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Research article

Global attractors for a nonlinear plate equation modeling the oscillations of suspension bridges

Yang Liu[1,2,*]

[1] College of Mathematics and Computer Science, Northwest Minzu University, Lanzhou 730030, China

[2] College of Mathematics, Sichuan University, Chengdu 610065, China

* **Correspondence:** Email: liuyangnufn@163.com; 183091699@xbmu.edu.cn.

Abstract: This paper is concerned with a nonlinear plate equation modeling the oscillations of suspension bridges. Under mixed boundary conditions consisting of simply supported and free boundary conditions, we obtain the global well-posedness of solutions in suitable function spaces. In addition, we use the perturbed energy method to prove the existence of a bounded absorbing set and establish a stabilizability estimate. Then, we derive the existence of a global attractor by verifying the asymptotic smoothness of the corresponding dissipative dynamical system.

Keywords: nonlinear plate equations; global well-posedness; global attractors; dissipative dynamical system; asymptotic smoothness

Mathematics Subject Classification: 35L35, 35A01, 35B41

1. Introduction

In this paper, we study the following two-dimensional nonlinear plate equation modeling the small amplitude oscillations of suspension bridges

$$u_{tt} + \Delta^2 u + a(x,y)u + \mu u_t + f(u) = g(x,y), \quad (x,y) \in \Omega, \ t > 0 \tag{1.1}$$

with initial conditions

$$u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), \quad (x,y) \in \Omega \tag{1.2}$$

and mixed boundary conditions consisting of simply supported and free boundary conditions (see [27, Section 2.5])

$$\begin{cases} u(0,y,t) = u_{xx}(0,y,t) = u(k,y,t) = u_{xx}(k,y,t) = 0, \ \ y \in (-l,l), \ t > 0, \\ u_{yy}(x,\pm l,t) + ru_{xx}(x,\pm l,t) = 0, \ \ x \in (0,k), \ t > 0, \\ u_{yyy}(x,\pm l,t) + (2-r)u_{xxy}(x,\pm l,t) = 0, \ \ x \in (0,k), \ t > 0. \end{cases} \tag{1.3}$$

Here, the unknown function $u = u(x,y,t)$ represents the deflection at time t of a filament having position (x,y) in Ω. The domain $\Omega = (0,k) \times (-l,l) \subset \mathbb{R}^2$ with $0 < l \ll k$ is the rectangular deck of a suspension bridge having the two short edges $y = \pm l$ hinged and the two long edges $x = 0, k$ free. The constant r is the Poisson ratio depending on the material composing the bridge deck and usually lies in $(0, 1/2)$. The term $a(x,y)u$ denotes the restoring force provided by the hangers and $a(x,y)$ is a sign-changing and bounded measurable function. The term μu_t represents the weak damping caused by the internal friction and $\mu > 0$ is the damping coefficient. In addition, $g \in L^2(\Omega)$ stands for the external force acting on the bridge deck. Concerning the nonlinear source term f, we always assume that there exist constants $b > 0$ and $p > 2$ such that

$$|f(\bar{u}) - f(u)| \le b\left(1 + |u|^{p-2} + |\bar{u}|^{p-2}\right)|\bar{u} - u|, \ \ u, \bar{u} \in \mathbb{R}. \tag{1.4}$$

Moreover, there exist constants $0 \le \eta < A_1/S_2^2$ and $\varrho > 0$ such that

$$F(u) \ge -\frac{\eta}{2}u^2 - \varrho \tag{1.5}$$

and

$$uf(u) \ge F(u) - \frac{\eta}{2}u^2 - \varrho \tag{1.6}$$

where

$$F(u) = \int_0^u f(s)\,\mathrm{d}s$$

and the constants A_1 and S_2 will be specified in Section 2.

Deformations and oscillations of suspension bridges have attracted a great deal of attention. The one-dimensional beams and rods were used to simulate deformations and oscillations of suspension bridges. Lazer and McKenna [14] considered a beam equation

$$u_{tt} + Ku_{xxxx} + au^+ = \sin\frac{\pi x}{k}(S + \varepsilon g(t)), \ \ x \in (0,k), \ t > 0$$

where $u^+ = \max\{u, 0\}$, $a > 0$ is the elastic coefficient of the hangers, K is the flexural rigidity of the bridge deck, k is the length of the suspension bridge, S is a large constant, ε is a small parameter and $g(t)$ is a periodic function. They obtained multiple periodic solutions. McKenna and Walter [21] investigated a beam equation of the form

$$u_{tt} + Ku_{xxxx} + au^+ = 1 + \varepsilon g \tag{1.7}$$

where $g = g(x,t)$ is a periodic function. They got multiple periodic solutions depending on the range of a. In particular, under the situation $K = 1$ and $\varepsilon g = 0$ McKenna and Walter [22] studied travelling wave solutions to Eq. (1.7). Lazer and McKenna [15] suggested several rod models and simulated the sudden transition from vertical to torsional oscillations which occurred in the Tacoma Narrows bridge collapse (see [9, 10, 24]). Arioli and Gazzola [3] proposed a multiple rods model and also showed the sudden appearance of torsional oscillations. Battisti et al. [4] studied a nonlinear beam equation with nonlocal term

$$u_{tt} + u_{xxxx} + \left(P - \frac{2}{k}\int_0^k u_x^2 \,\mathrm{d}x\right)u_{xx} = 0$$

where $P > 0$ is the compression parameter. They established the existence of periodic solutions and discussed the energy transfer from one oscillation mode to another. Moreover, they provided corresponding numerical experiments.

