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*Research article*

# On a partially synchronizable system for a coupled system of wave equations in one dimension

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**Abstract:** In this paper, we study a partially synchronizable system for a coupled system of wave equations with different wave speeds in the framework of classical solutions in one dimensional. A partially synchronizable system is defined as a system with at least one partial synchronized solutions. In fact, we cannot consider partial synchronization as the case that the system has the same wave speeds, because the influence of different wave speeds cause only some of the function in a given space being in a partially synchronized state, rather than all functions. Therefore, we can only consider under what conditions the coupled system can have partially synchronized solutions. We will consider it in two ways. On the one hand, under the necessary conditions, we obtain an unclosed characteristic equation associated with the partially synchronizable state. We add conditions to the wave speed matrix and coupling matrix to make the equation closed. From this, the characteristic function can be obtained, and all partially synchronized solutions are obtained; then we obtain the conditions under which the initial value should be satisfied. On the other hand, we consider a system of three variables first, where there are only two synchronized variables. By subtracting them to obtain a new variable, the problem can be transformed into the problem wherein the system that satisfies the new variable should have only zero solutions. Then solving this problem can lead to obtaining the conditions required for a partially synchronized solution. After extending it to the case of $N$ variables, similar conclusions can be obtained.

**Keywords:** partially synchronized solution; partially synchronizable system; coupled system of wave equations; partial exact boundary synchronization; different wave speeds
**Mathematics Subject Classification:** 93B05, 93C20

## 1. Introduction

The research on the exact boundary synchronization of coupled system of wave equations mainly focuses on the case of the same wave speed [1, 3–8]. One of the important issues is that the system can still maintain synchronization after removal of all of the boundary controls. However, it is clear that not all systems can have this property, and only the systems that satisfy the compatibility conditions have such properties.

In [2], Lei et al. put forward the concept of a synchronizable system, that is, if a system has a synchronized solution, then it is called a synchronizable system. They studied the coupled system of wave equations with different wave speeds and obtained all of the synchronized initial values which can make the system have synchronized solutions.

Previous studies on the synchronizable system focused on the framework of weak solutions in $N$-dimensions. Here, we consider the synchronizable system and the partially synchronizable system in the framework of classical solutions in one dimension. We mainly study the coupled system of wave equations with different wave speeds.

We consider the following coupled system with different wave speeds:

$$U_{tt} - \Lambda U_{xx} + AU = 0, \tag{1.1}$$

where $U = (u_1, \cdots, u_N)^T$ is the state variable and $\Lambda = \text{diag}(c_1^2, \cdots, c_N^2)$ is the wave speed matrix with $c_i > 0$ $(i = 1, \cdots, N)$, which are not all equal. $A = (a_{ij})$ is an $N \times N$ coupling matrix with constant elements. $U_t$ represents the partial derivative with respect to time $t$, and $U_x$ represents the partial derivative with respect to the spatial variable $x$.

The system (1.1) satisfies the following homogeneous Dirichlet boundary conditions

$$x = 0 : U = 0, \quad x = L : U = 0, \tag{1.2}$$

and it has the following initial data

$$t = 0 : (U, U_t) = (U_0(x), U_1(x)). \tag{1.3}$$

For any given $m (0 < m < N, m \in \mathbb{N})$, we give the following definition

**Definition 1.1.** *The system given by* (1.1)–(1.2) *is called a partially synchronizable system if there is an initial value* $(U_0, U_1)$ *such that the solution* $U = U(t, x)$ *to the problem given by* (1.1)–(1.3) *satisfies the following partial synchronization property:*

$$u_{m+1} = \cdots = u_N \stackrel{def.}{=} \tilde{\tilde{u}}, \tag{1.4}$$

*where* $\tilde{\tilde{u}}$ *is called the partially synchronizable state.*

The initial value $(U_0, U_1)$ of a partially synchronized solution also satisfy the following partial synchronization properties:

$$u_{m+1,0}(0, x) \equiv \cdots \equiv u_{N0}(0, x), \quad u_{m+1,1}(0, x) \equiv \cdots \equiv u_{N1}(0, x). \tag{1.5}$$

If $(U_0, U_1) \equiv (0, 0)$, then the system admits a solution $U \equiv 0$, which is, of course, a partially synchronized solution. Thus, in the following discussion, it is natural to exclude some trivial situations.

In Section 2, we consider the case in which both $\Lambda$ and $A$ satisfy the compatibility conditions. In Section 3, we study the partially synchronizable system without the compatibility conditions for some cases.

## 2. The case with compatibility

In this section, we consider the partially synchronizable system, that is the system possesses partially synchronized solutions, for a coupled system of wave equations with different wave speeds.

Let

$$C_{m+1} = \begin{pmatrix} 0 & \cdots & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

that is, consider an $(N-m-1) \times N$ full-row rank matrix. Let $\{e_1, \cdots, e_m, e_{m+1}\}$ be $N$-dimensional column vectors defined as follows: for $1 \le s \le m$,

$$(e_s)_j = \begin{cases} 1, & j = s, \\ 0, & otherwise, \end{cases}$$

and for $s = m+1$,

$$(e_{m+1})_j = \begin{cases} 1, & m+1 \le j \le N, \\ 0, & otherwise. \end{cases}$$

It is obvious that

$$\mathrm{Ker}(C_{m+1}) = \mathrm{Span}\{e_1, \cdots, e_m, e_{m+1}\}.$$

Let $u = (u_1, \cdots, u_m, u_{m+1})^T$; then, the partial synchronization condition is equivalent to

$$U = u_1 e_1 + \cdots + u_m e_m + u_{m+1} e_{m+1} = (e_1, \cdots, e_m, e_{m+1})u.$$

The component forms of the system given by(1.1)–(1.2) are as follows

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2} - c_i^2 \dfrac{\partial^2 u_i}{\partial x^2} + \sum\limits_{j=1}^{N} a_{ij} u_j = 0, \quad (i = 1, \cdots, N) \\ x = 0 : u_1 = \cdots = u_N = 0, \\ x = L : u_1 = \cdots = u_N = 0. \end{cases} \tag{2.1}$$

Assume that there exists the initial value $(U_0, U_1)$ such that the system (1.1) has a partially synchronized solution $U = U(t, x) : u_{m+1} = \cdots = u_N$. From (2.1), we get

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2} - c_i^2 \dfrac{\partial^2 u_i}{\partial x^2} + a_{i1} u_1 + \cdots + a_{im} u_m + \tilde{a}_{i,m+1} u_{m+1} = 0, (i = 1, \cdots, m) \\ \dfrac{\partial^2 u_{m+1}}{\partial t^2} - c_k^2 \dfrac{\partial^2 u_{m+1}}{\partial x^2} + a_{k1} u_1 + \cdots + a_{km} u_m + \tilde{a}_{k,m+1} u_{m+1} = 0, (k = m+1, \cdots, N) \\ x = 0 : \ u_1 = \cdots = u_m = u_{m+1} = 0, \\ x = L : \ u_1 = \cdots = u_m = u_{m+1} = 0, \end{cases} \tag{2.2}$$

where

$$\tilde{a}_{i,m+1} = \sum_{j=m+1}^{N} a_{ij}, (i = 1, \cdots, m), \ \tilde{a}_{k,m+1} = \sum_{j=m+1}^{N} a_{kj}, (k = m+1, \cdots, N). \tag{2.3}$$

We denote

$$\tilde{\Lambda}_{m+1} = \mathrm{diag}(c_1^2, \cdots, c_m^2, c_k^2), \quad \tilde{A}_{m+1} = \begin{pmatrix} a_{11} & \cdots & a_{1m} & \tilde{a}_{1,m+1} \\ \cdots & & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} & \tilde{a}_{m,m+1} \\ a_{k1} & \cdots & a_{km} & \tilde{a}_{k,m+1} \end{pmatrix}$$

where $\tilde{\Lambda}_{m+1}$ and $\tilde{A}_{m+1}$ are related to $k$. Then the system (2.2) can be written as

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \tilde{\Lambda}_{m+1}\dfrac{\partial^2 u}{\partial x^2} + \tilde{A}_{m+1}u = 0, \\ x = 0 : u = 0, \\ x = L : u = 0. \end{cases} \tag{2.4}$$

By (2.2), we have that

$$-c_k^2\frac{\partial^2 u_{m+1}}{\partial x^2} + a_{k1}u_1 + \cdots + a_{km}u_m + \tilde{a}_{k,m+1}u_{m+1} = 0, (k = m+1, \cdots, N)$$

is indepndent of $k$, even though $\tilde{\Lambda}_{m+1}$ and $\tilde{A}_{m+1}$ may be related to $k$.

