satisfies the equation

$$-\Delta u = u^{2^*-1}, \qquad u > 0 \qquad \text{in } \mathbb{R}^N.$$

Moreover,

$$\|\nabla U_{\varepsilon,y}\|_2^2 = \|U_{\varepsilon,y}\|_{2^*}^{2^*} = S^{\frac{N}{2}}.$$

Let $C_{N,p}$ be the optimal constant of Gagliardo-Nirenberg inequality. Then, we have

Lemma 2.2. *Let* $2 < p < 2^*$*, then* $\lim_{p\to 2^*} C_{N,p} = S^{-\frac{1}{2}}$.

Proof. Denoting by $u_\varepsilon := \varphi U_{\varepsilon,0} \in H^1(\mathbb{R}^N)$, where $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a radial cut-off function with

$$0 \leqslant \varphi \leqslant 1, \qquad \varphi = 1 \text{ in } B_1 \qquad \text{and} \qquad \varphi = 0 \text{ in } B_2^c.$$

By the classical results [20], we have

$$\|\nabla u_\varepsilon\|_2^2 = \|u_\varepsilon\|_{2^*}^{2^*} = S^{\frac{N}{2}} + o_\varepsilon(1).$$

Since

$$|u_\varepsilon(x)|^p \leqslant |u_\varepsilon(x)|^2 + |u_\varepsilon(x)|^{2^*} \qquad \forall x \in \mathbb{R}^N,$$

the Lebesgue dominated convergence theorem implies $\lim_{p\to 2^*}\|u_\varepsilon\|_p^p = \|u_\varepsilon\|_{2^*}^{2^*}$. Using the Gagliardo-Nirenberg inequality, we have

$$\|u_\varepsilon\|_p \leqslant C_{N,p}\|\nabla u_\varepsilon\|_2^{\gamma_p}\|u_\varepsilon\|_2^{1-\gamma_p}.$$

Taking $p \to 2^*$, we obtain

$$\|u_\varepsilon\|_{2^*} \leqslant \liminf_{p\to 2^*} C_{N,p}\|\nabla u_\varepsilon\|_2,$$

which implies $S^{-\frac{1}{2}} \leqslant \liminf_{p\to 2^*} C_{N,p}$.

For every $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, using the Hölder inequality and the Sobolev inequality, we have

$$\|u\|_p \leqslant \|u\|_{2^*}^{\gamma_p}\|u\|_2^{1-\gamma_p} \leqslant S^{-\frac{\gamma_p}{2}}\|\nabla u\|_2^{\gamma_p}\|u\|_2^{1-\gamma_p}$$

By the definition of $C_{N,p}$, we obtain $S^{-\frac{\gamma_p}{2}} \geqslant C_{N,p}$. Therefore, $S^{-\frac{1}{2}} \geqslant \limsup_{p\to 2^*} C_{N,p}$.

3. Proof of Theorem 1.1

For every $0 < \mu < a^{q(\gamma_q-1)}\alpha(N,q)$. In order to use the existence result of Sobolev subcritical case [9, Theorem 1.3], μ should satisfy

$$0 < \mu < a^{q(\gamma_q-1)+\frac{(1-\gamma_p)p(2-\gamma_q q)}{\gamma_p p-2}}\Big(\frac{p(2-\gamma_q q)}{2C_{N,p}^p(\gamma_p p-\gamma_q q)}\Big)^{\frac{2-\gamma_q q}{\gamma_p p-2}}\frac{q(\gamma_p p-2)}{2C_{N,q}^q(\gamma_p p-\gamma_q q)} := \mu_p. \tag{3.1}$$

By Lemma 2.2, is it not difficult to prove that

$$\mu_p \to a^{q(\gamma_q-1)}C' \geqslant a^{q(\gamma_q-1)}\alpha(N,q)$$

as $p \to 2^*$, where C' is defined by (1.4). Therefore, μ satisfies (3.1) as long as p is close enough to 2^*.