In order to pursue reliability and accuracy of the model, various evolution plate equations were employed to model deformations and oscillations of suspension bridges. Under simply supported boundary condition

$$u(x, y, t) = \Delta u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \ t > 0, \tag{1.8}$$

Zhong et al. [36] investigated a nonlinear plate equation

$$u_{tt} + \Delta^2 u + au^+ + \mu u_t + f(u) = g(x, y)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open set with a sufficiently smooth boundary $\partial\Omega$. Under certain assumptions on $f \in C^3(\mathbb{R})$, they obtained global existence and uniqueness of solutions with certain regularity and derived the existence of a global attractor by verifying the condition (C) [20], a compactness criterion. Park and Kang [23] studied the following nonlinear plate equation with a more general damping

$$u_{tt} + \Delta^2 u + au^+ + \mu(x, y)h(u_t) + f(u) = g(x, y)$$

subject to simply supported boundary condition (1.8) where $\mu \in L^\infty(\Omega)$, $\mu(x, y) > 0$ and $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$. Under some assumptions on $f \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, they got the existence of a global attractor by verifying the asymptotic compactness.

Under mixed boundary conditions (1.3), Ferrero and Gazzola [9] investigated the following plate equation with a more general restoring force

$$u_{tt} + \Delta^2 u + h(x, y, u) + \mu u_t = g$$

and established global existence, uniqueness and asymptotic behavior of solutions. Their main results showed that if $g \in L^2(\Omega)$ is independent of t then the unique global solution converges to the stationary solution as time tends to infinity. Based on the potential well theory (see e.g. [11,12,16,19,29–31,33,34]), Wang [28] dealt with the following plate equation with nonlinear source term

$$u_{tt} + \Delta^2 u + a(x, y)u + \mu u_t = |u|^{p-2}u \tag{1.9}$$

where $2 < p < \infty$. He obtained local and global existence, uniqueness, asymptotic behavior and finite time blow-up of solutions to problem (1.9), (1.2), (1.3) with subcritical initial energy. Xu et al. [32] considered the following plate equation with nonlinear damping

$$u_{tt} + \Delta^2 u + a(x, y)u + \mu|u_t|^{q-2}u_t = |u|^{p-2}u \tag{1.10}$$

where $2 < q < p < \infty$. In the framework of the potential well theory, they got local and global existence, uniqueness, asymptotic behavior and finite time blow-up of solutions to problem (1.10), (1.2), (1.3) with subcritical and critical initial energy, respectively. Moreover, they derived the finite time blow-up of solutions to problem (1.9), (1.2), (1.3) with supercritical initial energy. Liu et al. [18] also obtained this blow-up result. In the case of neglecting the effects of internal friction and external force, Berchio et al. [5] analyzed a plate equation of the form

$$u_{tt} + \gamma\Delta^2 u + \Upsilon(y)(u + u^3) = 0$$

subject to mixed boundary conditions (1.3) where $r = 1/5$, $l = k/150$, $\gamma > 0$ and Υ is the characteristic function of a set. With a finite dimensional approximation, they proved that the system remains stable at low energies while numerical results showed that for large energies the system becomes unstable. They analyzed the energy thresholds of instability and provided interesting remarks on several questions left open by the Tacoma Narrows bridge collapse. In the case of neglecting the effects of the restoring force, Ferreira Jr et al. [8] investigated a nonlocal plate equation

$$u_{tt} + \Delta^2 u + \left(P - S\int_\Omega u_x^2\, \mathrm{d}x\mathrm{d}y\right)u_{xx} + \mu u_t = g \tag{1.11}$$

where $S > 0$ depends on the elasticity of the material composing the bridge deck, $S\int_\Omega u_x^2\, \mathrm{d}x\mathrm{d}y$ measures the geometric nonlinearity of the deck due to its stretching and $P > 0$ is the prestressing constant. They proved global existence, uniqueness and asymptotic behavior of solutions to problem (1.11), (1.2), (1.3). Furthermore, they proved the results on stability and instability and complemented the theoretical results with some numerical experiments. Bonheure et al. [6] also studied problem (1.11), (1.2), (1.3) and proved the well-posedness of periodic solutions. They made the phase space be orthogonally split into two subspaces containing the longitudinal and the torsional movements of bridge decks, respectively. For the longitudinal component, they gave the sufficient conditions for the stability of periodic solutions and of solutions. Moreover, they performed a stability analysis and provided the corresponding numerical simulations in which instabilities may occur.