From (2.1), for $\forall i \in \{1, \cdots, m\}$ and $\forall k, l \in \{m+1, \cdots, N\}$, we get

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2} - c_i^2\dfrac{\partial^2 u_i}{\partial x^2} + a_{i1}u_1 + \cdots + a_{im}u_m + \tilde{a}_{i,m+1}u_{m+1} = 0, \\ (c_l^2 - c_k^2)\dfrac{\partial^2 u_{m+1}}{\partial x^2} + (a_{k1} - a_{l1})u_1 + \cdots + (a_{km} - a_{lm})u_m + (\tilde{a}_{k,m+1} - \tilde{a}_{l,m+1})u_{m+1} = 0, \\ x = 0 : \ u_1 = \cdots = u_m = u_{m+1} = 0, \\ x = L : \ u_1 = \cdots = u_m = u_{m+1} = 0, \end{cases} \tag{2.5}$$

When $c_{m+1} = \cdots = c_N$, we have

$$(a_{k1} - a_{l1})u_1 + \cdots + (a_{km} - a_{lm})u_m + (\tilde{a}_{k,m+1} - \tilde{a}_{l,m+1})u_{m+1} = 0.$$

If $u_1, \cdots, u_m, u_{m+1}$ are linearly independent, we get

$$a_{k1} = a_{l1}, \cdots, a_{km} = a_{lm}, \tilde{a}_{k,m+1} = \tilde{a}_{l,m+1}. \tag{2.6}$$

Therefore, we have the following conclusions

**Theorem 2.1.** *If the system given by* (1.1)–(1.2) *is a partially synchronizable system with $u_{m+1xx}$, where $u_1, \cdots, u_m, u_{m+1}$ are linearly independent, then matrices $\Lambda$ and $A$ satisfy the $C_{m+1}$-compatibility conditions $c_l = c_k$ ($\forall l, k \in \{m+1, \cdots, N\}$) and* (2.6)*, respectively. Conversely, if the system given by* (1.1)–(1.2) *satisfies the conditions $c_l = c_k$ ($\forall l, k \in \{m+1, \cdots, N\}$) and* (2.6)*, then for any initial value $(U_0, U_1) \in (C^2[0, L])^N \times (C^1[0, L])^N$ with the partial synchronization properties*

$$t = 0 : u_{m+1,0}(0, x) = \cdots = u_{N0}(0, x), \quad u_{m+1,1}(0, x) = \cdots = u_{N1}(0, x), \tag{2.7}$$

*there is a corresponding partially synchronized solution $U = (u_1, \cdots, u_m, u_{m+1}, \cdots, u_{m+1})^T$.*

Then consider the corresponding coupled system of wave equations with Dirichlet boundary conditions:

$$\begin{cases} U_{tt} - \Lambda U_{xx} + AU = 0, \\ x = 0 : U = H(t) \\ x = L : U = 0. \end{cases} \tag{2.8}$$

in which $H(t)$ is a boundary condition.

The system is a partially synchronizable system if there exists $T > 0$ such that, for any given initial value $(U_0, U_1) \in (C^2[0, L])^N \times (C^1[0, L])^N$, there exists a boundary conditions $H(t) \in (C^2[0, T])^N$ such that the solution to the problem (2.8) is $U = U(t, x)$ and (1.3) satisfies the following partial synchronization property:

$$t \geq T : u_{m+1}(t, x) \equiv \cdots \equiv u_N(t, x) \tag{2.9}$$

From Theorem 2.1, we have the following:

**Corollary 2.2.** *If matrices $\Lambda$ and $A$ satisfy the conditions $c_l = c_k$ $(\forall l, k \in \{m + 1, \cdots, N\})$ and* (2.6)*, then the partial exact boundary synchronization* (2.9) *of system* (2.8) *is equivalent to that for any given initial value $(U_0, U_1) \in (C^2[0, L])^N \times (C^1[0, L])^N$; also, there exists a boundary condition $H$ such that the corresponding solution $U = U(t, x)$ attains a partially synchronized state at time $t = T$:*

$$u_{m+1}(T, x) \equiv \cdots \equiv u_N(T, x), \quad u_{m+1t}(T, x) \equiv \cdots \equiv u_{Nt}(T, x). \tag{2.10}$$

## 3. The case without compatibility

When

$$c_l \neq c_k, \quad \forall l, k \in \{m + 1, \cdots, N\}, \tag{3.1}$$

from (2.5), we get

$$\begin{cases} -\dfrac{\partial^2 u_{m+1}}{\partial x^2} = \dfrac{a_{k1} - a_{l1}}{c_l^2 - c_k^2} u_1 + \cdots + \dfrac{a_{km} - a_{lm}}{c_l^2 - c_k^2} u_m + \dfrac{\tilde{a}_{k,m+1} - \tilde{a}_{l,m+1}}{c_l^2 - c_k^2} u_{m+1}, \\ x = 0 : \ u_1 = \cdots = u_m = u_{m+1} = 0, \\ x = L : \ u_1 = \cdots = u_m = u_{m+1} = 0. \end{cases} \tag{3.2}$$

Denote

$$\alpha_1 = \frac{a_{k1} - a_{l1}}{c_l^2 - c_k^2}, \cdots, \alpha_m = \frac{a_{km} - a_{lm}}{c_l^2 - c_k^2}, \quad \alpha_{m+1} = \frac{\tilde{a}_{k,m+1} - \tilde{a}_{l,m+1}}{c_l^2 - c_k^2}. \tag{3.3}$$

In (3.2), $-\dfrac{\partial^2 u_{m+1}}{\partial x^2}$ is uniquely expressed by $u_1, \cdots, u_m, u_{m+1}$; then, we know that $\alpha_1, \cdots, \alpha_m, \alpha_{m+1}$ are constants that are independent of $k, l$ $(k, l \in \{m + 1, \cdots, N\})$. This gives the relationship of the matrices $\Lambda$ and $A$: $\forall k, l \in \{m + 1, \cdots, N\}$:

$$\begin{cases} a_{l1} + \alpha_1 c_l^2 = a_{k1} + \alpha_1 c_k^2, \\ \cdots \\ a_{lm} + \alpha_m c_l^2 = a_{km} + \alpha_m c_k^2, \\ \tilde{a}_{l,m+1} + \alpha_{m+1} c_l^2 = \tilde{a}_{k,m+1} + \alpha_{m+1} c_k^2, \end{cases} \tag{3.4}$$

and (3.2) becomes

$$\begin{cases} -\dfrac{\partial^2 u_{m+1}}{\partial x^2} = \alpha_1 u_1 + \cdots + \alpha_m u_m + \alpha_{m+1} u_{m+1}, \\ x = 0 : \ u_1 = \cdots = u_m = u_{m+1} = 0, \\ x = L : \ u_1 = \cdots = u_m = u_{m+1} = 0. \end{cases} \tag{3.5}$$

Obviously, (3.5) is a necessary condition for the system given by (1.1)–(1.2) to have a partially synchronized solution. However, (3.5) is not a closed system, and it is not easy to obtain information about the partially synchrinizable state from it.

### *3.1. A special case*

An important special case is that (3.5) is a closed system. At this moment, $\alpha_1 = 0, \cdots, \alpha_m = 0$, that is, $\forall k, l \in \{m+1, \cdots, N\}$ and $a_{k1} = a_{l1}, \cdots, a_{km} = a_{lm}$, which means that

$$\begin{cases} a_{m+1,1} = \cdots = a_{N1}, \\ \cdots \\ a_{m+1,m} = \cdots = a_{Nm}. \end{cases} \tag{3.6}$$

Thus, (3.2) is reduced to a closed system

$$\begin{cases} -\dfrac{\partial^2 u_{m+1}}{\partial x^2} = \alpha_{m+1} u_{m+1}, \\ x = 0 : u_{m+1} = 0, \\ x = L : u_{m+1} = 0. \end{cases} \tag{3.7}$$

Hence, if (3.7) is ture, it must follow that

$$\alpha_{m+1} = \lambda \tag{3.8}$$

where $\lambda$ is the eigenvalue of $-\dfrac{d^2}{dx^2}$. We take

$$u_{m+1} = b(t) u_\lambda(x), \tag{3.9}$$

where $u_\lambda(x)$ is the eigenfunction of $-\dfrac{d^2}{dx^2}$ corresponding to the eigenvalue $\lambda$ and $u_\lambda(x) \not\equiv 0$. $u_\lambda(x)$ and $\lambda$ satisfy the following:

$$\begin{cases} -\dfrac{d^2 u_\lambda(x)}{dx^2} = \lambda u_\lambda(x), \\ x = 0 : u_\lambda(x) = 0, \\ x = L : u_\lambda(x) = 0. \end{cases}$$

From the previous discussion, we have that $\lambda = \dfrac{n^2\pi^2}{L^2}$ $(n \in \mathbb{N}_+)$. We also have the eigenfunction

$$u_\lambda(x) = C_2 \sin(\sqrt{\lambda}\, x) \tag{3.10}$$

where $C_2$ is a non-zero constant. Here, (3.3) becomes

$$\tilde{a}_{l,m+1} + \lambda c_l^2 = \tilde{a}_{k,m+1} + \lambda c_k^2, \quad (\forall l, k \in \{m+1, \cdots, N\}) \tag{3.11}$$

Substituting (3.10) into (2.2), we get that $b(t)$ satisfies

$$[\frac{\mathrm{d}^2 b(t)}{\mathrm{d}t^2} + (\tilde{a}_{k,m+1} + \lambda c_k^2) b(t)] u_\lambda(x) + a_{k1} u_1 + \cdots + a_{km} u_m = 0. \tag{3.12}$$

If $u_1, \cdots, u_m, u_\lambda(x)$ are linearly independent, we get

$$a_{k1} = 0, \cdots, a_{km} = 0, \quad \forall k \in \{m+1, \cdots, N\}. \tag{3.13}$$