Lemma 3.1. *We have*

$$\limsup_{p\to 2^*} m^-(a,p) \leqslant m^-(a,2^*).$$

Proof. For every $u \in S_a$, by [9, Lemma 5.3], there exists a unique $t_{p,u} \in \mathbb{R}$ such that $t_{p,u} \star u \in \mathcal{P}^-_{a,p}$, that is

$$e^{2t_{p,u}}\|\nabla u\|_2^2 = \mu\gamma_q e^{q\gamma_q t_{p,u}}\|u\|_q^q + \gamma_p e^{p\gamma_p t_{p,u}}\|u\|_p^p, \tag{3.2}$$

and

$$2e^{2t_{p,u}}\|\nabla u\|_2^2 < \mu q\gamma_q^2 e^{q\gamma_q t_{p,u}}\|u\|_q^q + p\gamma_p^2 e^{p\gamma_p t_{p,u}}\|u\|_p^p. \tag{3.3}$$

Since $q\gamma_q < 2$ and $p\gamma_p > 2$, by (3.2), we have

$$\left(\frac{\mu\gamma_q\|u\|_q^q}{\|\nabla u\|_2^2}\right)^{\frac{1}{2-q\gamma_q}} < e^{t_{p,u}} < \left(\frac{\|\nabla u\|_2^2}{\gamma_p\|u\|_p^p}\right)^{\frac{1}{p\gamma_p-2}}.$$

We know

$$|u(x)|^p \leqslant |u(x)|^2 + |u(x)|^{2^*} \quad \forall x \in \mathbb{R}^N,$$

the Lebesgue dominated convergence theorem implies $\lim_{p\to 2^*}\|u\|_p^p = \|u\|_{2^*}^{2^*}$. Therefore, there exists two constants $t_2 > t_1$ independent of p such that $t_{p,u} \in [t_1, t_2]$ when p close enough to 2^*. Up to a subsequence, we assume that $t_{p,u} \to t_u$ as $p \to 2^*$.

Let $p \to 2^*$, by (3.2) and (3.3), we obtain

$$e^{2t_u}\|\nabla u\|_2^2 = \mu\gamma_q e^{q\gamma_q t_u}\|u\|_q^q + e^{2^* t_u}\|u\|_{2^*}^{2^*},$$

and

$$2e^{2t_u}\|\nabla u\|_2^2 \leqslant \mu q\gamma_q^2 e^{q\gamma_q t_u}\|u\|_q^q + 2^* e^{2^* t_u}\|u\|_{2^*}^{2^*},$$

which implies $t_u \star u \in \mathcal{P}^-_{a,2^*} \cup \mathcal{P}^0_{a,2^*}$. From [10, Page 20], we know $\mathcal{P}^0_{a,2^*} = \emptyset$ and hence $t_u \star u \in \mathcal{P}^-_{a,2^*}$.

By the definition of $m^-(a,p)$, we have

$$m^-(a,p) \leqslant E_p(t_{p,u} \star u) = \frac{1}{2}e^{2t_{p,u}}\|\nabla u\|_2^2 - \frac{\mu}{q}e^{q\gamma_q t_{p,u}}\|u\|_q^q - \frac{1}{p}e^{p\gamma_p t_{p,u}}\|u\|_p^p,$$

which implies

$$\limsup_{p\to 2^*} m^-(a,p) \leqslant \limsup_{p\to 2^*} E_p(t_{p,u} \star u) = E_{2^*}(t_u \star u).$$

By the definition of $m^-(a,2^*)$ and the arbitrary of u, we know the conclusion holds.

The proof of the following two lemmas can be found in [11, Lemmas 3.1, 3.2].

Lemma 3.2. $0 < m^-(a,2^*) < m^+(a,2^*) + \frac{1}{N}S^{\frac{N}{2}}$.

Lemma 3.3. $m^\pm(a,2^*)$ *is non-increasing for* $0 < a < (\mu^{-1}\alpha(N,q))^{\frac{1}{q(1-\gamma_q)}}$.

Let $2+\frac{4}{N} < p_n < 2^*$ and $p_n \to 2^*$ as $n \to \infty$. By [9, Theorem 1.3 (ii)], there exists mountain-pass type solutions $\{u_n\} \in \mathcal{P}^-_{a,p_n}$ for $E_{p_n}|_{S_a}$ which are positive, radially symmetric such that $E_{p_n}(u_n) = m^-(a,p_n)$.