The above works lay a rich mathematical theory for deformations and oscillations of suspension bridges and provide important reference value for practical problems. In the present paper, we would like to study the long-time dynamics of solutions to problem (1.1)–(1.3) and the global attractor is an effective way to handle this issue. Although [23, 36] have already involved the existence of global attractors, our research object is actually distinct since mixed boundary conditions (1.3) are more realistic and complex so that the phase space, energy estimates and the compactness criterion are all different. On the other hand, our assumptions on the nonlinear source term f are weaker than those in [23].

The main results of our paper are stated as follows.

Theorem 1.1 (Global well-posedness). *Let $-\lambda_1 < a_1 \le a(x,y) \le a_2$, $u_0 \in H_*^2(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, for any $T > 0$, problem* (1.1)–(1.3) *admits a unique solution $u \in C\left([0,T]; H_*^2(\Omega)\right)$ with $u_t \in C\left([0,T]; L^2(\Omega)\right)$ which depends continuously on the initial data.*

Define an operator $S(t) : Z \to Z$ by

$$S(t)(u_0, u_1) := (u(t), u_t(t))$$

where the phase space $Z := H^2_*(\Omega) \times L^2(\Omega)$. Then, it is easy to see from Theorem 1.1 that $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup generated by problem (1.1)–(1.3).

Theorem 1.2 (Existence of global attractors)**.** *In addition to the conditions of Theorem 1.1, suppose that $a_1 \geq 0$. Then, the dynamical system $(Z, S(t))$ corresponding to problem* (1.1)–(1.3) *possesses a global attractor.*

In the above theorems, the constant λ_1 and the space $H^2_*(\Omega)$ will be stated in detail in Section 2.

The rest of this paper is organized as follows. In Section 2, we display some notations and prepare several preliminary definitions and conclusions related to problem (1.1)–(1.3). Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

2. Preliminaries

Throughout the paper, for the sake of simplicity, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}, \quad \|\cdot\| := \|\cdot\|_2, \quad (u, v) := \int_\Omega uv \,\mathrm{d}x\mathrm{d}y.$$

Moreover, C represents a generic positive constant that may be different even in the same formula and $C(\cdot, \cdots, \cdot)$ stands for a positive constant depending on the quantities appearing in the parenthesis.

As in [2, 9], we define a Hilbert space

$$H^2_*(\Omega) := \left\{ u \in H^2(\Omega) \middle| \, u = 0 \text{ on } \{0, k\} \times (-l, l) \right\}$$

endowed with the inner product

$$(u, v)_{*2} := (u, v)_{H^2_*(\Omega)} = \int_\Omega (\Delta u \Delta v + (1 - r)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\,\mathrm{d}x\mathrm{d}y$$

and the norm

$$\|u\|_{*2} := \|u\|_{H^2_*(\Omega)} = \left(\int_\Omega |\Delta u|^2 \mathrm{d}x\mathrm{d}y + 2(1 - r) \int_\Omega \left(u^2_{xy} - u_{xy}u_{yy} \right) \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}$$

which is equivalent to $\|\cdot\|_{H^2(\Omega)}$. Here, in terms of [1, Theorem 4.15], $\|\cdot\|_{H^2(\Omega)}$ can be defined by $\left(\|D^2 \cdot\|^2 + \|\cdot\|^2\right)^{1/2}$. According to [28], we have the following two inequalities.

Lemma 2.1 ([28])**.** *Assume that $1 \leq q < \infty$. Then, for any $u \in H^2_*(\Omega)$, there holds $\|u\|_q \leq S_q \|u\|_{*2}$ where*

$$S_q := \left(\frac{k}{2l} + \frac{\sqrt{2}}{2} \right) (2kl)^{\frac{q+2}{2q}} \left(\frac{1}{1 - r} \right)^{\frac{1}{2}}.$$

Lemma 2.2 ([28])**.** *Assume that $-\lambda_1 < a_1 \leq a(x, y) \leq a_2$ where $\{\lambda_j\}_{j=1}^\infty$ is the eigenvalue sequence of the eigenvalue problem*

$$\begin{cases} \Delta^2 u = \lambda u, \quad (x, y) \in \Omega, \\ u(0, y) = u_{xx}(0, y) = u(k, y) = u_{xx}(k, y) = 0, \quad y \in (-l, l), \\ u_{yy}(x, \pm l) + r u_{xx}(x, \pm l) = 0, \quad x \in (0, k), \\ u_{yyy}(x, \pm l) + (2 - r) u_{xxy}(x, \pm l) = 0, \quad x \in (0, k), \end{cases}$$

and $0 < \lambda_1 < 1$*. Then, for any* $u \in H^2_*(\Omega)$*, there holds*

$$A_1\|u\|^2_{*2} \leq \|u\|^2_{*2} + (au, u) \leq A_2\|u\|^2_{*2}$$

where

$$A_1 := \begin{cases} 1 + \dfrac{a_1}{\lambda_1}, & a_1 < 0, \\ 1, & a_1 \geq 0, \end{cases} \qquad A_2 := \begin{cases} 1, & a_2 < 0, \\ 1 + \dfrac{a_2}{\lambda_1}, & a_2 \geq 0. \end{cases}$$

