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Research article

Normalization and reduction of the Stark Hamiltonian

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Abstract: We detail a calculation of the second order normal form of the Stark effect Hamiltonian after regularization, using the Kustaanheimo-Stiefel mapping. After reduction, we obtain an integrable two degree of freedom system on $S^2_h \times S^2_h$, which we reduce again to obtain a one degree of freedom Hamiltonian system.

Keywords: Kustaanheimo-Stiefel normalization; Stark Hamiltonian; reduction of symmetries
Mathematics Subject Classification: 70K45

1. Introduction

In the study of perturbations of the three degree of freedom Kepler Hamiltonian pulling back, the regularized Hamiltonian by the Kustaanheimo-Stiefel (KS) map, gives a perturbation of the four degree of freedom harmonic oscillator Hamiltonian, when restricted to the zero level set of the KS symmetry. We use the formulation of the KS transformation in [6] which allows us to reduce the KS symmetry using invariant theory for the first time. As an illustration, we apply this procedure to the regularized Stark Hamiltonian, which is normalized after applying the KS transformation. We do not expect this Hamiltonian to be completely integrable (see Lagrange [4] and also [5]). Our treatment follows that of [3] and gives the full details of obtaining the second order normal form [1]. We use the notation of [2] and note that our procedure of regularization, pull back by the KS map, normalization and reduction may be used to study three degree of freedom perturbed Keplerian systems.

2. The basic set up

On $T_0\mathbb{R}^3 = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ with coordinates (x, y) and standard symplectic form $\omega_3 = \sum_{i=1}^{3} \mathrm{d}x_i \wedge \mathrm{d}y_i$ consider the Stark Hamiltonian

$$K(x, y) = \tfrac{1}{2}\langle y, y\rangle - \frac{1}{|x|} + f x_3. \tag{2.1}$$

Here, $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbb{R}^3$ with associated norm $|\ |$. On the negative energy level $-\frac{1}{2}k^2$ with $k > 0$ rescaling time $\mathrm{d}t \mapsto \frac{|x|}{k}\mathrm{d}s$, we obtain

$$0 = \tfrac{1}{2k}\big(|x|\langle y, y\rangle + k^2|x|\big) - \frac{1}{k} + f x_3 \frac{|x|}{k}. \tag{2.2}$$

In other words, (x, y) lies in the $\frac{1}{k}$ level set of

$$\widehat{K}(x, y) = \tfrac{1}{2k}|x|\big(\langle y, y\rangle + k^2|x|\big) + f x_3 \frac{|x|}{k}. \tag{2.3}$$

We assume that f is small, namely, $f = \varepsilon\beta$. After the symplectic coordinate change $(x, y) \mapsto (\frac{1}{k}x, ky)$ the Hamiltonian $\widehat{K}$ becomes the preregularized Hamiltonian

$$\mathcal{K}(x, y) = \tfrac{1}{2}|x|(\langle y, y\rangle + |x|) + \varepsilon\beta x_3|x| \tag{2.4}$$

on the level set $\mathcal{K}^{-1}(1)$.

Let $T_0\mathbb{R}^4 = (\mathbb{R}^4 \setminus \{0\}) \times \mathbb{R}^4$ have coordinates (q, p) and a symplectic form $\omega_4 = \sum_{i=1}^4 \mathrm{d}q_i \wedge \mathrm{d}p_i$. Pull back $\mathcal{K}$ by the Kustaanheimo-Stiefel mapping

$$\mathcal{KS} : T_0\mathbb{R}^4 \to T_0\mathbb{R}^3 : (q, p) \mapsto (x, y),$$

where

$$\begin{aligned}
x_1 &= 2(q_1q_3 + q_2q_4) = U_2 - K_1\\
x_2 &= 2(q_1q_4 - q_2q_3) = U_3 - K_2\\
x_3 &= q_1^2 + q_2^2 - q_3^2 - q_4^2 = U_4 - K_3\\
y_1 &= (\langle q, q\rangle)^{-1}(q_1p_3 + q_2p_4 + q_3p_1 + q_4p_2) = (H_2 + V_1)^{-1}V_2\\
y_2 &= (\langle q, q\rangle)^{-1}(q_1p_4 - q_2p_3 - q_3p_2 + q_4p_1) = (H_2 + V_1)^{-1}V_3\\
y_3 &= (\langle q, q\rangle)^{-1}(q_1p_1 + q_2p_2 - q_3p_3 - q_4p_4) = (H_2 + V_1)^{-1}V_4
\end{aligned}$$

and

$$\begin{aligned}
H_2 &= \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2)\\
\Xi &= q_1p_2 - q_2p_1 + q_3p_4 - q_4p_3,
\end{aligned}$$

to get the regularized Stark Hamiltonian

$$\mathcal{H} = H_2 + \varepsilon\beta(U_4V_1 + H_2U_4 - K_3V_1 - H_2K_3) \tag{2.5}$$

on $\Xi^{-1}(0)$, since $|x| = \langle q, q\rangle = H_2 + V_1$. Here

$$\begin{aligned}
K_1 &= -(q_1q_3 + q_2q_4 + p_1p_3 + p_2p_4)\\
K_2 &= -(q_1q_4 - q_2q_3 + p_1p_4 - p_2p_3)\\
K_3 &= \tfrac{1}{2}(q_3^2 + q_4^2 + p_3^2 + p_4^2 - q_1^2 - q_2^2 - p_1^2 - p_2^2)
\end{aligned}$$

$$\begin{aligned}
L_1 &= q_4p_1 - q_3p_2 + q_2p_3 - q_1p_4 \\
L_2 &= q_1p_3 + q_2p_4 - q_3p_1 - q_4p_2 \\
L_3 &= q_3p_4 - q_4p_3 + q_2p_1 - q_1p_2 \\
U_1 &= -(q_1p_1 + q_2p_2 + q_3p_3 + q_4p_4) \\
U_2 &= q_1q_3 + q_2q_4 - p_1p_3 - p_2p_4 \\
U_3 &= q_1q_4 - q_2q_3 + p_2p_3 - p_1p_4 \\
U_4 &= \tfrac{1}{2}(q_1^2 + q_2^2 - q_3^2 - q_4^2 + p_3^2 + p_4^2 - p_1^2 - p_2^2) \\
V_1 &= \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2 - p_1^2 - p_2^2 - p_3^2 - p_4^2) \\
V_2 &= q_1p_3 + q_2p_4 + q_3p_1 + q_4p_2 \\
V_3 &= q_1p_4 - q_2p_3 - q_3p_2 + q_4p_1 \\
V_4 &= q_1p_1 + q_2p_2 - q_3p_3 - q_4p_4.
\end{aligned}$$

generate the algebra of polynomials invariant under the S^1 action φ_s^{Ξ} given by the flow of X_{Ξ} on $(T\mathbb{R}^4 = \mathbb{R}^8, \omega_4)$. The Hamiltonian $\mathcal{H}$ (2.5) is invariant under this S^1 action and thus is a smooth function on the orbit space $\Xi^{-1}(0)/S^1 \subseteq \mathbb{R}^{16}$ with coordinates $(K, L, H, \Xi; U, V)$.