Lemma 3.4. $\{u_n\}$ *is bounded in* $H^1(\mathbb{R}^N)$.

Proof. By Lemma 3.1, we have

$$m^-(a,2^*)+1 \geqslant E_{p_n}(u_n) = \Big(\frac{1}{2}-\frac{1}{p_n\gamma_{p_n}}\Big)\|\nabla u_n\|_2^2 - \mu\gamma_q\Big(\frac{1}{q\gamma_q}-\frac{1}{p_n\gamma_{p_n}}\Big)\|u_n\|_q^q$$
$$\geqslant \Big(\frac{1}{2}-\frac{1}{p_n\gamma_{p_n}}\Big)\|\nabla u_n\|_2^2 - \mu\gamma_q\Big(\frac{1}{q\gamma_q}-\frac{1}{p_n\gamma_{p_n}}\Big)C_{N,q}^q a^{q(1-\gamma_q)}\|\nabla u_n\|_2^{q\gamma_q}$$

for n sufficiently large. Since $q\gamma_q < 2$, we know $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Up to a subsequence, there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, $u_n \to u$ in $L^r(\mathbb{R}^N)$ with $r \in (2,2^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Our main goal is to prove that u is the mountain-pass type solution of $E_{2^*}|_{S_a}$. Next, we prove u satisfies (1.2).

Lemma 3.5. *There exists $\lambda \leqslant 0$ such that*

$$-\Delta u = \lambda u + \mu u^{q-1} + u^{2^*-1}, \tag{3.4}$$

and $\lambda = 0$ if and only if $u \equiv 0$.

Proof. By [9, Theorem 1.3], there exists $\lambda_n < 0$ such that

$$-\Delta u_n = \lambda_n u_n + \mu u_n^{q-1} + u_n^{p_n-1}, \tag{3.5}$$

which together with $u_n \in \mathcal{P}^-_{a,p_n}$, implies

$$\lambda_n a^2 = \mu(\gamma_q - 1)\|u_n\|_q^q + (\gamma_{p_n} - 1)\|u_n\|_{p_n}^{p_n}. \tag{3.6}$$

Let $n \to \infty$, by (3.6), we have that there exists a $\lambda \leqslant 0$ such that $\lambda_n \to \lambda$ and

$$\lambda a^2 = \mu(\gamma_q - 1)\|u\|_q^q.$$

Therefore, $\lambda = 0$ if and only if $u \equiv 0$.

For every $\psi \in H^1(\mathbb{R}^N)$, since $\{u_n^{2^*-1}\}$ is bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$ and $\{u_n^{q-1}\}$ is bounded in $L^{\frac{q}{q-1}}(\mathbb{R}^N)$, by weak convergence, we have

$$\int_{\mathbb{R}^N} u_n^{2^*-1}\psi dx \to \int_{\mathbb{R}^N} u^{2^*-1}\psi dx \qquad \text{and} \qquad \int_{\mathbb{R}^N} u_n^{q-1}\psi dx \to \int_{\mathbb{R}^N} u^{q-1}\psi dx$$

as $n \to \infty$. We know that

$$|u_n(x)|^{p_n-1}|\psi(x)| \leqslant |u_n(x)|^{q-1}|\psi(x)| + |u_n(x)|^{2^*-1}|\psi(x)| \qquad \forall x \in \mathbb{R}^N.$$

Therefore, the Lebesgue dominated convergence theorem implies

$$\int_{\mathbb{R}^N} u_n^{p_n-1}\psi dx \to \int_{\mathbb{R}^N} u^{2^*-1}\psi dx \qquad \text{as } n \to \infty.$$

By (3.5), we have

$$0 = \int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\psi - \lambda_n u_n\psi - \mu u_n^{q-1}\psi - u_n^{p_n-1}\psi)dx$$
$$\to \int_{\mathbb{R}^N}(\nabla u\cdot\nabla\psi - \lambda u\psi - \mu u^{q-1}\psi - u^{2^*-1}\psi)dx,$$

as $n \to \infty$, which implies u satisfies (3.4).