Following [2], we introduce a space

$$H^1_*(\Omega) := \left\{ u \in H^1(\Omega) \middle| u = 0 \text{ on } \{0, k\} \times (-l, l) \right\}$$

which is defined as the closure of $C^\infty_*(\Omega)$ with respect to the norm

$$\|u\|_{*1} := \|u\|_{H^1_*(\Omega)} = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}. \tag{2.1}$$

Here,

$$C^\infty_*(\Omega) := \left\{ u \in C^\infty\left(\overline{\Omega}\right) \middle| \; \exists \varepsilon > 0, \; u(x, y) = 0 \text{ if } x \in [0, \varepsilon] \cup [k - \varepsilon, k] \right\}$$

is a normed space equipped with the norm $\|\cdot\|_{*1}$. The inner product in $H^1_*(\Omega)$ is defined by

$$(u, v)_{*1} := (u, v)_{H^1_*(\Omega)} = \int_\Omega \nabla u \nabla v \, \mathrm{d}x\mathrm{d}y, \quad u, v \in H^1_*(\Omega).$$

Moreover, the embedding $H^2_*(\Omega) \hookrightarrow H^1_*(\Omega)$ is compact.

Next, we prove the equivalence between $\|\cdot\|_{*1}$ and $\|\cdot\|_{H^1(\Omega)}$.

Lemma 2.3. *For any* $u \in H^1_*(\Omega)$*, the two norms* $\|u\|_{*1}$ *and* $\|u\|_{H^1(\Omega)}$ *are equivalent.*

Proof. For any $u \in H^1_*(\Omega)$, we have

$$|u(x, y)| = \left| \int_0^x u_s(s, y) \, \mathrm{d}s \right| \leq \int_0^k |u_x(x, y)| \, \mathrm{d}x.$$

By virtue of Schwarz's inequality, we further obtain

$$|u(x, y)| \leq k^{\frac{1}{2}} \left(\int_0^k u_x^2(x, y) \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Hence,

$$\int_\Omega |u(x, y)|^2 \, \mathrm{d}x\mathrm{d}y \leq k^2 \int_\Omega u_x^2(x, y) \, \mathrm{d}x\mathrm{d}y$$

which implies

$$\|u\|^2 \leq k^2 \|Du\|^2.$$

Thus,

$$\|Du\|^2 \leq \|u\|^2_{H^1(\Omega)} \leq (1 + k^2)\|Du\|^2$$

which means that $\|Du\|$ is equivalent to $\|u\|_{H^1(\Omega)}$. On the other hand, it is easy to see from (2.1) that $\|u\|_{*1}$ is equivalent to $\|Du\|$. Accordingly, the proof of this lemma is finished.

In terms of Lemma 2.3, we can draw the following conclusion which will be applied in the proofs of our main results.

Corollary 2.4. *Assume that* $2 \leq q < \infty$*. Then, for any* $u \in H^1_*(\Omega)$*, there exists a constant* $\mathfrak{C} := \mathfrak{C}(\Omega, q)$ *such that* $\|u\|_q \leq \mathfrak{C}\|u\|_{*1}$.

Definition 2.5 (Weak solutions)**.** *For given* $T > 0$*, a function* $u \in C\left([0,T]; H^2_*(\Omega)\right)$ *with* $u_t \in C\left([0,T]; L^2(\Omega)\right)$ *is called a weak solution to problem* (1.1)–(1.3) *in* $\Omega \times [0,T]$*, if* $u(0) = u_0$ *in* $H^2_*(\Omega)$*,* $u_t(0) = u_1$ *in* $L^2(\Omega)$ *and*

$$\begin{aligned} (u_t(t), v) + \int_0^t (u(\tau), v)_* \,\mathrm{d}\tau + \int_0^t (au(\tau), v)\,\mathrm{d}\tau + \mu(u(t), v) \\ + \int_0^t (f(u(\tau)), v)\,\mathrm{d}\tau = \int_0^t (g, v)\,\mathrm{d}\tau + (u_1, v) + \mu(u_0, v) \end{aligned} \tag{2.2}$$

for any $v \in H^2_*(\Omega)$ *and* $t \in (0,T]$.

Remark 2.6. *Eq.* (2.2) *implies that*

$$\langle u_{tt}(t), v\rangle + (u(t), v)_* + (au(t), v) + \mu(u_t(t), v) + (f(u(t)), v) = (g, v) \tag{2.3}$$

for a.e. $t \in (0,T]$ *where* $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $H^2_*(\Omega)$ *and its dual space* $\mathcal{H}(\Omega)$.

3. Proof of Theorem 1.1

In the proof of Theorem 1.1, we shall see that $u \in L^\infty\left(0,T; H^2_*(\Omega)\right)$ with $u_t \in L^\infty\left(0,T; L^2(\Omega)\right)$. In order to further demonstrate $u \in C\left([0,T]; H^2_*(\Omega)\right)$ with $u_t \in C\left([0,T]; L^2(\Omega)\right)$, we need the following two lemmas.