Additionally,

$$\frac{\mathrm{d}^2 b(t)}{\mathrm{d}t^2} + (\tilde{a}_{k,m+1} + \lambda c_k^2) b(t) = 0, \tag{3.14}$$

where $\tilde{a}_{k,m+1} + \lambda c_k^2$ is a constant that is independent of $k$; it is denoted by

$$d = \tilde{a}_{k,m+1} + \lambda c_k^2. \tag{3.15}$$

Therefore, for any given initial conditions

$$t = 0: \quad b(0) = b_0, \;\; b_t(0) = b_1,$$

$b(t)$ satisfies the following:

$$b(t) = \begin{cases} b_0 \cosh(\sqrt{-d}\, t) + \dfrac{b_1}{\sqrt{-d}} \sinh(\sqrt{-d}\, t), & d < 0, \\ b_0 + b_1 t, & d = 0 \\ b_0 \cos(\sqrt{d}\; t) + \dfrac{b_1}{\sqrt{d}} \sin(\sqrt{d}\; t), & d > 0. \end{cases} \tag{3.16}$$

For any given $k$ ($k = m+1, \cdots, N$), $u_{m+1}$ satisfies

$$\begin{cases} \dfrac{\partial^2 u_{m+1}}{\partial t^2} - c_k^2 \dfrac{\partial^2 u_{m+1}}{\partial x^2} + \tilde{a}_{k,m+1} u_{m+1} = 0, \\ x = 0: u_{m+1} = 0, \quad x = L: u_{m+1} = 0, \\ t = 0: u_{m+1} = b_0 u_\lambda(x), \quad u_{m+1t}(0) = b_1 u_\lambda(x). \end{cases} \tag{3.17}$$

Conversely, if (3.8) and (3.13) holds, then the system is a partially synchoronizable system and it has at least one partially synchronized solution.

In fact, consider (3.17); its solution $u_{m+1}$ is given by (3.10) and (3.16).

Let $U = (u_1, \cdots, u_m, u_{m+1}, \cdots, u_{m+1})$; we get

$$\begin{cases} U_{tt} - \Lambda U_{xx} + AU = 0, \\ x = 0: U = 0, \quad x = L: U = 0, \\ t = 0: U_0(x) = \sum\limits_{i=1}^{m} u_{i0}(0, x) e_i + b_0 u_\lambda(x) e_{m+1}, \; U_1(x) = \sum\limits_{i=1}^{m} u_{i1}(0, x) e_i + b_1 u_\lambda(x) e_{m+1}. \end{cases}$$

Then, $U$ is a required partially synchronized solution.

If the system given by (1.1)–(1.2) satisfies the condition $c_l \neq c_k (\forall l, k \in \{m+1, \cdots, N\})$, and (3.13), then not all of the initial values with the partial synchronization property can have partially synchronized solutions. Therefore, when $c_l \neq c_k$ ($\forall l, k \in \{m+1, \cdots, N\}$), even if (1.1)–(1.2) represents a partially synchronizable system, it does not mean that all of the partially synchronized initial values of the system can have a partial synchronization solutions.

From the above, we can get the following theorem.

**Theorem 3.1.** *If the system given by* (1.1)–(1.2) *is a partially synchronizable system with* $c_l \neq c_k$ $(\forall l, k \in \{m+1, \cdots, N\})$ *and* (3.13)*, then the matrices* $\Lambda$ *and* $A$ *satisfy* (3.11)*. And, for the following initial conditions with the partial synchronization property*

$$t = 0 : U = \sum_{i=1}^{m} u_{i0}(0, x)e_i + b_0 u_\lambda e_{m+1}, \quad U_t = \sum_{i=1}^{m} u_{i1}(0, x)e_i + b_1 u_\lambda e_{m+1}, \tag{3.18}$$

*where* $u_\lambda$ *is given by* (3.10) *and* $(b_0, b_1) \neq 0$*, the system given by* (1.1)–(1.2) *has a solution with the partial synchronization property*

$$U = \sum_{i=1}^{m} u_i e_i + b(t)u_\lambda e_{m+1}, \tag{3.19}$$

*where* $b(t)$ *is given by the following equation*

$$\begin{cases} \dfrac{\mathrm{d}^2 b(t)}{\mathrm{d}t^2} + db(t) = 0, \\ t = 0 : \ b = b_0, \ b_t = b_1. \end{cases}$$

*Proof.* By (3.6), (3.2) reduces to (3.7), then condition (3.3) becomes

$$\alpha_{m+1} = \frac{\tilde{a}_{k,m+1} - \tilde{a}_{l,m+1}}{c_l^2 - c_k^2}.$$

If (3.7) is ture, $\alpha_{m+1}$ can only take the eigenvalue $\lambda$ of $-\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$; then, $\Lambda$ and $A$ satisfy (3.11). Moreover, by (3.7), $u_{m+1}$ has the form of (3.9). Substituting (3.9) into (2.2), we get

$$[\frac{\mathrm{d}^2 b(t)}{\mathrm{d}t^2} + (\tilde{a}_{k,m+1} + \lambda c_k^2)b(t)]u_\lambda(x) + a_{k1}u_1 + \cdots + a_{km}u_m = 0.$$

Then, since (3.13) holds and $u_\lambda(x) \not\equiv 0$, we get (3.14).

Conversely, if the system given by (1.1)–(1.2) has the partially synchronized initial condition (3.18), it is easy to verify that the solution (3.19) satisfies the conditions of the system given by (1.1)–(1.2) and the initial condition (3.18); thus, it is the partially synchronized solution.

### *3.2. A model*

We consider a sample system when $N = 3, m = 1$, as follows:

$$\begin{cases} u_{1tt} - c_1^2 u_{1xx} = 0, \\ u_{2tt} - c_2^2 u_{2xx} + a_{21}u_1 = 0, \\ u_{3tt} - c_3^2 u_{3xx} + a_{31}u_1 = 0, \end{cases} \tag{3.20}$$

with the boundary conditions

$$x = 0 : u_1 = u_2 = u_3 = 0, \quad x = L : u_1 = u_2 = u_3 = 0, \tag{3.21}$$

as well as the partial synchronization initial conditions

$$t = 0 : u_2(0, x) = u_3(0, x), \quad u_{2t}(0, x) = u_{3t}(0, x). \tag{3.22}$$

Since $u_2 = u_3$, we get

$$\begin{cases} u_{1tt} - c_1^2 u_{1xx} = 0, \\ u_{2tt} - c_2^2 u_{2xx} + a_{21} u_1 = 0, \\ u_{2tt} - c_3^2 u_{2xx} + a_{31} u_1 = 0. \end{cases}$$

The system is equivalent to

$$\begin{cases} u_{1tt} - c_1^2 u_{1xx} = 0, \\ u_{2tt} - c_2^2 u_{2xx} + a_{21} u_1 = 0, \\ -(c_2^2 - c_3^2) u_{2xx} + (a_{21} - a_{31}) u_1 = 0. \end{cases} \tag{3.23}$$

If $c_2 = c_3$, we suppose that $u_1$ and $u_2$ are linearly indepedent; then, there exist $a_{21} = a_{31}$ and $a_{22} + a_{23} = a_{32} + a_{33}$. We have discussed this situation in Theorem 2.1.

If $c_2 \neq c_3$, we get

$$-u_{2xx} + \frac{a_{21} - a_{31}}{c_2^2 - c_3^2} u_1 = 0. \tag{3.24}$$

Substituting this formula into (3.23), we have

$$u_{2tt} + \left( \frac{c_2^2 a_{31} - c_3^2 a_{21}}{c_2^2 - c_3^2} \right) u_1 = 0. \tag{3.25}$$

From (3.24) and (3.25), we can get

$$u_{2xxtt} = \frac{a_{21} - a_{31}}{c_2^2 - c_3^2} u_{1tt}$$

and

$$u_{2ttxx} = \left( \frac{c_3^2 a_{21} - c_2^2 a_{31}}{c_2^2 - c_3^2} \right) u_{1xx}.$$

Hence, from (3.23),

$$u_{1tt} = c_1^2 u_{1xx}.$$

Then, we get

$$u_{2xxtt} = \frac{a_{21} - a_{31}}{c_2^2 - c_3^2} u_{1tt} = \frac{c_1^2 (a_{21} - a_{31})}{c_2^2 - c_3^2} u_{1xx}.$$

Assume that $u_2$ is smooth enough; then, $u_{2ttxx} = u_{2xxtt}$. Thus, we have

$$\frac{c_1^2 a_{21} - c_1^2 a_{31}}{c_2^2 - c_3^2} = \frac{c_3^2 a_{21} - c_2^2 a_{31}}{c_2^2 - c_3^2},$$

that is,

$$\frac{a_{21}}{c_2^2 - c_1^2} = \frac{a_{31}}{c_3^2 - c_1^2}. \tag{3.26}$$

This means that matrices $\Lambda$ and $A$ satisfy the compatibility condition (3.26). Obviously, this is a necessary condition.

Conversely, we consider whether the system (3.20) is a partially synchronizable system when condition (3.26) holds. Let $w = u_2 - u_3$; from (3.20), we get

$$w_{tt} - c_3^2 w_{xx} + (c_3^2 - c_2^2)u_{2xx} + (a_{21} - a_{31})u_1 = 0. \tag{3.27}$$

From the boundary condition (3.21) and initial condition (3.22), we have

$$x = 0 : w = 0, \quad x = L : w = 0,$$

and

$$t = 0 : w = 0, \quad w_t = 0.$$

If

$$(c_3^2 - c_2^2)u_{2xx} + (a_{21} - a_{31})u_1 = 0,$$

i.e., if (3.24) holds, we get that $w \equiv 0$; then, the system (3.20) is a partially synchronizable system. Therefore, (3.24) is a necessary and sufficient condition for system (3.20) to realize partial synchronization.