3. The first order normal form on $\Xi^{-1}(0)/S^1$

The harmonic oscillator vector field X_{H_2} on $(T\mathbb{R}^4, \omega_4)$ induces the vector field $Y_{H_2} = \sum_{i=1}^4 (2V_i \frac{\partial}{\partial U_i} - 2U_i \frac{\partial}{\partial V_i})$ on the orbit space $\mathbb{R}^8/S^1 \subseteq \mathbb{R}^{16}$, which leaves $\Xi^{-1}(0)/S^1$ invariant.

We now compute the first order normal form of the Hamiltonian $\mathcal{H}$ (2.5) on the reduced space $\Xi^{-1}(0)/S^1 \subseteq \mathbb{R}^8/S^1$.

The average of $H_2U_4 - K_3V_1$ over the flow

$$\varphi_t^{Y_{H_2}}(K, L, H_2, \Xi; U, V) = (K, L, H_2, \Xi; U\cos 2t + V\sin 2t, -U\sin 2t + V\cos 2t)$$

of Y_{H_2} is

$$\begin{aligned}
\overline{H_2U_4 - K_3V_1} &= \tfrac{1}{\pi}\int_0^{\pi} (\varphi_t^{Y_{H_2}})^*(H_2U_4 - K_3V_1)\,\mathrm{d}t \\
&= \tfrac{1}{\pi}\int_0^{\pi} (U_4\cos 2t + V_4\sin 2t)\mathrm{d}t\, H_2 - \tfrac{1}{\pi}\int_0^{\pi} (-U_1\sin 2t + V_1\cos 2t)\mathrm{d}t\, K_3 = 0.
\end{aligned}$$

The second equality above follows because $L_{X_{H_2}}K_3 = 0$ and the third because $\overline{\cos 2t} = \overline{\sin 2t} = 0$. The average of U_4V_1 over the flow of Y_{H_2} on $\Xi^{-1}(0)/S^1$ is

$$\begin{aligned}
\overline{U_4V_1} &= \tfrac{1}{\pi}\int_0^{\pi} (\varphi_t^{Y_{H_2}})^*(U_4V_1)\,\mathrm{d}t \\
&= -\tfrac{1}{2}U_1U_4\,\overline{\sin 4t} + U_4V_1\,\overline{\cos^2 2t} - U_1V_4\,\overline{\sin^2 2t} + \tfrac{1}{2}V_1V_4\,\overline{\sin 4t} \\
&= \tfrac{1}{2}(U_4V_1 - U_1V_4) = -\tfrac{1}{2}H_2K_3,
\end{aligned}$$

since $\overline{\cos^2 2t} = \overline{\sin^2 2t} = \frac{1}{2}$ and $\overline{\sin 4t} = 0$. The last equality above follows from the explicit description of the orbit space $\mathbb{R}^8/S^1$ as the semialgebraic variety in $\mathbb{R}^{16}$ with coodinates $(K, L, U, V; H_2, \Xi)$ given by

$$\begin{aligned}
\langle U, U\rangle = U_1^2 + U_2^2 + U_3^2 + U_4^2 &= H_2^2 - \Xi^2 \geq 0 \;\; H_2 \geq 0 \\
\langle V, V\rangle = V_1^2 + V_2^2 + V_3^2 + V_4^2 &= H_2^2 - \Xi^2 \geq 0 \\
\langle U, V\rangle = U_1V_1 + U_2V_2 + U_3V_3 + U_4V_4 &= 0 \\
U_2V_1 - U_1V_2 &= L_1\Xi - K_1H_2 \\
U_3V_1 - U_1V_3 &= L_2\Xi - K_2H_2 \\
U_4V_1 - U_1V_4 &= L_3\Xi - K_3H_2 \\
U_4V_3 - U_3V_4 &= K_1\Xi - L_1H_2 \\
U_2V_4 - U_4V_2 &= K_2\Xi - L_2H_2 \\
U_3V_2 - U_2V_3 &= K_3\Xi - L_3H_2.
\end{aligned} \tag{3.1}$$

So the average of $U_4V_1 + H_2U_4 - K_3V_1 - H_2K_3$ over the flow of Y_{H_2} is $-\frac{3}{2}H_2K_3$ on $\Xi^{-1}(0)/S^1$. Thus the first order normal form of the regularized Stark Hamiltonian $\mathcal{H}$ (2.5) on $\Xi^{-1}(0)/S^1$ is

$$\mathcal{H}^{(1)}_{\mathrm{nf}} = H_2 - \tfrac{3}{2}\beta\varepsilon H_2K_3. \tag{3.2}$$

4. The second order normal form on $\Xi^{-1}(0)/S^1$

In order to compute the second order normal form of the Hamiltonian $\mathcal{H}$ on $\Xi^{-1}(0)/S^1$, we need to find a function F on $\mathbb{R}^{16}$ such that changing coordinates by the time ε value of the flow of the Hamiltonian vector field Y_F brings the regularized Hamiltonian $\mathcal{H}$ (2.5) into first order normal form. Choose F so that

$$L_{Y_F}H_2 = \beta\big(- U_4V_1 - \tfrac{1}{2}H_2K_3 - H_2U_4 + K_3V_1\big). \tag{4.1}$$

The following calculation shows that this choice does the job.