Lemma 3.1 ([26])**.** *Let* U *and* V *be two Banach spaces such that* $U \subset V$ *with a continuous injection. If a function* $u \in L^\infty(0,T;U)$ *and* $u \in C_w([0,T];V)$ *then* $u \in C_w([0,T];U)$*. Here,* $C_w([0,T];W)$ *means the subspace of* $L^\infty(0,T;W)$ *consisting of those functions which are almost everywhere equal to weakly continuous functions with values in a Banach space* W.

Lemma 3.2. *Under the conditions of Theorem 1.1, if a function* $u \in L^2\left(0,T; H^2_*(\Omega)\right)$ *with* $u_t \in L^2\left(0,T; L^2(\Omega)\right)$ *is a solution to problem* (1.1)–(1.3) *then*

$$\left(u_{tt}(t) + \Delta^2 u(t), u_t(t)\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t(t)\|^2 + \|u(t)\|^2_{*2}\right).$$

Proof. Extend u to be zero outside $(0,T]$. Let $\tilde{u} := \zeta u$ where the truncation function $\zeta : \mathbb{R} \to [0,1]$ is 0 on $\mathbb{R} \setminus (0,T]$, 1 on $(\delta, T-\delta]$ with small $\delta > 0$ and otherwise linear. Set the mollification of v to be

$$\tilde{u}_\varepsilon := \eta_\varepsilon * \tilde{u}$$

where the mollifier $\eta_\varepsilon(t) := 1/\varepsilon\eta(t/\varepsilon)$, $\varepsilon > 0$ and $\eta(t)$ is a nonnegative even C^∞-function on the real line with compact support and integral one. According to the regularization theory, we have $\tilde{u}_\varepsilon \in C^\infty\left(\mathbb{R}; H^2_*(\Omega)\right)$. $\varepsilon \to 0$,

$$\tilde{u}_\varepsilon \to \tilde{u} \text{ in } L^2\left(\mathbb{R}; H^2_*(\Omega)\right),$$

$$\tilde{u}_{\varepsilon t} \to \tilde{u}_t \text{ in } L^2\left(\mathbb{R}; L^2(\Omega)\right).$$

Note that

$$\left(\tilde{u}_{\varepsilon tt}(t) + \Delta^2 \tilde{u}_\varepsilon(t), \tilde{u}_{\varepsilon t}(t)\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_{\varepsilon t}(t)\|^2 + \|\tilde{u}_\varepsilon(t)\|_{*2}^2\right).$$

Taking $\varepsilon \to 0$ first, the above relation still holds for $\tilde{u}$. Finally, by taking $\delta \to 0$ and restriction to $(0, T]$ we finish the proof of Lemma 3.2.

Proof of Theorem 1.1. We divide the proof of this theorem into four steps.

Step I. Galerkin approximations.

Let $\{w_j\}_{j=1}^\infty$ be the eigenfunctions of the eigenvalue problem in Lemma 2.2. Then, according to [9, Theorem 3.4] $\{w_j\}_{j=1}^\infty$ is an orthogonal basis of $H_*^2(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. Denote $W_n := \{\omega_1, \omega_2, \cdots, \omega_n\}$. Set

$$u_{0n} := \sum_{j=1}^n (u_0, \omega_j)\omega_j$$

and

$$u_{1n} := \sum_{j=1}^n (u_1, \omega_j)\omega_j$$

such that

$$u_{0n} \to u_0 \text{ in } H_*^2(\Omega) \tag{3.1}$$

and

$$u_{1n} \to u_1 \text{ in } L^2(\Omega) \tag{3.2}$$

as $n \to \infty$. For all $n \geq 1$ we seek n functions $\xi_{1n}, \xi_{2n}, \cdots, \xi_{nn} \in C^2[0, T]$ to construct the approximate solutions to problem (1.1)–(1.3)

$$u_n(t) := \sum_{j=1}^n \xi_{jn}(t)\omega_j, \quad n = 1, 2, \cdots \tag{3.3}$$

which satisfy

$$\left(u_{ntt}(t) + \Delta^2 u_n(t) + a u_n(t) + \mu u_{nt}(t) + f(u_n(t)) - g, v\right) = 0, \quad t > 0, \tag{3.4}$$

$$u_n(0) = u_{0n}, \quad u_{nt}(0) = u_{1n}, \tag{3.5}$$

for any $v \in W_n$. Let $\xi_n(t) := (\xi_{1n}(t), \xi_{2n}(t), \cdots, \xi_{nn}(t))^T$. Then, by taking $v = \omega_i$ $(i = 1, 2, \cdots, n)$ in (3.4) the vector function ξ_n solves

$$\xi_n''(t) + \mu\xi_n'(t) + \mathcal{L}_n(\xi_n(t)) = 0, \quad t > 0, \tag{3.6}$$

$$\xi_n(0) = ((u_0, \omega_1), (u_0, \omega_2), \cdots, (u_0, \omega_n))^T, \tag{3.7}$$