Let $W = w_{tt} - c_3^2 w_{xx}$; from (3.27), we have

$$W_{tt} + (c_3^2 - c_2^2)u_{2xxtt} + (a_{21} - a_{31})u_{1tt} = 0,$$

and

$$W_{xx} + (c_3^2 - c_2^2)u_{2xxxx} + (a_{21} - a_{31})u_{1xx} = 0.$$

Then, we get

$$W_{tt} - c_2^2 W_{xx} + (c_3^2 - c_2^2)(u_{2tt} - c_2^2 u_{2xx})_{xx} + (a_{21} - a_{31})(u_{1tt} - c_1^2 u_{1xx} + (c_1^2 - c_2^2)u_{1xx}) = 0.$$

From (3.20), we have

$$u_{2tt} - c_2^2 u_{2xx} = -a_{21}u_1, \quad u_{1tt} - c_1^2 u_{1xx} = 0;$$

then

$$(u_{2tt} - c_2^2 u_{2xx})_{xx} = -a_{21}u_{1xx}.$$

Hence,

$$W_{tt} - c_2^2 W_{xx} + [-(c_3^2 - c_2^2)a_{21} + (a_{21} - a_{31})(c_1^2 - c_2^2)]u_{1xx} = 0.$$

According to the condition (3.26), it is easy to see that

$$-(c_3^2 - c_2^2)a_{21} + (a_{21} - a_{31})(c_1^2 - c_2^2) = (c_1^2 - c_3^2)a_{21} - (c_1^2 - c_2^2)a_{31} = 0.$$

Hence,

$$W_{tt} - c_2^2 W_{xx} = 0. \tag{3.28}$$

If the initial and boundary conditions of $W$ satisfy

$$t = 0 : W(0, x) = 0, \quad W_t(0, x) = 0,$$

and

$$x = 0 : W(t, 0) = 0, \quad x = L : W(t, L) = 0,$$

then $W \equiv 0$; we immediately get $w \equiv 0$. Hence, (3.20) is a partially synchronizable system.

From (3.27), $W = -[(c_3^2 - c_2^2)u_{2xx} - (a_{21} - a_{31})u_1]$; then, $W_t = -[(c_3^2 - c_2^2)u_{2xxt} - (a_{21} - a_{31})u_{1t}]$; we need the following:

$$W(0, x) = -[(c_3^2 - c_2^2)u_{2xx} - (a_{21} - a_{31})u_1](0, x) = 0$$

and

$$W_t(0, x) = -[(c_3^2 - c_2^2)u_{2xxt} - (a_{21} - a_{31})u_{1t}](0, x) = 0.$$

Then, we have

$$u_{2xx}(0, x) = \frac{a_{21} - a_{31}}{c_2^2 - c_3^2} u_1(0, x), \tag{3.29}$$

and

$$u_{2xxt}(0, x) = \frac{a_{21} - a_{31}}{c_2^2 - c_3^2} u_{1t}(0, x). \tag{3.30}$$

Regarding the boundary conditions

$$\begin{aligned} x = 0 :& W(t, 0) = -(c_3^2 - c_2^2)u_{2xx}(t, 0) + (a_{21} - a_{31})u_1(t, 0), \\ x = L :& W(t, L) = -(c_3^2 - c_2^2)u_{2xx}(t, L) + (a_{21} - a_{31})u_1(t, L). \end{aligned}$$

From (3.20), we have

$$u_{2xx}(t, 0) = \frac{u_{2tt} + a_{21}u_1}{c_2^2}(t, 0), \quad u_{2xx}(t, L) = \frac{u_{2tt} + a_{21}u_1}{c_2^2}(t, L).$$

By (3.21), $u_{2tt}(t, 0) = 0$ and $u_{2tt}(t, L) = 0$, using (3.21) again, we have

$$x = 0 : W(t, 0) = 0, \quad x = L : W(t, L) = 0.$$

After the above discussion, we have the following conclusions.

**Theorem 3.2.** *If the system given by* (3.20)–(3.21) *is a partially synchronizable system, then matrices* $\Lambda$ *and* $A$ *satisfy the compatibility condition* (3.26). *Conversely, if* (3.26) *is satisfied for the initial value* $(U_0, U_1)$ *with the partial synchronization property* (3.22) *which also satisfies the conditions* (3.29)–(3.30)*, then the system given by* (3.20)–(3.21) *has a solution* $U = U(t, x)$ *satisfying the partial synchronazition condition*

$$u_2 = u_3.$$

*3.3. A general case of N=3*

When $N = 3$, $m = 1$, the system (1.1) can be written as

$$\begin{cases} u_{1tt} - c_1^2 u_{1xx} + a_{11}u_1 + a_{12}u_2 + a_{13}u_3 = 0, \\ u_{2tt} - c_2^2 u_{2xx} + a_{21}u_1 + a_{22}u_2 + a_{23}u_3 = 0, \\ u_{3tt} - c_3^2 u_{3xx} + a_{31}u_1 + a_{32}u_2 + a_{33}u_3 = 0, \end{cases} \tag{3.31}$$

with the boundary conditions

$$x = 0 : u_1 = u_2 = u_3 = 0, \quad x = L : u_1 = u_2 = u_3 = 0, \tag{3.32}$$

as well as the initial conditions with the partial synchronization property:

$$t = 0 : u_2(0, x) = u_3(0, x), \quad u_{2t}(0, x) = u_{3t}(0, x). \tag{3.33}$$

And, $m = 1$ means that $u_2 = u_3$; then, the system (3.31) becomes

$$\begin{cases} u_{1tt} - c_1^2 u_{1xx} + a_{11}u_1 + (a_{12} + a_{13})u_2 = 0, \\ u_{2tt} - c_2^2 u_{2xx} + a_{21}u_1 + (a_{22} + a_{23})u_2 = 0, \\ u_{2tt} - c_3^2 u_{2xx} + a_{31}u_1 + (a_{32} + a_{33})u_2 = 0. \end{cases} \tag{3.34}$$

Therefore, we get a system that is equivalent to (3.34), as follows:

$$\begin{cases} u_{1tt} - c_1^2 u_{1xx} + a_{11}u_1 + (a_{12} + a_{13})u_2 = 0, \\ u_{2tt} - c_2^2 u_{2xx} + a_{21}u_1 + (a_{22} + a_{23})u_2 = 0, \\ -(c_2^2 - c_3^2)u_{2xx} + (a_{21} - a_{31})u_1 + (a_{22} + a_{23} - a_{32} - a_{33})u_2 = 0. \end{cases} \tag{3.35}$$

When $c_2 = c_3$, we require that $u_1$ and $u_2$ be linearly independent. From system (3.34), we can get

$$a_{21} = a_{31}, \quad a_{22} + a_{23} = a_{32} + a_{33}.$$

We have discussed this situation in Theorem 2.1.

When $c_2 \neq c_3$, let $u_2 = u_3$; from the third equation in (3.35), we get

$$u_{2xx} - \frac{a_{21} - a_{31}}{c_2^2 - c_3^2}u_1 - \frac{a_{22} + a_{23} - a_{32} - a_{33}}{c_2^2 - c_3^2}u_2 = 0, \tag{3.36}$$

and, substituting (3.36) into the second equation in (3.35), we obtain

$$u_{2tt} + \frac{c_2^2 a_{31} - c_3^2 a_{21}}{c_2^2 - c_3^2}u_1 + \frac{c_2^2(a_{32} + a_{33}) - c_3^2(a_{22} + a_{23})}{c_2^2 - c_3^2}u_2 = 0. \tag{3.37}$$

From (3.36) and (3.37), we have

$$u_{2xxtt} = \frac{a_{21} - a_{31}}{c_2^2 - c_3^2}u_{1tt} + \frac{a_{22} + a_{23} - a_{32} - a_{33}}{c_2^2 - c_3^2}u_{2tt},$$

$$u_{2ttxx} = -\frac{c_2^2 a_{31} - c_3^2 a_{21}}{c_2^2 - c_3^2}u_{1xx} - \frac{c_2^2(a_{32} + a_{33}) - c_3^2(a_{22} + a_{23})}{c_2^2 - c_3^2}u_{2xx}.$$

Assume that $u_2$ is smooth enough, then, $u_{2xxtt} = u_{2ttxx}$; thus, we get

$$\begin{aligned} &\frac{a_{21} - a_{31}}{c_2^2 - c_3^2}u_{1tt} + \frac{a_{22} + a_{23} - a_{32} - a_{33}}{c_2^2 - c_3^2}u_{2tt} + \frac{c_2^2 a_{31} - c_3^2 a_{21}}{c_2^2 - c_3^2}u_{1xx} \\ &+ \frac{c_2^2(a_{32} + a_{33}) - c_3^2(a_{22} + a_{23})}{c_2^2 - c_3^2}u_{2xx} = 0. \end{aligned}$$

Substituting (3.31), (3.36) and (3.37) into the above equation, we get

$$\begin{aligned}&\frac{a_{21}(c_1^2-c_3^2)-a_{31}(c_1^2-c_2^2)}{c_2^2-c_3^2}u_{1xx}-\frac{(a_{11}-a_{32}-a_{33})a_{21}-(a_{11}-a_{22}-a_{23})a_{31}}{c_2^2-c_3^2}u_1\\&-\frac{(a_{21}-a_{31})(a_{12}+a_{13})}{c_2^2-c_3^2}u_2=0.\end{aligned}\tag{3.38}$$

It can be assumed that $u_{1xx}$, $u_1$ and $u_2$ are linearly independent. Then, we obtain

$$\frac{a_{21}(c_1^2-c_3^2)-a_{31}(c_1^2-c_2^2)}{c_2^2-c_3^2}=0,$$

$$\frac{(a_{11}-a_{32}-a_{33})a_{21}-(a_{11}-a_{22}-a_{23})a_{31}}{c_2^2-c_3^2}=0$$

and

$$\frac{(a_{21}-a_{31})(a_{12}+a_{13})}{c_2^2-c_3^2}=0.$$

From the above, and under the assumption that $c_1$, $c_2$ and $c_3$ are not equal to each other, we get

$$\frac{a_{21}}{a_{31}}=\frac{c_2^2-c_1^2}{c_3^2-c_1^2},\tag{3.39}$$

$$\frac{a_{21}}{a_{31}}=\frac{a_{22}+a_{23}-a_{11}}{a_{32}+a_{33}-a_{11}}\tag{3.40}$$

and

$$a_{12}+a_{13}=0.\tag{3.41}$$

Obviously, (3.39)–(3.41) are necessary conditions for the system (3.31) to be a partially synchronizable system.