$$\begin{aligned}
(\varphi^{Y_F}_\varepsilon)^*\mathcal{H} &= \mathcal{H} + \varepsilon L_{Y_F}\mathcal{H} + \tfrac{1}{2}\varepsilon^2 L^2_{Y_F}\mathcal{H} + \mathrm{O}(\varepsilon^3) \\
&= H_2 + \varepsilon\beta(U_4V_1 + H_2U_4 - K_3V_1 - H_2K_3) + \varepsilon L_{Y_F}H_2 \\
&\qquad + \varepsilon^2\beta L_{Y_F}(U_4V_1 + H_2U_4 - K_3V_1 - H_2K_3) + \tfrac{1}{2}\varepsilon^2 L^2_{Y_F}H_2 + \mathrm{O}(\varepsilon^3) \\
&= H_2 + \varepsilon\beta(U_4V_1 + H_2U_4 - K_3V_1 - H_2K_3) \\
&\qquad + \varepsilon\beta\big(- U_4V_1 - \tfrac{1}{2}H_2K_3 - H_2U_4 + K_3V_1\big) \\
&\qquad\quad + \varepsilon^2\big[L_{Y_F}\big(- L_{Y_F}H_2 - \tfrac{3}{2}\beta H_2K_3\big) + \tfrac{1}{2}L^2_{Y_F}H_2\big] + \mathrm{O}(\varepsilon^3) \\
&= H_2 - \tfrac{3}{2}\varepsilon\beta H_2K_3 - \tfrac{1}{2}\varepsilon^2\big(L^2_{Y_F}H_2 + 3\beta L_{Y_F}(H_2K_3)\big) + \mathrm{O}(\varepsilon^3).
\end{aligned} \tag{4.2}$$

To determine the function F, we solve equation (4.1). Write $F = F_1 + F_2$, where $L_{Y_{H_2}}F_1 = \beta(U_4V_1 + \frac{1}{2}H_2K_3)$ and $L_{Y_{H_2}}F_2 = \beta(H_2U_4 - K_3V_1)$. Then

$$\begin{aligned}
L_{Y_F}H_2 &= -L_{Y_{H_2}}F = -L_{Y_{H_2}}F_1 - L_{Y_{H_2}}F_2 \\
&= -\beta(U_4V_1 + \tfrac{1}{2}H_2K_3) - \beta(H_2U_4 - K_3V_1).
\end{aligned} \tag{4.3}$$

Since $L_{Y_{H_2}} V_4 = -2U_4$ and $L_{Y_{H_2}} U_1 = 2V_1$, it follows that

$$F_2 = -\tfrac{\beta}{2}(H_2V_4 + K_3U_1). \tag{4.4a}$$

Now

$$F_1 = \tfrac{\beta}{\pi}\int_0^{\pi} t(\varphi_t^{Y_{H_2}})^*(U_4V_1 + \tfrac{1}{2}H_2K_3)\,\mathrm{d}t = \tfrac{\beta}{\pi}\int_0^{\pi} t(\varphi_t^{Y_{H_2}})^*(U_4V_1)\mathrm{d}t + \tfrac{\pi\beta}{4}H_2K_3,$$

see [1], and

$$\begin{aligned}
&\tfrac{\beta}{\pi}\int_0^{\pi} t(\varphi_t^{Y_{H_2}})^*(U_4V_1)\mathrm{d}t = \\
&= -\tfrac{\beta}{2}\,(U_1U_4)\,\tfrac{1}{\pi}\int_0^{\pi} t\,\sin 4t\mathrm{d}t + \beta(U_4V_1)\,\tfrac{1}{\pi}\int_0^{\pi} t\cos^2 2t\mathrm{d}t \\
&\qquad - \beta(U_1V_4)\,\tfrac{1}{\pi}\int_0^{\pi} t\sin^2 2t\mathrm{d}t + \tfrac{\beta}{2}(V_1V_4)\,\tfrac{1}{\pi}\int_0^{\pi} t\,\sin 4t\mathrm{d}t \\
&= \tfrac{\beta}{8}(U_1U_4 - V_1V_4) + \tfrac{\pi\beta}{4}(U_4V_1 - U_1V_4),
\end{aligned}$$

since $\frac{1}{\pi}\int_0^{\pi} t\,\sin 4t\mathrm{d}t = -\frac{1}{4}$ and $\frac{1}{\pi}\int_0^{\pi} t\sin^2 2t\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} t\cos^2 2t\mathrm{d}t = \frac{\pi}{4}$. Thus

$$F_1 = \tfrac{\beta}{8}(U_1U_4 - V_1V_4) + \tfrac{\pi\beta}{4}(U_4V_1 - U_1V_4 + H_2K_3) = \tfrac{\beta}{8}(U_1U_4 - V_1V_4)$$

on $\Xi^{-1}(0)/S^1$, see (3.1). Hence on $\Xi^{-1}(0)/S^1$

$$F = F_1 + F_2 = \tfrac{\beta}{8}(U_1U_4 - V_1V_4) - \tfrac{\beta}{2}(H_2V_4 + K_3U_1). \tag{4.5}$$

We now calculate the average over the flow of Y_{H_2} of

$$-\tfrac{3}{2}\beta L_{Y_F}(H_2K_3) - \tfrac{1}{2}L^2_{Y_F}H_2, \tag{4.6}$$

which is the ε^2 term in the transformed Hamiltonian $(\varphi_\varepsilon^{Y_F})^*\mathcal{H}$, see (4.2). This determines the second order normal form of $\mathcal{H}$ on $\Xi^{-1}(0)/S^1$. We begin with the term

$$-\tfrac{3}{2}\beta L_{Y_F}(H_2K_3) = -\tfrac{3}{2}\beta[K_3(L_{Y_F}H_2) - H_2(L_{Y_{K_3}}F)].$$

The average of

$$-\tfrac{3}{2}beta\,K_3(L_{Y_F}H_2) = \tfrac{3}{2}\beta^2\,K_3(U_4V_1 + \tfrac{1}{2}H_2K_3 + H_2U_4 - K_3V_1)$$

vanishes on $\Xi^{-1}(0)/S^1$. The term

$$\begin{aligned}
&\tfrac{3}{2}\beta H_2(L_{Y_{K_3}}F) = \tfrac{3}{2}\beta^2 H_2\,L_{Y_{K_3}}\big(\tfrac{1}{8}(U_1U_4 - V_1V_4) - \tfrac{1}{2}(H_2V_4 + K_3U_1)\big) \\
&= \tfrac{3}{2}\beta^2 H_2[(- 2L_2\frac{\partial}{\partial K_1} + 2L_1\frac{\partial}{\partial K_2} - 2K_2\frac{\partial}{\partial L_1} + 2K_1\frac{\partial}{\partial L_2} \\
&\quad - 2U_4\frac{\partial}{\partial U_1} + 2U_1\frac{\partial}{\partial U_4} - 2V_4\frac{\partial}{\partial V_1} + 2V_1\frac{\partial}{\partial V_4})](\tfrac{1}{8}(U_1U_4 - V_1V_4) \\
&\qquad - \tfrac{1}{2}(H_2V_4 + K_3U_1)), \text{ see [2, table 1]}
\end{aligned}$$

$$= \tfrac{3}{2}\beta^2 H_2[\tfrac{1}{4}(-U_4^2 + U_1^2 + V_4^2 - V_1^2) - H_2V_1 + K_3U_4].$$