$$\xi_n'(0) = ((u_1, \omega_1), (u_1, \omega_2), \cdots, (u_1, \omega_n))^T \tag{3.8}$$

where $\mathcal{L}_n : \mathbb{R}^n \to \mathbb{R}^n$ is the map defined by

$$\mathcal{L}_n(\xi_n(t)) := (\mathcal{L}_{1n}(\xi_n(t)), \mathcal{L}_{2n}(\xi_n(t)), \cdots, \mathcal{L}_{nn}(\xi_n(t)))^T,$$

$$\mathcal{L}_{in}(\xi_n(t)) := \left(\sum_{j=1}^{n} \xi_{jn}(t)\Delta^2\omega_j + a\sum_{j=1}^{n} \xi_{jn}(t)\omega_j + f\left(\sum_{j=1}^{n} \xi_{jn}(t)\omega_j\right) - g, \omega_i\right), \quad i = 1, 2, \cdots, n.$$

In terms of the standard theory for ordinary differential equations, the Cauchy problem (3.6)–(3.8) admits a unique local solution $\xi_n \in C^2[0, T_n)$ with $T_n \le T$. In turn, this gives a solution $u_n(t)$ defined by (3.3) and satisfying (3.4), (3.5).

Step II. A priori estimates.

Taking $v = u_{nt}(t)$ in (3.4), we obtain

$$E_n'(t) + \mu \|u_{nt}(t)\|^2 = 0 \tag{3.9}$$

where

$$E_n(t) := \frac{1}{2}\|u_{nt}(t)\|^2 + \frac{1}{2}\|u_n(t)\|_{*2}^2 + \frac{1}{2}(au_n(t), u_n(t)) + \int_\Omega F(u_n(t))\,\mathrm{d}x\mathrm{d}y - (g, u_n(t)). \tag{3.10}$$

Integrating (3.9) with respect to t from 0 to t, we reach

$$E_n(t) + \mu\int_0^t \|u_{n\tau}(\tau)\|^2\,\mathrm{d}\tau = E_n(0), \quad t \in [0, T_n). \tag{3.11}$$

For the fourth term on the right-hand side of (3.10), it follows from (1.5) that

$$\int_\Omega F(u_n(t))\,\mathrm{d}x\mathrm{d}y \ge -\frac{\eta}{2}\|u_n(t)\|^2 - 2\varrho kl.$$

In light of Lemma 2.1, we get

$$\int_\Omega F(u_n(t))\,\mathrm{d}x\mathrm{d}y \ge -\frac{\eta S_2^2}{2}\|u_n(t)\|_{*2}^2 - 2\varrho kl. \tag{3.12}$$

For the fifth term on the right-hand side of (3.10), we deduce from Schwarz's inequality, Lemma 2.1 and Cauchy's inequality with $\epsilon > 0$ that

$$\begin{aligned}(g, u_n(t)) \le& \|g\|\|u_n(t)\| \\ \le& \epsilon S_2^2\|u_n(t)\|_{*2}^2 + \frac{1}{4\epsilon}\|g\|^2.\end{aligned} \tag{3.13}$$

Consequently, by substituting (3.12) and (3.13) into (3.10), applying Lemma 2.2 and choosing sufficiently small ϵ such that

$$\delta := \frac{A_1}{2} - \frac{\eta S_2^2}{2} - \epsilon S_2^2 > 0,$$

we obtain

$$E_n(t) \ge \frac{1}{2}\|u_{nt}(t)\|^2 + \delta\|u_n(t)\|_{*2}^2 - C\left(\|g\|^2 + 2kl\right), \quad t \in [0, T_n). \tag{3.14}$$

Therefore, from (3.11), (3.14), (3.1), (3.2) and $g \in L^2(\Omega)$, it follows that

$$\|u_{nt}(t)\|^2 + \|u_n(t)\|_{*2}^2 \le C, \quad t \in [0, T_n) \tag{3.15}$$

where C is independent of n. Estimate (3.15) allows us to extend the approximate solutions to problem (1.1)–(1.3) to the interval $[0, T]$ for any $T > 0$.

Step III. Passage to the limit.

From estimate (3.15) we learn that there exist a subsequence of $\{u_n\}$ (still denoted by the same notation) and a function u such that as $n \to \infty$,

$$u_n \rightharpoonup u \text{ weakly star in } L^\infty\left(0, T; H_*^2(\Omega)\right)$$

and

$$u_{nt} \rightharpoonup u_t \text{ weakly star in } L^\infty\left(0, T; L^2(\Omega)\right)$$

for any $T > 0$. Since the embedding $H_*^2(\Omega) \hookrightarrow H_*^1(\Omega)$ is compact, we infer from the Simon-Aubin compact embedding [25] that up to a further subsequence

$$u_n \to u \text{ in } C\left([0, T]; H_*^1(\Omega)\right). \tag{3.16}$$

We claim that for all $t \in [0, T]$,

$$\int_0^t (f(u_n(\tau)), v)\, \mathrm{d}\tau \to \int_0^t (f(u(\tau)), v)\, \mathrm{d}\tau$$