Set $\|u\|_2 = c \leqslant a$. By Pohozaev identity, we know $u \in \mathcal{P}_{c,2^*}$. Thus, by [10, (4.2)],

$$E_{2^*}(u) \geqslant m(c, 2^*) = m^+(c, 2^*)$$

Lemma 3.6. *We have $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$ as $n \to \infty$.*

Proof. Let $v_n = u_n - u \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The Brézis-Lieb Lemma [22] implies

$$\|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla v_n\|_2^2 + o_n(1), \qquad \|u_n\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|v_n\|_{2^*}^{2^*} + o_n(1),$$

and

$$\|u_n\|_q^q = \|u\|_q^q + \|v_n\|_q^q + o_n(1) = \|u\|_q^q + o_n(1).$$

Since $u_n \in \mathcal{P}^-_{a,p_n}$, by the Young inequality, we know

$$\begin{aligned}
\|\nabla u_n\|_2^2 &= \mu\gamma_q\|u_n\|_q^q + \gamma_{p_n}\|u_n\|_{p_n}^{p_n} \\
&\leqslant \mu\gamma_q\|u_n\|_q^q + \gamma_{p_n}\Big(\frac{2^* - p_n}{2^* - q}\|u_n\|_q^q + \frac{p_n - q}{2^* - q}\|u_n\|_{2^*}^{2^*}\Big) \\
&= \mu\gamma_q\|u_n\|_q^q + \|u_n\|_{2^*}^{2^*} + o_n(1).
\end{aligned}$$

Therefore,

$$\|\nabla v_n\|_2^2 \leqslant \|v_n\|_{2^*}^{2^*} + o(1) \leqslant S^{-\frac{2^*}{2}}\|\nabla v_n\|_2^{2^*} + o_n(1). \tag{3.7}$$

We assume that $\|\nabla v_n\|_2^2 \to l$ as $n \to \infty$. By (3.7), we know $l = 0$ or $l \geqslant S^{\frac{N}{2}}$. If $l \geqslant S^{\frac{N}{2}}$, by Lemmas 3.1 and 3.3, we have

$$\begin{aligned}
m^-(a, 2^*) &\geqslant \limsup_{n\to\infty} m^-(a, p_n) = \limsup_{n\to\infty} E_{p_n}(u_n) \\
&= \limsup_{n\to\infty}\Big[\Big(\frac{1}{2} - \frac{1}{p_n\gamma_{p_n}}\Big)\|\nabla u_n\|_2^2 - \mu\gamma_q\Big(\frac{1}{q\gamma_q} - \frac{1}{p_n\gamma_{p_n}}\Big)\|u_n\|_q^q\Big] \\
&= \limsup_{n\to\infty}\Big[\Big(\frac{1}{2} - \frac{1}{2^*}\Big)\|\nabla u_n\|_2^2 - \mu\gamma_q\Big(\frac{1}{q\gamma_q} - \frac{1}{2^*}\Big)\|u_n\|_q^q\Big] \\
&= \limsup_{n\to\infty}\Big(\frac{1}{2} - \frac{1}{2^*}\Big)\|\nabla v_n\|_2^2 + \Big[\Big(\frac{1}{2} - \frac{1}{2^*}\Big)\|\nabla u\|_2^2 - \mu\gamma_q\Big(\frac{1}{q\gamma_q} - \frac{1}{2^*}\Big)\|u\|_q^q\Big] \\
&= \limsup_{n\to\infty}\Big(\frac{1}{2} - \frac{1}{2^*}\Big)\|\nabla v_n\|_2^2 + E_{2^*}(u) \\
&\geqslant \limsup_{n\to\infty}\Big(\frac{1}{2} - \frac{1}{2^*}\Big)\|\nabla v_n\|_2^2 + m^+(c, 2^*) \\
&\geqslant \frac{1}{N}S^{\frac{N}{2}} + m^+(a, 2^*),
\end{aligned}$$

which contradicts with Lemma 3.2. Thus, we obtain $l = 0$ which implies $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$.