Next we calculate $\frac{3}{2}\beta\overline{H_2(L_{Y_{K_3}}F)}$. Since $\overline{H_2V_1} = 0 = \overline{K_3U_4}$ we need only calculate the average of U_1^2, U_4^2, V_1^2 and V_4^2. We get $\overline{U_1^2} = \frac{1}{2}(U_1^2 + V_1^2) = \overline{V_1^2}$ and $\overline{U_4^2} = \frac{1}{2}(U_4^2 + V_4^2) = \overline{V_4^2}$. Thus $\frac{3}{2}\beta\overline{H_2(L_{Y_{K_3}}F)} = 0$. So the average $-\frac{3}{2}\beta\overline{L_{Y_F}(H_2K_3)}$ of the first term in expression (4.6) vanishes on $\Xi^{-1}(0)/S^1$.

Next we calculate the average of the term $L^2_{Y_F}H_2$ in expression (4.6) on $\Xi^{-1}(0)/S^1$. We have

$$\begin{aligned}
L^2_{Y_F}H_2 &= -L_{Y_F}(L_{Y_{H_2}}F)\\
&= -\beta L_{Y_F}(U_4V_1 + \tfrac{1}{2}H_2K_3 + H_2U_4 - K_3V_1), \text{ using (4.3)}\\
&= \beta[\overbrace{(L_{Y_{U_4}}F)V_1}^{I} + \overbrace{U_4(L_{Y_{V_1}}F)}^{II} - \tfrac{1}{2}\overbrace{(L_{Y_F}H_2)K_3}^{III} + \tfrac{1}{2}\overbrace{H_2(L_{Y_{K_3}}F)}^{IV}\\
&\qquad - \overbrace{(L_{Y_F}H_2)U_4}^{V} + \overbrace{H_2(L_{Y_{U_4}}F)}^{VI} - \overbrace{(L_{Y_{K_3}}F)V_1}^{VII} - \overbrace{K_3(L_{Y_{V_1}}F)}^{VIII}].
\end{aligned}$$

We begin by finding

$$\begin{aligned}
L_{Y_{H_2}}F &= \beta[(2V_1\frac{\partial}{\partial U_1} + 2V_2\frac{\partial}{\partial U_2} + 2V_3\frac{\partial}{\partial U_3} + 2V_4\frac{\partial}{\partial U_4} - 2U_1\frac{\partial}{\partial V_1} - 2U_2\frac{\partial}{\partial V_2}\\
&\qquad - 2U_3\frac{\partial}{\partial V_3} - 2U_4\frac{\partial}{\partial V_4})](\tfrac{1}{8}(U_1U_4 - V_1V_4) - \tfrac{1}{2}(H_2V_4 + K_3U_1))\\
&= \beta[\tfrac{1}{2}(V_1U_4 + U_1V_4) + H_2U_4 - K_3V_1];\\
L_{Y_{K_3}}F &= \beta((-2L_2\frac{\partial}{\partial K_1} + 2L_1\frac{\partial}{\partial K_2} - 2K_2\frac{\partial}{\partial L_1} + 2K_1\frac{\partial}{\partial L_2} - 2U_4\frac{\partial}{\partial U_1} + 2U_1\frac{\partial}{\partial U_4}\\
&\qquad - 2V_4\frac{\partial}{\partial V_1} + 2V_1\frac{\partial}{\partial V_4})(\tfrac{1}{8}(U_1U_4 - V_1V_4) - \tfrac{1}{2}(H_2V_4 + K_3U_1)))\\
&= \beta[\tfrac{1}{4}(-U_4^2 + U_1^2 + V_4^2 - V_1^2) - H_2V_1 + K_3U_4];\\
L_{Y_{U_4}}F &= \beta[(-2U_1\frac{\partial}{\partial K_3} - 2U_3\frac{\partial}{\partial L_1} + 2U_2\frac{\partial}{\partial L_2} - 2V_4\frac{\partial}{\partial H_2} - 2K_3\frac{\partial}{\partial U_1} + 2L_2\frac{\partial}{\partial U_2}\\
&\qquad + 2V_3\frac{\partial}{\partial U_3} - 2H_2\frac{\partial}{\partial V_4})(\tfrac{1}{8}(U_1U_4 - V_1V_4) - \tfrac{1}{2}(H_2V_4 + K_3U_1))]\\
&= \beta[V_4^2 + K_3^2 - \tfrac{1}{4}(K_3U_4 - H_2V_1) + H_2^2 + U_1^2];\\
L_{Y_{V_1}}F &= \beta[(2V_2\frac{\partial}{\partial K_1} + 2V_3\frac{\partial}{\partial K_2} + 2V_4\frac{\partial}{\partial K_3} + 2U_1\frac{\partial}{\partial H_2} + 2H_2\frac{\partial}{\partial U_1} + 2K_1\frac{\partial}{\partial V_2}\\
&\qquad + 2K_2\frac{\partial}{\partial V_3} + 2K_3\frac{\partial}{\partial V_4})(\tfrac{1}{8}(U_1U_4 - V_1V_4) - \tfrac{1}{2}(H_2V_4 + K_3U_1))]\\
&= \beta[-2(U_1V_4 + H_2K_3) + \tfrac{1}{4}(H_2U_4 - K_3V_1)].
\end{aligned}$$