Conversely, we consider whether the system (3.31) can realize partial synchronization when conditions (3.39)–(3.41) are satisfied. Denote $w=u_2-u_3$; then, $w(0,x)=0$, $w_t(0,x)=0$ and $w$ satisfies

$$w_{tt}-c_3^2w_{xx}-(a_{23}-a_{33})w+(c_3^2-c_2^2)u_{2xx}+(a_{21}-a_{31})u_1+(a_{22}+a_{23}-a_{32}-a_{33})u_2=0,$$

with the boundary conditions

$$x=0:w(t,0)=0,\quad x=L:w(t,L)=0.$$

If

$$(c_3^2-c_2^2)u_{2xx}+(a_{21}-a_{31})u_1+(a_{22}+a_{23}-a_{32}-a_{33})u_2=0,\tag{3.42}$$

then we have that $w(t,x)=0$; hence, (3.31) is a partially synchronizable system. Therefore, (3.42) is not only a necessary condition for the system (3.31) to realize partial synchronization, but it is also a sufficient condition. However, (3.42) is not a self-closed equation, which is difficult to solve. We want to obtain the algebraic conditions for the coupling matrix $A$ and the wave speed matrix $\Lambda$.

Denote $W = w_{tt} - c_3^2 w_{xx} - (a_{23} - a_{33})w$; then,

$$W + (c_3^2 - c_2^2)u_{2xx} + (a_{21} - a_{31})u_1 + (a_{22} + a_{23} - a_{32} - a_{33})u_2 = 0. \tag{3.43}$$

We can get

$$W_{tt} + (c_3^2 - c_2^2)u_{2xxtt} + (a_{21} - a_{31})u_{1tt} + (a_{22} + a_{23} - a_{32} - a_{33})u_{2tt} = 0$$

and

$$W_{xx} + (c_3^2 - c_2^2)u_{2xxxx} + (a_{21} - a_{31})u_{1xx} + (a_{22} + a_{23} - a_{32} - a_{33})u_{2xx} = 0.$$

From this we have

$$\begin{aligned} &W_{tt} - c_2^2 W_{xx} + (c_3^2 - c_2^2)(u_{2tt} - c_2^2 u_{2xx})_{xx} + (a_{21} - a_{31})(u_{1tt} - c_1^2 u_{1xx} + (c_1^2 - c_2^2)u_{1xx}) \\ &(a_{23} + a_{33} - a_{32} - a_{33})(u_{2tt} - c_2^2 u_{2xx}) = 0. \end{aligned} \tag{3.44}$$

From (3.31), it follows that

$$\begin{aligned} u_{2tt} - c_2^2 u_{2xx} &= -(a_{21}u_1 + a_{22}u_2 + a_{23}(u_2 - w)) \\ &= -a_{21}u_1 - (a_{22} + a_{23})u_2 + a_{23}w, \end{aligned}$$

$$(u_{2tt} - c_2^2 u_{2xx})_{xx} = -a_{21}u_{1xx} - (a_{22} + a_{23})u_{2xx} - a_{23}w_{xx},$$

and

$$\begin{aligned} u_{1tt} - c_1^2 u_{1xx} &= -(a_{11}u_1 + a_{12}u_2 + a_{13}(u_2 - w)) \\ &= -a_{11}u_1 - (a_{12} + a_{13})u_2 + a_{13}w. \end{aligned}$$

Substituting the above formulas into (3.44), we get

$$\begin{aligned} W_{tt} - c_2^2 W_{xx} &+ (c_3^2 - c_2^2)[-a_{21}u_{1xx} - (a_{22} + a_{23})u_{2xx} + a_{23}w_{xx}] \\ &+ (a_{21} - a_{31})[-a_{11}u_1 - (a_{12} + a_{13})u_2 + a_{13}w + (c_1^2 - c_2^2)u_{1xx}] \\ &+ (a_{23} + a_{33} - a_{32} - a_{33})[-a_{21}u_1 - (a_{22} + a_{23})u_2 + a_{23}w] = 0, \end{aligned}$$

i.e.,

$$\begin{aligned} W_{tt} - c_2^2 W_{xx} &+ (c_3^2 - c_2^2)a_{23}w_{xx} + [(a_{21} - a_{31})a_{13} + (a_{23} + a_{33} - a_{32} - a_{33})a_{23}]w \\ &+ [-(c_3^2 - c_2^2)a_{21} + (a_{21} - a_{31})(c_1^2 - c_2^2)]u_{1xx} - (c_3^2 - c_2^2)(a_{22} + a_{23})u_{2xx} \\ &- [(a_{21} - a_{31})a_{11} + (a_{23} + a_{33} - a_{32} - a_{33})a_{21}]u_1 \\ &- [(a_{21} - a_{31})(a_{12} + a_{13}) + (a_{23} + a_{33} - a_{32} - a_{33})(a_{23} + a_{33})]u_2 = 0. \end{aligned} \tag{3.45}$$

By (3.39)–(3.41), the above equation can be simplified as follows:

$$\begin{aligned} W_{tt} - c_2^2 W_{xx} &+ (c_3^2 - c_2^2)a_{23}w_{xx} + [(a_{21} - a_{31})a_{13} + (a_{23} + a_{33} - a_{32} - a_{33})a_{23}]w \\ &- (c_3^2 - c_2^2)(a_{22} + a_{23})u_{2xx} - [(a_{21} - a_{31})(a_{23} + a_{33})]u_1 \\ &- [(a_{23} + a_{33} - a_{32} - a_{33})(a_{23} + a_{33})]u_2 = 0. \end{aligned}$$

Then, conditions (3.39)–(3.41) are not sufficient to close the equation (3.45). We want $W$ to satisfy a self-closed system. For this purpose, we can assume that

$$a_{23} = 0, a_{13} = 0, a_{22} + a_{23} = 0. \tag{3.46}$$

Combining this with (3.39)–(3.41), we get

$$a_{12} = 0,\ a_{13} = 0,\ a_{22} = 0,\ a_{23} = 0,\ \ \frac{a_{21}}{a_{31}} = \frac{c_2^2 - c_1^2}{c_3^2 - c_1^2},\ \ \frac{a_{21}}{a_{31}} = \frac{-a_{11}}{a_{32} + a_{33} - a_{11}}; \tag{3.47}$$

then, $W$ satisfies the conditions for a self-closed system:

$$W_{tt} - c_2^2 W_{xx} = 0.$$

Obviously, the requirements of (3.47) are stronger than those of (3.39)–(3.41).

If the initial and boundary conditions of $W$ satisfy

$$t = 0 : W(0, x) = 0,\ \ W_t(0, x) = 0$$

and

$$x = 0 : W(t, 0) = 0,\ \ x = L : W(t, L) = 0,$$

then $W \equiv 0$; we immediately get $w \equiv 0$. Hence, (3.31) is a partially synchronizable system.

From (3.43), we get that $W = -(c_3^2 - c_2^2)u_{2xx} - (a_{21} - a_{31})u_1 + (a_{32} + a_{33})u_2$; then,

$$W_t = -(c_3^2 - c_2^2)u_{2xxt} - (a_{21} - a_{31})u_{1t} + (a_{32} + a_{33})u_{2t};$$

we must have

$$W(0, x) = -(c_3^2 - c_2^2)u_{2xx}(0, x) - (a_{21} - a_{31})u_1(0, x) + (a_{32} + a_{33})u_2(0, x) = 0$$

and

$$W_t(0, x) = -(c_3^2 - c_2^2)u_{2xxt}(0, x) - (a_{21} - a_{31})u_{1t}(0, x) + (a_{32} + a_{33})u_{2t}(0, x) = 0.$$

Thus,

$$u_{2xx}(0, x) = -\frac{a_{21} - a_{31}}{c_3^2 - c_2^2}u_1(0, x) + \frac{a_{32} + a_{33}}{c_3^2 - c_2^2}u_2(0, x), \tag{3.48}$$

and

$$u_{2xxt}(0, x) = -\frac{a_{21} - a_{31}}{c_3^2 - c_2^2}u_{1t}(0, x) + \frac{a_{32} + a_{33}}{c_3^2 - c_2^2}u_{2t}(0, x). \tag{3.49}$$

Regarding the boundary conditions

$$\begin{aligned} x = 0 :& W(t, 0) = -(c_3^2 - c_2^2)u_{2xx}(t, 0) - (a_{21} - a_{31})u_1(t, 0) + (a_{32} + a_{33})u_2(t, 0), \\ x = L :& W(t, L) = -(c_3^2 - c_2^2)u_{2xx}(t, L) - (a_{21} - a_{31})u_1(t, L) + (a_{32} + a_{33})u_2(t, L). \end{aligned}$$

From (3.31) and (3.46), we have

$$u_{2xx}(t, 0) = \frac{u_{2tt} + a_{21}u_1}{c_2^2}(t, 0), \quad u_{2xx}(t, L) = \frac{u_{2tt} + a_{21}u_1}{c_2^2}(t, L).$$

By (3.21), $u_{2tt}(t, 0) = 0$ and $u_{2tt}(t, L) = 0$, using (3.21) agian, we have

$$x = 0 : W(t, 0) = 0,\ \ x = L : W(t, L) = 0.$$

From the above discussion, we get the following theorem.