Lemma 3.7. *We have $u \not\equiv 0$.*

Proof. Since $u_n \in \mathcal{P}^-_{a,p_n}$, we have

$$\|\nabla u_n\|_2^2 = \mu\gamma_q\|u_n\|_q^q + \gamma_{p_n}\|u_n\|_{p_n}^{p_n}, \tag{3.8}$$

and

$$2\|\nabla u_n\|_2^2 < \mu q\gamma_q^2\|u_n\|_q^q + p_n\gamma_{p_n}^2\|u_n\|_{p_n}^{p_n}. \tag{3.9}$$

Combining (3.8) and (3.9), there is

$$\begin{aligned}\left(2 - q\gamma_q\right)\|\nabla u_n\|_2^2 &< \gamma_{p_n}\left(p_n\gamma_{p_n} - q\gamma_q\right)\|u_n\|_{p_n}^{p_n} \\ &\leqslant \gamma_{p_n}\left(p_n\gamma_{p_n} - q\gamma_q\right)C_{N,p_n}^{p_n}a^{p_n(1-\gamma_{p_n})}\|\nabla u_n\|_2^{p_n\gamma_{p_n}}.\end{aligned}$$

That is

$$2 - q\gamma_q \leqslant \gamma_{p_n}\left(p_n\gamma_{p_n} - q\gamma_q\right)C_{N,p_n}^{p_n}a^{p_n(1-\gamma_{p_n})}\|\nabla u_n\|_2^{p_n\gamma_{p_n}-2}.$$

Let $n \to \infty$, by Lemma 2.2, we obtain

$$2 - q\gamma_q \leqslant \left(2^* - q\gamma_q\right)S^{-\frac{2^*}{2}}\|\nabla u\|_2^{2^*-2},$$

which implies $u \not\equiv 0$.

Remark 3.1. By Lemma 3.5, we have $\lambda < 0$.

Lemma 3.8. *$u_n \to u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$ and hence $u \in S_a$.*

Proof. The idea of this proof comes from the proof of [10, Proposition 3.1]. Multiplying $u_n - u$ on both sides of (3.4) and (3.5), integrating and then subtract, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(u_n - u)|^2dx - \int_{\mathbb{R}^N}(\lambda_n u_n - \lambda u)(u_n - u)dx = \\ &\int_{\mathbb{R}^N}\mu(|u_n|^{q-2}u_n dx - |u|^{q-2}u)(u_n - u)dx + \int_{\mathbb{R}^N}(|u_n|^{p_n-2}u_n - |u|^{2^*-2}u)(u_n - u)dx.\end{aligned} \tag{3.10}$$

By Lemma 3.6, since $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$, the first, third and fourth integrals of (3.10) tend to 0 as $n \to \infty$. Therefore,

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^N}(\lambda_n u_n - \lambda u)(u_n - u)dx = \lambda\lim_{n\to\infty}\int_{\mathbb{R}^N}(u_n - u)^2dx,$$

which implies $u_n \to u$ in $L^2(\mathbb{R}^N)$.

Remark 3.2. From Lemma 3.6, we get that $u_n \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$.

Proof of Theorem 1.1. By Lemma 3.5 and Remark 3.2, we just need to prove that $E_{2^*}(u) = m^-(a, 2^*)$ and $u \in \mathcal{P}_{a,2^*}^-$. Since $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$, by the Sobolev inequality, $u_n \to u$ in $L^{2^*}(\mathbb{R}^N)$. Therefore, combining Lemma 3.1, we have

$$m^-(a, 2^*) \leqslant E_{2^*}(u) = \lim_{n\to\infty}E_{p_n}(u_n) = \lim_{n\to\infty}m^-(a, p_n) \leqslant m^-(a, 2^*), \tag{3.11}$$

which implies $E_{2^*}(u) = m^-(a, 2^*)$. Let $n \to \infty$, by (3.8) and (3.9), we know that $u \in \mathcal{P}_{a,2^*}^- \cup \mathcal{P}_{a,2^*}^0$. Since $\mathcal{P}_{a,2^*}^0 = \emptyset$ (see [10, Page 20]), there is $u \in \mathcal{P}_{a,2^*}^-$.

Remark 3.3. From (3.11), we can get that $\lim_{p\to 2^*} m^-(a, p) = m^-(a, 2^*)$.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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