So the average of term I on $\Xi^{-1}(0)/S^1$ is

$$\begin{aligned}
\beta\overline{(L_{Y_{U_4}}F)V_1} &= \beta^2(\overline{V_1V_4^2} + \overline{K_3^2V_1} - \tfrac{1}{4}\overline{K_3U_4V_1} + \tfrac{1}{4}\overline{H_2V_1^2} + \overline{H_2^2V_1} + \overline{U_1^2V_1})\\
&= \beta^2(\tfrac{1}{8}H_2K_3^2 + \tfrac{1}{8}H_2(U_1^2 + V_1^2)), \qquad (4.7a)
\end{aligned}$$

since the average of $V_1V_4^2$, $K_3^2V_1$, $H_2^2V_1$, and $U_1^2V_1$ are each 0, $\overline{U_4V_1} = -\frac{1}{2}H_2K_3$ and $\overline{V_1^2} = \frac{1}{2}(U_1^2 + V_1^2)$.
Term II is

$$\beta U_4(L_{Y_{V_1}}F) = \beta^2(- 2U_1U_4V_4 - 2H_2K_3U_4 + \tfrac{1}{4}H_2U_4^2 - \tfrac{1}{4}K_3U_4V_1).$$

So

$$\begin{aligned}\beta\overline{U_4(L_{Y_{V_1}}F)} &= \tfrac{1}{4}\beta^2H_2\overline{U_4^2} - \tfrac{1}{4}\beta^2\, K_3\overline{U_4V_1}\\ &= \tfrac{1}{8}\beta^2H_2(U_4^2 + V_4^2) + \tfrac{1}{8}\beta^2H_2K_3^2.\end{aligned} \tag{4.7b}$$

For term III, we have already shown that

$$-\tfrac{\beta}{2}\overline{(L_{Y_F}H_2)K_3} = 0. \tag{4.7c}$$

and for term IV we have already shown that

$$\tfrac{\beta}{2}\overline{H_2(L_{Y_{K_3}}F)} = 0. \tag{4.7d}$$

Term V is

$$-\beta(L_{Y_F}H_2)U_4 = \beta^2(\tfrac{1}{2}U_4^2V_1 + \tfrac{1}{2}U_1U_4V_1 + H_2U_4^2 - K_3U_4V_1).$$

So

$$\begin{aligned}-\beta\overline{(L_{Y_F}H_2)U_4} &= \beta^2(\tfrac{1}{2}\overline{U_4^2V_1} + \tfrac{1}{2}\overline{U_1U_4V_1} + \overline{H_2U_4^2} - \overline{K_3U_4V_1})\\ &= \tfrac{\beta^2}{2}H_2(U_4^2 + V_4^2) - \tfrac{\beta^2}{2}K_3(U_4V_1 - U_1V_4),\end{aligned} \tag{4.7e}$$

since the average of $U_4^2V_1$ and $U_1U_4V_1$ vanish; while $\overline{U_4^2} = \frac{1}{2}(U_4^2 + V_4^2)$ and $\overline{U_4V_1} = \frac{1}{2}(U_4V_1 - U_1V_4)$.
Term VI is

$$\beta H_2(L_{Y_{U_4}}F) = \beta^2H_2(V_4^2 + K_3^2 - \tfrac{1}{4}K_3U_4 + \tfrac{1}{4}H_2V_1 + H_2^2 + U_1^2).$$

So

$$\begin{aligned}\beta\overline{H_2(L_{Y_{U_4}}F)} &= \beta^2\, H_2\overline{V_4^2} + \beta^2\, H_2K_3^2 + \beta^2\, H_2^3 + \beta^2\, H_2\overline{U_1^2}\\ &= \tfrac{1}{2}\beta^2\, H_2(U_4^2 + V_4^2) + \beta^2H_2K_3^2 + \beta^2H_2^3 + \tfrac{1}{2}\beta^2\, H_2(U_1^2 + V_1^2),\end{aligned} \tag{4.7f}$$

since $\overline{K_2U_4} = 0 = \overline{H_2V_1}$.
Term VII is

$$-\beta(L_{Y_{K_3}}F)V_1 = \beta^2(\tfrac{1}{4}[U_4^2V_1 - U_1^2V_1 - V_1V_4^2 + V_1^3] + H_2V_1^2 - K_3U_4V_1).$$

So

$$\begin{aligned}-\beta\overline{(L_{Y_{K_3}}F)V_1} &= \beta^2H_2\overline{V_1^2} - \beta^2K_3\overline{U_4V_1}\\ &= \tfrac{\beta^2}{2}H_2(U_1^2 + V_1^2) - \tfrac{\beta^2}{2}K_3(U_4V_1 - U_1V_4).\end{aligned} \tag{4.7g}$$

Term VIII is

$$-\beta K_3(L_{Y_{V_1}}F) = \beta^2(2K_3U_1V_4 + 2H_2K_3^2 - \tfrac{1}{4}H_2K_3U_4 + \tfrac{1}{4}K_3^2V_1).$$

So

$$-\beta\overline{K_3(L_{Y_{V_1}}F)} = 2\beta^2 K_3\overline{U_1V_4} + 2\beta^2 H_2K_3^2 = 3\beta^2 H_2K_3^2, \tag{4.7h}$$

since $\overline{U_1V_4} = \frac{1}{2}H_2K_3$. Collecting together the results of all the above term calculations gives

$$\begin{aligned}
\overline{L^2_{Y_F}H_2} &= \beta\overline{(L_{Y_{U_4}}F)V_1} + \beta\overline{U_4(L_{Y_{V_1}}F)} - \beta\overline{(L_{Y_F}H_2)U_4} + \beta\overline{H_2(L_{Y_{U_4}}F)} \\
&\quad - \beta\overline{(L_{Y_{K_3}}F)V_1} - \beta\overline{K_3(L_{Y_{V_1}}F)} \\
&= \beta^2([\tfrac{1}{8}H_2K_3^2 + \tfrac{1}{8}H_2(U_1^2+V_1^2)] + [\tfrac{1}{8}H_2(U_4^2+V_4^2) + \tfrac{1}{8}H_2K_3^2] \\
&\quad + [\tfrac{1}{2}H_2(U_4^2+V_4^2) - \tfrac{1}{2}K_3(U_4V_1 - U_1V_4)] + [\tfrac{1}{2}H_2(U_4^2+V_4^2) + H_2K_3^2 + H_2^3 \\
&\quad + \tfrac{1}{2}H_2(U_1^2+V_1^2)] + [\tfrac{1}{2}H_2(U_1^2+V_1^2) - \tfrac{1}{2}K_3(U_4V_1 - U_1V_4)] + 3H_2K_3^2) \\
&= \beta^2[\tfrac{21}{4}H_2K_3^2 + H_2^3 + \tfrac{9}{8}H_2(U_1^2+V_1^2) + \tfrac{9}{8}H_2(U_4^2+V_4^2)],
\end{aligned}$$