as $n \to \infty$. From (1.4), Hölder's inequality with $(p-2)/(2p-2) + 1/(2p-2) + 1/2 = 1$, Minkowski's inequality, Lemma 2.1, Corollary 2.4 and estimate (3.15), we deduce that

$$\begin{aligned}
|(f(u_n(t)) - f(u(t)), v)| \le & b \left|\left(\left(1 + |u_n(t)|^{p-2} + |u(t)|^{p-2}\right) |u_n(t) - u(t)|, v\right)\right| \\
\le & b \left((2kl)^{\frac{p-2}{2p-2}} + \|u_n(t)\|_{2p-2}^{p-2} + \|u(t)\|_{2p-2}^{p-2}\right) \|u_n(t) - u(t)\|_{2p-2} \|v\| \\
\le & C \|u_n(t) - u(t)\|_{*1}.
\end{aligned} \tag{3.17}$$

Consequently,

$$\left|\int_0^t (f(u_n(\tau)) - f(u(\tau)), v)\, \mathrm{d}\tau\right| \le C \int_0^t \|u_n(\tau) - u(\tau)\|_{*1}\, \mathrm{d}\tau.$$

Thus, the assertion follows from (3.16).

As a consequence, integrating (3.4) with respect to t and passing to the limit as $n \to \infty$, we arrive at (2.2). In view of $u \in L^\infty\left(0, T; H_*^2(\Omega)\right)$ and $u_t \in L^\infty\left(0, T; L^2(\Omega)\right)$, we have $u \in C_w\left([0, T]; L^2(\Omega)\right)$. Hence, by Lemma 3.1 we derive $u \in C_w\left([0, T]; H_*^2(\Omega)\right)$. We infer from Remark 2.6 that $u_{tt} \in L^2\left(0, T; \mathcal{H}(\Omega)\right)$. Thus, $u_t \in C_w\left([0, T]; \mathcal{H}(\Omega)\right)$. Again, by Lemma 3.1 we get $u_t \in C_w\left([0, T]; L^2(\Omega)\right)$. Thanks to Lemma 3.2, we see that the function

$$t \mapsto \|u_t(t)\|^2 + \|u(t)\|_{*2}^2$$

is continuous on $[0, T]$. Hence, $u \in C\left([0, T]; H_*^2(\Omega)\right)$ and $u_t \in C\left([0, T]; L^2(\Omega)\right)$. Moreover, we infer from (3.1) and (3.2) that $u(0) = u_0$ in $H_*^2(\Omega)$ and $u_t(0) = u_1$ in $L^2(\Omega)$. Therefore, u is a global solution to problem (1.1)–(1.3) in the sense of Definition 2.5.

Step IV. Continuous dependence and uniqueness.

Suppose that u and $\bar{u}$ are two solutions to problem (1.1)–(1.3) with initial data u_0, u_1 and $\bar{u}_0$, $\bar{u}_1$, respectively. Set $\tilde{u} := \bar{u} - u$. Then, $\tilde{u}$ is a solution to the following equation

$$\tilde{u}_{tt} + \Delta^2\tilde{u} + a(x,y)\tilde{u} + \mu\tilde{u}_t + f(\bar{u}) - f(u) = 0 \tag{3.18}$$

with

$$\tilde{u}(0) = \tilde{u}_0 := \bar{u}_0 - u_0, \ \ \tilde{u}_t(0) = \tilde{u}_1 := \bar{u}_1 - u_1.$$

By the analogous arguments in Lemma 3.2, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t(t)\|^2 + \|\tilde{u}(t)\|_{*2}^2 + (a\tilde{u}(t), \tilde{u}(t))\right) + \mu\|\tilde{u}_t(t)\|^2 \\ =& -(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)). \end{aligned} \tag{3.19}$$

By the arguments similar to the proof of (3.17), we can obtain

$$\begin{aligned} -(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)) \le& b\left((2kl)^{\frac{p-2}{2p-2}} + \|u(t)\|_{2p-2}^{p-2} + \|\bar{u}(t)\|_{2p-2}^{p-2}\right)\|\tilde{u}(t)\|_{2p-2}\|\tilde{u}_t(t)\| \\ \le& C\|\tilde{u}(t)\|_{*2}\|\tilde{u}_t(t)\|. \end{aligned} \tag{3.20}$$

Applying Cauchy's inequality with $\epsilon > 0$, we get

$$-(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)) \le C(\epsilon)\|\tilde{u}(t)\|_{*2}^2 + \epsilon\|\tilde{u}_t(t)\|^2.$$

Hence, by taking $\epsilon = \mu$ we deduce from (3.19) and Lemma 2.2 that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t(t)\|^2 + \|\tilde{u}(t)\|_{*2}^2 + (a\tilde{u}(t), \tilde{u}(t))\right) \le C\left(\|\tilde{u}_t(t)\|^2 + \|\tilde{u}(t)\|_{*2}^2 + (a\tilde{u}(t), \tilde{u}(t))\right).$$