**Theorem 3.3.** *If the system given by* (3.31)–(3.32) *is a partially synchronizable system in which* $u_{1xx}$*,* $u_1$ *and* $u_2$ *are linearly independent then matrix* $\Lambda$ *and matrix* $A$ *satisfy the conditions* (3.39)–(3.41)*. Conversely, if condition* (3.47) *holds for the initial value* $(U_0, U_1)$ *with the partial synchronization property* (3.33)*, which satisfies the conditions* (3.48)–(3.49)*, then the system given by* (3.31)–(3.32) *has a solution* $U = U(t, x)$ *satisfying the partial synchronazition condition*

$$u_2 = u_3.$$

**Remark 3.4.** *The status of variables* $u_2$ *and* $u_3$ *in* (3.31) *are equal. In the previous processing step, we retained* $u_2$*, eliminated* $u_3$ *and obtained the condition* (3.47)*. In fact, if we keep* $u_3$ *and eliminate* $u_2$*, we can obtain the following conditions:*

$$a_{12} = 0,\ a_{13} = 0,\ a_{32} = 0,\ a_{33} = 0,\ \frac{a_{21}}{a_{31}} = \frac{c_2^2 - c_1^2}{c_3^2 - c_1^2},\ \frac{a_{21}}{a_{31}} = \frac{a_{22} + a_{23} - a_{11}}{-a_{11}}. \tag{3.50}$$

*Compared with condition (3.47), it can be seen that, with the exception that*

$$a_{12} = 0,\ a_{13} = 0,\ \frac{a_{21}}{a_{31}} = \frac{c_2^2 - c_1^2}{c_3^2 - c_1^2}, \tag{3.51}$$

*remain unchanged, the conditions*

$$a_{32} = 0,\ a_{33} = 0,\ \frac{a_{21}}{a_{31}} = \frac{a_{22} + a_{23} - a_{11}}{-a_{11}}, \tag{3.52}$$

$$a_{22} = 0,\ a_{23} = 0,\ \frac{a_{21}}{a_{31}} = \frac{-a_{11}}{a_{32} + a_{33} - a_{11}}, \tag{3.53}$$

*exhibit a symmetrical state. Therefore, whether the condition* (3.51) *is combined with the condition* (3.52) *or* (3.53)*, the system* (3.31) *can have partially synchronized solutions.*

**Remark 3.5.** *Actually, if we do not have conditions* (3.39)–(3.41) *and directly require* (3.45) *to be a self-closed system, we have*

$$\begin{cases} a_{23} = 0 \\ (a_{21} - a_{31})a_{13} + (a_{22} + a_{23} - a_{32} - a_{33})a_{23} = 0, \\ -(c_3^2 - c_2^2)a_{21} + (a_{21} - a_{31})(c_1^2 - c_2^2) = 0, \\ a_{22} + a_{23} = 0, \\ (a_{21} - a_{31})a_{11} + (a_{22} + a_{23} - a_{32} - a_{33})a_{21} = 0, \\ (a_{21} - a_{31})(a_{12} + a_{13}) + (a_{22} + a_{23} - a_{32} - a_{33})(a_{32} + a_{33}) = 0. \end{cases}$$

*Thus, we can directly get* (3.47)*. Therefore, the previous steps provide us a method to realize the partial synchronization of the system given by* (3.31)–(3.32)*.*

### 3.4. The general case with N variables

For the case of $N$ variables in which the first $m$ variables do not require synchronization, we rewrite the equation (1.1) as follows:

$$u_{itt} - c_i^2 u_{ixx} + \sum_{j=1}^{N} a_{ij} u_j = 0, \quad i = 1, \cdots, N \tag{3.54}$$

with the boundary conditions

$$x = 0 : u_1(t, 0) = \cdots = u_N(t, 0) = 0, \quad x = L : u_1(t, L) = \cdots = u_N(t, L) = 0, \tag{3.55}$$

and the initial conditions with the partial synchronization property

$$t = 0 : \; u_{m+1}(0, x) = \cdots = u_N(0, x), \;\; u_{m+1t}(0, x) = \cdots = u_{Nt}(0, x). \tag{3.56}$$

We want to get

$$u_{m+1} = \cdots = u_N. \tag{3.57}$$

For $c_{m+1} = \cdots = c_N$, we have discussed this situation in Theorem 2.1.

When $c_{m+1}, \cdots, c_N$ are different from each other, if we require (3.57) to hold from the previous discussion, we get condition (3.5).

Conversely, we want to find the conditions that make the system (3.57) be a partially synchronizable system. Here, we use the same method as the case when $N = 3, m = 1$. Let

$$w_1 = u_{m+1} - u_N, \; w_2 = u_{m+2} - u_N, \; \cdots, \; w_{N-m-1} = u_{N-1} - u_N.$$

Obviously, $(w_1, \cdots, w_{N-m-1}) \equiv 0$ is equivalent to condition (3.57).

Next, we want to find the condition which can realize $(w_1, \cdots, w_{N-m-1}) \equiv 0$. Let the $(m+1)$-th equation in (3.54) be subtracted from the $N$-th equation in (3.54), and, consistent with the mark in (2.3), we denote

$$\tilde{a}_{i,m+1} = \sum_{j=m+1}^{N} a_{ij}, (i = 1, \cdots, N); \tag{3.58}$$

we get

$$\begin{aligned} & w_{1tt} - c_{m+1}^2 w_{1xx} + (c_N^2 - c_{m+1}^2) u_{Nxx} + (a_{m+1,1} - a_{N1}) u_1 + \cdots + (a_{m+1,m} - a_{Nm}) u_m \\ & + (a_{m+1,m+1} - a_{N,m+1}) w_1 + \cdots + (a_{m+1,N-1} - a_{N,N-1}) w_{N-m-1} \\ & + (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1}) u_N = 0. \end{aligned} \tag{3.59}$$

The equations of $w_2, \cdots, w_{N-m-1}$ can be obtained in the same way. For example,

$$\begin{aligned} & w_{N-m-1tt} - c_{N-1}^2 w_{N-m-1xx} + (c_N^2 - c_{N-1}^2) u_{Nxx} + (a_{N-1,1} - a_{N1}) u_1 + \cdots \\ & + (a_{N-1,m} - a_{Nm}) u_m + (a_{N-1,m+1} - a_{N,m+1}) w_1 + \cdots + (a_{N-1,N-1} - a_{N,N-1}) w_{N-m-1} \\ & + (\tilde{a}_{N-1,m+1} - \tilde{a}_{N,m+1}) u_N = 0. \end{aligned}$$

Thus, we get the system of $(w_1, \cdots, w_{N-m-1})$ with the following boundary conditions

$$x = 0 : w_1 = \cdots = w_{N-m-1} = 0, \quad x = L : w_1 = \cdots = w_{N-m-1} = 0,$$

and the initial conditions

$$t = 0 : w_1 = \cdots = w_{N-m-1} = 0, \quad w_{1t} = \cdots = w_{N-m-1t} = 0.$$

If the conditions

$$\begin{cases} (c_N^2 - c_{m+1}^2)u_{Nxx} + (a_{m+1,1} - a_{N1})u_1 + \cdots + (a_{m+1,m} - a_{Nm})u_m \\ +(\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})u_N = 0, \\ \cdots \\ (c_N^2 - c_{N-1}^2)u_{Nxx} + (a_{N-1,1} - a_{N1})u_1 + \cdots + (a_{N-1,m} - a_{Nm})u_m \\ +(\tilde{a}_{N-1,m+1} - \tilde{a}_{N,m+1})u_N = 0, \end{cases} \tag{3.60}$$

hold, then the vector $(w_1, \cdots, w_{N-m-1})$ satisfies the conditions of a self-closed system. We get that $(w_1, \cdots, w_{N-m-1}) \equiv 0$. Hence, the system (3.54) is a partially synchronizable system. Obviously, this is a sufficient condition. However, (3.60) is not a self-closed equation, so it is different to solve. We want to get the algebraic conditions of matrices $A$ and $\Lambda$.