using $U_4V_1 - U_1V_4 = -K_3H_2$. Thus the second order normal form of the regularized Stark Hamiltonian $\mathcal{H}$ on $\Xi^{-1}(0)/S^1$ is

$$\begin{aligned}
\mathcal{H}^{(2)}_{\mathrm{nf}} &= H_2 - \tfrac{3}{2}\varepsilon\beta H_2K_3 - \tfrac{1}{2}\varepsilon^2\overline{L^2_{Y_F}H_2} = H_2 - \tfrac{3}{2}\varepsilon\beta H_2K_3 \\
&\quad - \tfrac{1}{2}\varepsilon^2\beta^2 H_2[\tfrac{21}{4}K_3^2 + H_2^2 + \tfrac{9}{8}(U_1^2+V_1^2) + \tfrac{9}{8}(U_4^2+V_4^2)].
\end{aligned} \tag{4.8}$$

5. The first order normal form of $\mathcal{H}^{(2)}_{\mathrm{nf}}$ on $T_hS^3_1$

Since $L_{X_{H_2}}\mathcal{H}^{(2)}_{\mathrm{nf}} = 0$ by construction, the second order normal form $\mathcal{H}^{(2)}_{\mathrm{nf}}$ (4.8) is a smooth function on $(H_2^{-1}(h)\cap\Xi^{-1}(0))/S^1 = T_hS^3_1$, the tangent h-sphere bundle of the unit 3-sphere S^3_1, given by

$$\begin{aligned}
\widetilde{\mathcal{H}} &= h - \tfrac{1}{2}\varepsilon^2\beta^2h^3 - \tfrac{3}{2}\varepsilon\beta hK_3 \\
&\quad - \tfrac{1}{2}\varepsilon^2\beta^2h[\tfrac{21}{4}K_3^2 + \tfrac{9}{8}(U_1^2+V_1^2) + \tfrac{9}{8}(U_4^2+V_4^2)].
\end{aligned} \tag{5.1}$$

We now show that the Hamiltonian $\widetilde{\mathcal{H}}$ (5.1) on $T_hS^3_1$ can be normalized again. On $(T\mathbb{R}^4, \omega_4)$ the Hamiltonian

$$K_3(q,p) = \tfrac{1}{2}(q_3^2 + q_4^2 + p_3^2 + p_4^2 - q_1^2 - q_2^2 - p_1^2 - p_2^2)$$

gives rise to the Hamiltonian vector field X_{K_3}, whose flow $\varphi_t^{X_{K_3}}(q,p)$ is

$$\begin{aligned}
&(q_1\cos t - p_1\sin t, q_2\cos t - p_2\sin t, q_3\cos t + p_3\sin t, q_4\cos t + p_4\sin t, \\
&\quad q_1\sin t + p_1\cos t, q_2\sin t + p_2\cos t, -q_3\sin t + p_3\cos t, -q_4\sin t + p_4\cos t),
\end{aligned}$$

which is periodic of period 2π.

The vector field X_{K_3} on $T\mathbb{R}^4$ induces the vector field

$$\begin{aligned}
Y_{K_3} &= -2L_2\frac{\partial}{\partial K_1} + 2L_1\frac{\partial}{\partial K_2} - 2K_2\frac{\partial}{\partial L_1} + 2K_1\frac{\partial}{\partial L_2} \\
&\quad - 2U_4\frac{\partial}{\partial U_1} + 2U_1\frac{\partial}{\partial U_4} - 2V_4\frac{\partial}{\partial V_1} + 2V_1\frac{\partial}{\partial V_4},
\end{aligned}$$

on $\Xi^{-1}(0)/S^1 \subseteq \mathbb{R}^{16}$ with coordinates $(K, L, H_2, \Xi; U, V)$, whose flow

$$\begin{aligned}\varphi_s^{Y_{K_3}}(K, L, H_2, \Xi; U, V) = (&-L_2 \sin 2s + K_1 \cos 2s, L_1 \sin 2s + K_2 \cos 2s, K_3 \\ &- K_2 \sin 2s + L_1 \cos 2s, K_1 \sin 2s + L_2 \cos 2s, L_3, H_2, \Xi; U_1 \cos 2s - U_4 \sin 2s, \\ &U_2, U_3, U_1 \sin 2s + U_4 \cos 2s, V_1 \cos 2s - V_4 \sin 2s, V_2, V_3, V_1 \sin 2s + V_4 \cos 2s)\end{aligned}$$

is periodic of period π. Since $L_{Y_{K_3}}$ maps the ideal of smooth functions which vanish identically on $\Xi^{-1}(0)/S^1$ into itself, Y_{K_3} is a vector field on $\Xi^{-1}(0)/S^1$. Since $L_{X_{K_3}} H_2 = 0$, it follows that Y_{K_3} induces a vector field on $T_h S_1^3$ with periodic flow. So we can normalize again.

To compute the normal form of the Hamiltonian $\widetilde{\mathcal{H}}$ (5.1) on $T_h S_1^3$ we need only calculate the average of the term

$$T = \tfrac{21}{4} K_3^2 + \tfrac{9}{8}(U_1^2 + V_1^2) + \tfrac{9}{8}(U_4^2 + V_4^2)$$

over the flow $\varphi_s^{Y_{K_3}}$. Since $L_{Y_{K_3}} K_3 = 0$, we need only calculate $\overline{U_1^2}$, $\overline{U_4^2}$, $\overline{V_1^2}$, and $\overline{V_4^2}$. Now

$$\begin{aligned}\overline{U_1^2} &= \tfrac{1}{\pi} \int_0^{\pi} (U_1 \cos 2s - U_4 \sin 2s)^2 \, \mathrm{d}s \\ &= \tfrac{1}{\pi} \int_0^{\pi} (U_1^2 \cos^2 2s - U_1 U_4 \sin 4s + U_4^2 \sin^2 2s) \mathrm{d}s = \tfrac{1}{2}(U_1^2 + U_4^2).\end{aligned}$$