Let

$$\begin{aligned} W_1 =& w_{1tt} - c_{m+1}^2 w_{1xx} + \sum_{j=m+1}^{N-1} (a_{m+1,j} - a_{Nj})w_{j-m}, \\ W_2 =& w_{2tt} - c_{m+2}^2 w_{2xx} + \sum_{j=m+1}^{N-1} (a_{m+2,j} - a_{Nj})w_{j-m}, \\ &\cdots \\ W_{N-m-1} =& w_{N-m-1tt} - c_{N-1}^2 w_{N-m-1xx} + \sum_{j=m+1}^{N-1} (a_{N-1,j} - a_{Nj})w_{j-m}. \end{aligned}$$

From (3.59), we get

$$\begin{aligned} &W_1 + (c_N^2 - c_{m+1}^2)u_{Nxx} + (a_{m+1,1} - a_{N1})u_1 + \cdots + (a_{m+1,m} - a_{Nm})u_m \\ &+ (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})u_N = 0. \end{aligned} \tag{3.61}$$

Then, we have

$$\begin{aligned} &W_{1tt} + (c_N^2 - c_{m+1}^2)u_{Nxxtt} + (a_{m+1,1} - a_{N1})u_{1tt} + \cdots + (a_{m+1,m} - a_{Nm})u_{mtt} \\ &+ (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})u_{Ntt} = 0 \end{aligned}$$

and

$$W_{1xx} + (c_N^2 - c_{m+1}^2)u_{Nxxxx} + (a_{m+1,1} - a_{N1})u_{1xx} + \cdots + (a_{m+1,m} - a_{Nm})u_{mxx}$$

$$+ (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})u_{Nxx} = 0.$$

From the above two equations, we get

$$\begin{aligned} & W_{1tt} - c_N^2 W_{1xx} + (c_N^2 - c_{m+1}^2)(u_{Ntt} - c_N^2 u_{Nxx})_{xx} + (a_{m+1,1} - a_{N1})(u_{1tt} - c_N^2 u_{1xx}) + \cdots \\ & + (a_{m+1,m} - a_{Nm})(u_{mtt} - c_N^2 u_{mxx}) + (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})(u_{Ntt} - c_N^2 u_{Nxx}) = 0. \end{aligned} \tag{3.62}$$

We want (3.62) be a self-closed system of $W_1$; thus, we simplify the above formula as follows. Using system (3.54), we have

$$\begin{aligned} u_{Ntt} - c_N^2 u_{Nxx} &= -(\sum_{j=1}^{m} a_{Nj}u_j + \sum_{j=m+1}^{N-1} a_{Nj}(w_{j-m} + u_N) + a_{NN}u_N) \\ &= -(\sum_{j=1}^{m} a_{Nj}u_j + \sum_{j=m+1}^{N-1} a_{Nj}w_{j-m} + \tilde{a}_{N,m+1}u_N) \end{aligned}$$

$$(u_{Ntt} - c_N^2 u_{Nxx})_{xx} = -(\sum_{j=1}^{m} a_{Nj}u_{j_{xx}} + \sum_{j=m+1}^{N-1} a_{Nj}w_{j-m_{xx}} + \tilde{a}_{N,m+1}u_{Nxx})$$

and

$$\begin{aligned} u_{1tt} - c_N^2 u_{1xx} =& u_{1tt} - c_1^2 u_{1xx} + (c_1^2 - c_N^2)u_{1xx} \\ =& -(\sum_{j=1}^{m} a_{1j}u_j + \sum_{j=m+1}^{N-1} a_{1j}(w_{j-m} + u_N) + a_{NN}u_N) + (c_1^2 - c_N^2)u_{1xx} \\ =& -(\sum_{j=1}^{m} a_{1j}u_j + \sum_{j=m+1}^{N-1} a_{1j}w_{j-m} + \tilde{a}_{1,m+1}u_N) + (c_1^2 - c_N^2)u_{1xx}. \end{aligned}$$

Then, $u_{2tt} - c_N^2 u_{2xx}, \cdots, u_{mtt} - c_N^2 u_{mxx}$ can be obtained similarly. For example,

$$\begin{aligned} u_{mtt} - c_N^2 u_{mxx} =& u_{mtt} - c_m^2 u_{mxx} + (c_m^2 - c_N^2)u_{mxx} \\ =& -(\sum_{j=1}^{m} a_{mj}u_j + \sum_{j=m+1}^{N-1} a_{mj}w_{j-m} + \tilde{a}_{m,m+1}u_N) + (c_m^2 - c_N^2)u_{mxx}. \end{aligned}$$

Substituting the above formula into (3.62), we get

$$\begin{aligned} & W_{1tt} - c_N^2 W_{1xx} - (c_N^2 - c_{m+1}^2)[(\sum_{j=1}^{m} a_{Nj}u_{j_{xx}} + \sum_{j=m+1}^{N-1} a_{Nj}w_{j-m_{xx}} + \tilde{a}_{N,m+1}u_{Nxx})] \\ & + (a_{m+1,1} - a_{N1})(c_1^2 - c_N^2)u_{1xx} + \cdots + (a_{m+1,m} - a_{Nm})(c_m^2 - c_N^2)u_{mxx} \\ & - (a_{m+1,1} - a_{N1})[(\sum_{j=1}^{m} a_{1j}u_j + \sum_{j=m+1}^{N-1} a_{1j}w_{j-m} + \tilde{a}_{1,m+1}u_N)] \\ & \cdots \end{aligned}$$

$$- (a_{m+1,m} - a_{Nm})[(\sum_{j=1}^{m} a_{mj}u_j + \sum_{j=m+1}^{N-1} a_{mj}w_{j-m} + \tilde{a}_{m,m+1}u_N)]$$
$$- (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})[(\sum_{j=1}^{m} a_{Nj}u_j + \sum_{j=m+1}^{N-1} a_{Nj}w_{j-m} + \tilde{a}_{N,m+1}u_N)].$$

After simplification, we obtain

$$\begin{aligned}
&W_{1tt} - c_N^2 W_{1xx} + [(c_{m+1}^2 - c_1^2)a_{N1} + (c_1^2 - c_N^2)a_{m+1,1}]u_{1xx} + \cdots\\
&+ [(c_{m+1}^2 - c_m^2)a_{Nm} + (c_m^2 - c_N^2)a_{m+1,m}]u_{mxx} + (c_N^2 - c_{m+1}^2)\tilde{a}_{N,m+1}u_{Nxx}\\
&- (c_N^2 - c_{m+1}^2)(a_{N,m+1}w_{1xx} + \cdots + a_{N,N-1}w_{N-m-1\,xx})\\
&- [\sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})a_{j1} + (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})a_{N1}]u_1\\
&\cdots\\
&- [\sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})a_{jm} + (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})a_{Nm}]u_m\\
&- [\sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})\tilde{a}_{j,m+1} + (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})\tilde{a}_{N,m+1}]u_N\\
&- [\sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})a_{j,m+1} + (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})a_{N,m+1}]w_1\\
&\cdots\\
&- [\sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})a_{j,N-1} + (\tilde{a}_{m+1,m+1} - \tilde{a}_{N,m+1})a_{N,N-1}]w_{N-m-1}\\
&= 0.
\end{aligned}$$

Assume that the matrices $A$ and $\Lambda$ satisfy the following conditions:

$$\begin{aligned}
&\frac{a_{m+1,1}}{c_{m+1}^2 - c_1^2} = \frac{a_{N1}}{c_N^2 - c_1^2}, \cdots, \frac{a_{m+1,m}}{c_{m+1}^2 - c_m^2} = \frac{a_{Nm}}{c_N^2 - c_m^2},\\
&a_{N,m+1} = 0, \cdots, a_{N,N-1} = 0, a_{NN} = 0,\\
&(a_{m+1,1} - a_{N1})a_{11} + \cdots + (a_{m+1,m} - a_{Nm})a_{m1} + \tilde{a}_{m+1,m+1}a_{N1} = 0,\\
&\cdots\\
&(a_{m+1,1} - a_{N1})a_{1m} + \cdots + (a_{m+1,m} - a_{Nm})a_{mm} + \tilde{a}_{m+1,m+1}a_{Nm} = 0,\\
&(a_{m+1,1} - a_{N1})a_{1,m+1} + \cdots + (a_{m+1,m} - a_{Nm})a_{m,m+1} = 0,\\
&\cdots\\
&(a_{m+1,1} - a_{N1})a_{1,N-1} + \cdots + (a_{m+1,m} - a_{Nm})a_{m,N-1} = 0,\\
&(a_{m+1,1} - a_{N1})\tilde{a}_{1,m+1} + \cdots + (a_{m+1,m} - a_{Nm})\tilde{a}_{m,m+1} = 0.
\end{aligned} \tag{3.63}$$

For short

$$\frac{a_{m+1,1}}{c_{m+1}^2 - c_1^2} = \frac{a_{N1}}{c_N^2 - c_1^2}, \cdots, \frac{a_{m+1,m}}{c_{m+1}^2 - c_m^2} = \frac{a_{Nm}}{c_N^2 - c_m^2},$$

$$
\begin{aligned}
&a_{N,m+1} = 0, \cdots, a_{N,N-1} = 0, a_{NN} = 0,\\
&(a_{m+1,1} - a_{N1})a_{1k} + \cdots + (a_{m+1,m} - a_{Nm})a_{mk} + \tilde{a}_{m+1,m+1}a_{Nk} = 0,\ k = 1, \cdots, m,\\
&(a_{m+1,1} - a_{N1})a_{1l} + \cdots + (a_{m+1,m} - a_{Nm})a_{ml} = 0,\ l = m+1, \cdots, N.
\end{aligned}
\tag{3.64}
$$

Then, (3.62) becomes $W_{1tt} - c_N^2 W_{1xx} = 0$. Using the same method to calculate the equations of $W_2, \cdots, W_{N-m-1}$, we get the conditions as follows: for all $i = m+1, \cdots, N-1$,

$$
\begin{aligned}
&\frac{a_{i1}}{c_i^2 - c_1^2} = \frac{a_{N1}}{c_N^2 - c_1^2}, \cdots, \frac{a_{im}}{c_i^2 - c_m^2} = \frac{a_{Nm}}{c_N^2 - c_m^2},\\
&a_{N,m+1} = 0, \cdots, a_{N,N-1} = 0, a_{NN} = 0,\\
&(a_{i1} - a_{N1})a_{11} + \cdots + (a_{im} - a_{Nm})a_{m1} + \tilde{a}_{i,m+1}a_{N1} = 0,\\
&\cdots\\
&(a_{i1} - a_{N1})a_{1m} + \cdots + (a_{im} - a_{Nm})a_{mm} + \tilde{a}_{i,m+1}a_{Nm} = 0,\\
&(a_{i1} - a_{N1})a_{1,m+1} + \cdots + (a_{im} - a_{Nm})a_{m,m+1} = 0,\\
&\cdots\\
&(a_{i1} - a_{N1})a_{1,N-1} + \cdots + (a_{im} - a_{Nm})a_{m,N-1} = 0,\\
&(a_{i1} - a_{N1})\tilde{a}_{1,m+1} + \cdots + (a_{im} - a_{Nm})\tilde{a}_{m,m+1} = 0.
\end{aligned}
\tag{3.65}
$$

This can be abbreviated as follows: for all $i = m+1, \cdots, N-1$,

$$
\begin{aligned}
&\frac{a_{i1}}{c_i^2 - c_1^2} = \frac{a_{N1}}{c_N^2 - c_1^2}, \cdots, \frac{a_{im}}{c_i^2 - c_m^2} = \frac{a_{Nm}}{c_N^2 - c_m^2},\\
&a_{N,m+1} = 0, \cdots, a_{N,N-1} = 0, a_{NN} = 0,\\
&(a_{i1} - a_{N1})a_{1k} + \cdots + (a_{im} - a_{Nm})a_{mk} + \tilde{a}_{i,m+1}a_{Nk} = 0,\ k = 1, \cdots, m,\\
&(a_{i1} - a_{N1})a_{1l} + \cdots + (a_{im} - a_{Nm})a_{ml} = 0,\ l = m+1, \cdots, N.
\end{aligned}
\tag{3.66}
$$

If condition (3.65) holds, then we get the self-closed systems of $W_1, \cdots, W_{N-m-1}$:

$$
\begin{aligned}
&W_{1tt} - c_N^2 W_{1xx} = 0,\\
&W_{2tt} - c_N^2 W_{2xx} = 0,\\
&\cdots\\
&W_{N-m-1tt} - c_N^2 W_{N-m-1xx} = 0.
\end{aligned}
$$

Assume that the initial and boundary conditions of $(W_1, \cdots, W_{N-m-1})$ satisfy

$$
x = 0 : W_1(t, 0) = \cdots = W_{N-m-1}(t, 0) = 0, \quad x = L : W_1(t, 0) = \cdots = W_{N-m-1}(t, 0) = 0,
$$

$$
t = 0 : W_1(0, x) = \cdots = W_{N-m-1}(0, x) = 0, \quad W_{1t}(0, x) = \cdots = W_{N-m-1t}(0, x) = 0.
$$

Then, $(W_1, \cdots, W_{N-m-1}) \equiv 0$. We immediately get that $(w_1, \cdots, w_{N-m-1}) \equiv 0$. This implies that (3.54) is a partially synchronizable system.

Now, we consider the initial and boundary conditions of $W_1, \cdots, W_{N-m-1}$. For $W_1$, from (3.61), we get

$$
W_1 = -[(c_N^2 - c_{m+1}^2)u_{Nxx} + \sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})u_j + \tilde{a}_{m+1,m+1}u_N];
$$

then,

$$W_{1t} = -[(c_N^2 - c_{m+1}^2)u_{Nxxt} + \sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})u_{j_t} + \tilde{a}_{m+1,m+1}u_{Nt}].$$

We must have

$$W_1(0,x) = -(c_N^2 - c_{m+1}^2)u_{Nxx}(0,x) - \sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})u_j(0,x) - \tilde{a}_{m+1,m+1}u_N(0,x) = 0$$

and

$$W_{1t}(0,x) = -(c_N^2 - c_{m+1}^2)u_{Nxxt}(0,x) - \sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})u_{j_t}(0,x) - \tilde{a}_{m+1,m+1}u_{Nt}(0,x) = 0;$$

thus, we require that

$$u_{Nxx}(0,x) = \frac{a_{m+1,1} - a_{N1}}{c_{m+1}^2 - c_N^2}u_1(0,x) + \cdots + \frac{a_{m+1,m} - a_{Nm}}{c_{m+1}^2 - c_N^2}u_m(0,x) + + \frac{\tilde{a}_{m+1,m+1}}{c_{m+1}^2 - c_N^2}u_N(0,x) \tag{3.67}$$

and

$$u_{Nxxt}(0,x) = \frac{a_{m+1,1} - a_{N1}}{c_{m+1}^2 - c_N^2}u_{1t}(0,x) + \cdots + \frac{\tilde{a}_{m+1,N}}{c_{m+1}^2 - c_N^2}u_{mt}(0,x) + \frac{\tilde{a}_{m+1,m+1}}{c_{m+1}^2 - c_N^2}u_{Nt}(0,x). \tag{3.68}$$

Regarding the boundary conditions of $W_1$,

$$x = 0 : W_1(t,0) = -(c_N^2 - c_{m+1}^2)u_{Nxx}(t,0) - \sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})u_j(t,0) - \tilde{a}_{m+1,m+1}u_N(t,0),$$
$$x = L : W_1(t,L) = -(c_N^2 - c_{m+1}^2)u_{Nxx}(t,L) - \sum_{j=1}^{m}(a_{m+1,j} - a_{Nj})u_j(t,L) - \tilde{a}_{m+1,m+1}u_N(t,L).$$

From (3.54), we have

$$u_{Nxx}(t,0) = \frac{u_{Ntt}(t,0) + a_{N1}u_1(t,0) + \cdots + a_{NN}u_N(t,0)}{c_N^2},$$
$$u_{Nxx}(t,L) = \frac{u_{Ntt}(t,L) + a_{N1}u_1(t,L) + \cdots + a_{NN}u_N(t,L)}{c_N^2}.$$

From (3.55), we have that $u_{Ntt}(t,0) = 0$ and $u_{Ntt}(t,L) = 0$. Using (3.55) again, we get

$$x = 0 : W_1(t,0) = 0, \quad x = L : W_1(t,L) = 0.$$

For $W_2, \cdots, W_{N-m-1}$, we can get similar conclusions for the initial value: for $i = m+1, \cdots, N-1$,

$$u_{Nxx}(0,x) = \frac{a_{i1} - a_{N1}}{c_i^2 - c_N^2}u_{1xx}(0,x) + \cdots + \frac{a_{im} - a_{Nm}}{c_i^2 - c_N^2}u_{mxx}(0,x) + \frac{\tilde{a}_{i,m+1}}{c_i^2 - c_N^2}u_{Nxx}(0,x) \tag{3.69}$$

and

$$u_{Nxxt}(0,x) = \frac{a_{i1} - a_{N1}}{c_i^2 - c_N^2}u_{1xxt}(0,x) + \cdots + \frac{a_{im} - a_{Nm}}{c_i^2 - c_N^2}u_{mxxt}(0,x) + \frac{\tilde{a}_{i,m+1}}{c_i^2 - c_N^2}u_{Nxxt}(0,x). \tag{3.70}$$

**Remark 3.6.** *By* (3.2)–(3.3)*, we know that conditions* (3.69)–(3.70) *are equivalent to*

$$u_{Nxx}(0, x) = -\alpha_1 u_{1xx}(0, x) - \cdots - \alpha_m u_{mxx}(0, x) - \alpha_{m+1} u_{Nxx}(0, x) \tag{3.71}$$

*and*

$$u_{Nxxt}(0, x) = -\alpha_1 u_{1xxt}(0, x) - \cdots - \alpha_m u_{mxxt}(0, x) - \alpha_{m+1} u_{Nxxt}(0, x). \tag{3.72}$$

*Hence,* (3.69)–(3.70) *are not* $2(N - m - 1)$ *conditions, but only two conditions.*

To sum up, we get the conclusion:

**Theorem 3.7.** *If the system given by* (3.54)–(3.55) *satisfies the compatibility condition* (3.65)*, the initial value* $(U_0, U_1)$ *has the partial synchronization property* (3.56)*, which satisfy conditions* (3.71)–(3.72)*, then the system given by* (3.54)–(3.55) *has a corresponding solution* $U = U(t, x)$ *that satisfies the partial synchronization condition* (3.57)*.*

**Remark 3.8.** *Under the assumption of Theorem 3.7, not all partially synchronized initial values* $(U_0, U_1)$ *have corresponding partially synchronized solutions,as only the initial values satisfying* (3.71)–(3.72) *can have partially synchronized solutions. Therefore, even if the system with different wave speeds is a partially synchronizable system, it does not mean that, for any given initial value satisfying the partial synchronization property, there is a corresponding partially synchronized solution.*

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

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