Similarly, $\overline{V_1^2} = \frac{1}{2}(V_1^2 + V_4^2)$, $\overline{U_4^2} = \frac{1}{2}(U_1^2 + U_4^2)$, and $\overline{V_4^2} = \frac{1}{2}(V_1^2 + V_4^2)$. Thus

$$\overline{T} = \tfrac{21}{4} K_3^2 + \tfrac{9}{8} h (U_1^2 + V_1^2 + U_4^2 + V_4^2), \tag{5.2}$$

which is no surprise since $L_{Y_{K_3}} T = 0$. So the first order normal form of $\widetilde{\mathcal{H}}$ (5.1) on $T_h S_1^3$ is

$$\begin{aligned}\widetilde{\mathcal{H}}_{\mathrm{nf}}^{(1)} = h &- \tfrac{1}{2}\varepsilon^2 \beta^2 h^3 - \tfrac{3}{2}\varepsilon \beta h K_3 \\ &- \varepsilon^2 \beta^2 h [\tfrac{21}{8} K_3^2 + \tfrac{9}{16}(U_1^2 + V_1^2 + U_4^2 + V_4^2)].\end{aligned} \tag{5.3}$$

6. The reduced Hamiltonian $\widetilde{\mathcal{H}}_{\mathrm{nf}}^{(1)}$ on $S_h^2 \times S_h^2$

The polynomial $U_1^2 + V_1^2 + U_4^2 + V_4^2$ is invariant under the flows $\varphi_t^{X_{H_2}}$, $\varphi_u^{X_\Xi}$, and thus is a polynomial on the orbit space $T_h S_1^3 / S^1 = S_h^2 \times S_h^2$, defined by

$$\begin{aligned}K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 &= h^2 \\ K_1 L_1 + K_2 L_2 + K_3 L_3 &= 0.\end{aligned}$$

We now find this polynomial. From the explicit description of $\mathbb{R}^8/S^1$ in (3.1) it follows that on $(H_2^{-1}(h) \cap \Xi^{-1}(0))/S^1$

$$\begin{aligned}U_2 V_1 - U_1 V_2 &= -h K_1 \\ U_3 V_1 - U_1 V_3 &= -h K_2 \\ U_4 V_1 - U_2 V_4 &= -h K_3.\end{aligned}$$

So on $S_h^2 \times S_h^2$

$$\begin{aligned} h^2(K_1^2 + K_2^2 + K_3^2) &= (U_2V_1 - U_1V_2)^2 + (U_3V_1 - U_1V_3)^2 + (U_4V_1 - U_1V_4)^2 \\ &= (U_1^2 + U_2^2 + U_3^2 + U_4^2)V_1^2 - U_1^2V_1^2 \\ &\quad - 2(U_1V_1)(U_1V_1 + U_2V_2 + U_3V_3 + U_4V_4) + 2U_1^2V_1^2 \\ &\quad + (V_1^2 + V_2^2 + V_3^2 + V_4^2)U_1^2 - U_1^2V_1^2 \\ &= h^2(V_1^2 + U_1^2), \end{aligned}$$

since $\langle U, U\rangle = h^2$, $\langle V, V\rangle = h^2$, and $\langle U, V\rangle = 0$. Thus $U_1^2 + V_1^2 = K_1^2 + K_2^2 + K_3^2$. Again from the explicit description of $(H_2^{-1}(h) \cap \Xi^{-1}(0))/S^1$ we have

$$\begin{aligned} U_4V_3 - U_3V_4 &= -hL_1 \\ U_4V_2 - U_2V_4 &= hL_2 \\ U_4V_1 - U_1V_4 &= -hK_3. \end{aligned}$$

So on $S_h^2 \times S_h^2$

$$\begin{aligned} h^2(L_1^2 + L_2^2 + K_3^2) &= (U_4V_3 - U_3V_4)^2 + (U_4V_2 - U_2V_4)^2 + (U_4V_1 - U_1V_4)^2 \\ &= (V_1^2 + V_2^2 + V_3^2 + V_4^2)U_4^2 - U_4^2V_4^2 \\ &\quad - 2(U_4V_4)(U_1V_1 + U_2V_2 + U_3V_3 + U_4V_4) + 2U_4^2V_4^2 \\ &\quad + (U_1^2 + U_2^2 + U_3^2 + U_4^2)V_4^2 - U_4^2V_4^2 \\ &= h^2(U_4^2 + V_4^2). \end{aligned}$$

Thus $U_4^2 + V_4^2 = L_1^2 + L_2^2 + K_3^2$. Consequently

$$\begin{aligned} U_1^2 + V_1^2 + U_4^2 + V_4^2 &= K_1^2 + K_2^2 + 2K_3^2 + L_1^2 + L_2^2 \\ &= K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 + K_3^2 - L_3^2 \\ &= 2h^2 + K_3^2 - L_3^2. \end{aligned}$$

on $S_h^2 \times S_h^2$. Hence on $S_h^2 \times S_h^2$

$$\widehat{\mathcal{H}} = \widetilde{\mathcal{H}}^{(1)}_{\mathrm{nf}} = h - \tfrac{13}{8}\varepsilon^2\beta^2h^3 - \tfrac{3}{2}\varepsilon\beta hK_3 - \tfrac{51}{16}\varepsilon^2\beta^2hK_3^2 + \tfrac{9}{16}\varepsilon^2\beta^2hL_3^2. \tag{6.1}$$

7. The Hamiltonian system $(\widehat{\mathcal{H}}, S_h^2 \times S_h^2, \{\ ,\ \})$

Using the coordinates $(\xi, \eta) = ((K + L)/2, (K - L)/2)$ on $\mathbb{R}^3 \times \mathbb{R}^3$, the space of smooth functions on the reduced space $S_h^2 \times S_h^2$, defined by

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = h^2 \;\text{ and }\; \eta_1^2 + \eta_2^2 + \eta_3^2 = h^2,$$

has a Poisson structure with bracket relations

$$\{\xi_i, \xi_j\} = \sum_{k=1}^{3} \epsilon_{ijk}\xi_k, \;\; \{\eta_i, \eta_j\} = -\sum_{k=1}^{3} \epsilon_{ijk}\eta_k, \;\; \{\xi_i, \eta_j\} = 0.$$

Since $\{K_3, L_3\} = 0$, it follows that $\{K_3, \widehat{\mathcal{H}}\} = 0$. Thus the flow $\varphi_r^{Z_{K_3}}$ of the Hamiltonian vector field Z_{K_3} on $(S_h^2 \times S_h^2, \{\ ,\ \})$ generates an S^1 symmetry of the Hamiltonian system $(\widehat{\mathcal{H}}, S_h^2 \times S_h^2, \{\ ,\ \})$. So this system is completely integrable.

We reduce this S^1 symmetry as follows. Consider the vector field Z_{K_3} on $\mathbb{R}^3 \times \mathbb{R}^3$ corresponding to the Hamiltonian $K_3 = \frac{1}{2}(\xi_3 + \eta_3)$. Its integral curves satisfy

$$\begin{aligned}
\dot{\xi}_1 &= \{\xi_1, K_3\} = \tfrac{1}{2}\{\xi_1, \xi_3\} = -\tfrac{1}{2}\xi_2 \\
\dot{\xi}_2 &= \{\xi_2, K_3\} = \tfrac{1}{2}\{\xi_2, \xi_3\} = \tfrac{1}{2}\xi_1 \\
\dot{\xi}_3 &= \{\xi_3, K_3\} = 0 \\
\dot{\eta}_1 &= \{\eta_1, K_3\} = \tfrac{1}{2}\{\eta_1, \eta_3\} = \tfrac{1}{2}\eta_2 \\
\dot{\eta}_2 &= \{\eta_2, K_3\} = \tfrac{1}{2}\{\eta_2, \eta_3\} = -\tfrac{1}{2}\eta_1 \\
\dot{\xi}_3 &= \{\eta_3, K_3\} = 0.
\end{aligned}$$

Thus the flow of Z_{K_3} on $\mathbb{R}^3 \times \mathbb{R}^3$ is

$$\begin{aligned}
\varphi_t^{Z_{K_3}}(\xi, \eta) = (&\xi_1 \cos t/2 - \xi_2 \sin t/2, \xi_1 \sin t/2 + \xi_2 \cos t/2, \xi_3, \\
&\eta_1 \cos t/2 + \eta_2 \sin t/2, \eta_1 \sin t/2 - \eta_2 \cos t/2, \eta_3),
\end{aligned}$$

which preserves $S_h^2 \times S_h^2$ and is periodic of period 4π.

We now determine the space $(S_h^2 \times S_h^2)/S^1$ of orbits of the vector field Z_{K_3}. We use invariant theory. The algebra of polynomials on $\mathbb{R}^3 \times \mathbb{R}^3$, which are invariant under the S^1 action given by the flow $\varphi_t^{Z_{K_3}}$, is generated by

$$\begin{array}{lll}
\sigma_1 = \xi_1^2 + \xi_2^2 & \sigma_2 = \eta_1^2 + \eta_2^2 & \sigma_3 = \xi_1\eta_2 - \xi_2\eta_1 \\
\sigma_4 = \xi_1\eta_1 + \xi_2\eta_2 & \sigma_5 = \frac{1}{2}(\xi_3 + \eta_3) & \sigma_6 = \frac{1}{2}(\xi_3 - \eta_3),
\end{array}$$

which are subject to the relation

$$\begin{aligned}
\sigma_3^2 + \sigma_4^2 &= (\xi_1\eta_2 - \xi_2\eta_1)^2 + (\xi_1\eta_1 + \xi_2\eta_2)^2 \\
&= (\xi_1^2 + \xi_2^2)(\eta_1^2 + \eta_2^2) = \sigma_1\sigma_2, \quad \sigma_1 \geq 0 \ \& \ \sigma_2 \geq 0.
\end{aligned} \tag{7.1a}$$

In terms of invariants the defining equations of $S_h^2 \times S_h^2$ become

$$\sigma_1 + (\sigma_5 + \sigma_6)^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 = h^2 \tag{7.1b}$$

$$\sigma_2 + (\sigma_5 - \sigma_6)^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 = h^2. \tag{7.1c}$$

Eliminating σ_1 and σ_2 from (7.1a) using (7.1b) and (7.1c) gives

$$\sigma_3^2 + \sigma_4^2 = (h^2 - (\sigma_5 + \sigma_6)^2)(h^2 - (\sigma_5 - \sigma_6)^2), \ |\sigma_5 + \sigma_6| \leq h \ \& \ |\sigma_5 - \sigma_6| \leq h, \tag{7.2a}$$

which defines $(S_h^2 \times S_h^2)/S^1$ as a semialgebraic variety in $\mathbb{R}^4$ with coordinates $(\sigma_3, \sigma_4, \sigma_5, \sigma_6)$. Thus the reduced space $(K_3^{-1}(2k) \cap (S_h^2 \times S_h^2))/S^1$ is defined by (7.2a) and

$$\sigma_5 = \tfrac{1}{2}(\xi_3 + \eta_3) = \tfrac{1}{2}K_3 = k. \tag{7.2b}$$

Consequently, $(K_3^{-1}(2k) \cap (S_h^2 \times S_h^2))/S^1$ is the semialgebraic variety

$$\begin{aligned}\sigma_3^2 + \sigma_4^2 &= (h^2 - (k+\sigma_6)^2)(h^2 - (k-\sigma_6)^2) \\ &= ((h-k)^2 - \sigma_6^2)((h+k)^2 - \sigma_6^2), \ \ |\sigma_6| \le h - |\ell| \end{aligned} \tag{7.3}$$

in $\mathbb{R}^3$ with coordinates $(\sigma_3, \sigma_4, \sigma_6)$. When $0 < |k| < h$ the reduced space (7.3) is a smooth 2-sphere. When $|k| = h$ it is a point. When $k = 0$ it is a topological 2-sphere with conical singular points at $(0, 0, \pm h)$. These singular points correspond to the fixed points $h(0, 0, \pm 1, 0, 0, \mp 1)$ of the S^1 action on $S_h^2 \times S_h^2$ generated by the flow of the vector field Z_{K_3}.

By (6.1) the reduced Hamiltonian on $(K_3^{-1}(2k) \cap (S_h^2 \times S_h^2))/S^1$ is

$$\widehat{\mathcal{H}}_{\mathrm{red}} = \tfrac{9}{4}\varepsilon^2\beta^2 h\, \sigma_6^2, \tag{7.4}$$

using $L_3 = \xi_3 - \eta_3 = 2\sigma_6$, having dropped the constant $h - \frac{13}{8}\varepsilon^2\beta^2h^3 - \frac{3}{2}\varepsilon\beta hk - \frac{51}{16}\varepsilon^2\beta^2hk^2$.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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