Consequently, we conclude from Gronwall's inequality that

$$\|\tilde{u}_t(t)\|^2 + \|\tilde{u}(t)\|_{*2}^2 + (a\tilde{u}(t), \tilde{u}(t)) \le C\left(\|\tilde{u}_1\|^2 + \|\tilde{u}_0\|_{*2}^2 + (a\tilde{u}_0, \tilde{u}_0)\right) \tag{3.21}$$

for all $t \in [0, T]$. By Lemma 2.2, we have

$$\|\tilde{u}(t)\|_{*2}^2 + (a\tilde{u}(t), \tilde{u}(t)) \ge A_1\|\tilde{u}(t)\|_{*2}^2 \tag{3.22}$$

and

$$\|\tilde{u}_0\|_{*2}^2 + (a\tilde{u}_0, \tilde{u}_0) \le A_2\|\tilde{u}_0\|_{*2}^2. \tag{3.23}$$

Combining (3.21)–(3.23), we arrive at

$$\|\tilde{u}_t(t)\|^2 + \|\tilde{u}(t)\|_{*2}^2 \le C\left(\|\tilde{u}_1\|^2 + \|\tilde{u}_0\|_{*2}^2\right)$$

for all $t \in [0, T]$.

In particular, by taking $u_0 = \bar{u}_0$ and $u_1 = \bar{u}_1$, it is clear that u is the unique solution to problem (1.1)–(1.3).

Thus, the proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

In this section, we write

$$\|S(t)z\|_Z^2 := \|u(t)\|_{*2}^2 + \|u_t(t)\|^2, \quad z := (u_0, u_1).$$

According to [7, Theorem 2.3] and [7, Proposition 2.10], we will prove the existence of a global attractor for problem (1.1)–(1.3) by verifying dissipativity and asymptotic smoothness of the corresponding dynamical system $(Z, S(t))$. For the convenience of the reader we display the two abstract results from [7].

Theorem 4.1 ([7]). *Let $(Z, S(t))$ be a dissipative dynamical system in a complete metric space Z. Then, $(Z, S(t))$ possesses a compact global attractor if and only if $(Z, S(t))$ is asymptotically smooth.*

Proposition 4.2 ([7]). *Let $(Z, S(t))$ be a dynamical system on a complete metric space Z endowed with a metric d. Assume that for any bounded positively invariant set B in Z and $\varsigma > 0$, there exists $T = T(\varsigma, B)$ such that*

$$d(S(T)z, S(T)\bar{z}) \leq \varsigma + \Phi_T(z, \bar{z}), \quad z, \bar{z} \in B$$

where $\Phi_T(z, \bar{z})$ is a function defined on $B \times B$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} \Phi_T(z_n, z_m) = 0$$

for any sequence $\{z_n\}$ in B. Then $(Z, S(t))$ is an asymptotically smooth dynamical system.

In order to demonstrate the dissipativity of the dynamical system $(Z, S(t))$, we first define the total energy function associated with problem (1.1)-(1.3)

$$E(t) := \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|u(t)\|_{*2}^2 + \frac{1}{2}(au(t), u(t)) + \int_\Omega F(u(t))\,\mathrm{d}x\mathrm{d}y - (g, u(t)). \tag{4.1}$$

The following lemma provides the properties of $E(t)$.

Lemma 4.3. *Under the conditions of Theorem 1.1,*

$$E'(t) = -\mu\|u_t(t)\|^2. \tag{4.2}$$

Moreover, there exist two constants $M_1, M_2 > 0$ such that

$$E(t) \geq M_1\left(\|u(t)\|_{*2}^2 + \|u_t(t)\|^2\right) - M_2\left(\|g\|^2 + 2kl\right). \tag{4.3}$$

Proof. In terms of Lemma 3.2, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t(t)\|^2 + \|u(t)\|_{*2}^2 + (au(t), u(t))\right) + \mu\|u_t(t)\|^2\\ =& -(f(u(t)), u_t(t)) + (g, u_t(t))\end{aligned}$$

and so (4.2) is obtained immediately. Moreover, it is easy to see from the arguments similar to the proof of (3.14) that (4.3) holds.

Now we employ the perturbed energy method [13, 17, 35] with some modifications to show that the dynamical system $(Z, S(t))$ corresponding to problem (1.1)–(1.3) is dissipative.

Lemma 4.4 (Absorbing set). *Under the conditions of Theorem 1.2, the semigroup $S(t)$ has a bounded absorbing set in Z.*

Proof. We perform a suitable modification of the total energy function as follows

$$\Psi(t) := E(t) + \varepsilon\psi(t) \tag{4.4}$$

where

$$\psi(t) := (u(t), u_t(t))$$

and $\varepsilon > 0$ is a constant to be determined later.

We first claim that there exist four constants γ_i $(i = 1, 2, 3, 4)$ depending on ε such that

$$\gamma_1 E(t) - \gamma_2 \left(\|g\|^2 + 2kl\right) \le \Psi(t) \le \gamma_3 E(t) + \gamma_4 \left(\|g\|^2 + 2kl\right). \tag{4.5}$$

Indeed, from Schwarz's and Cauchy's inequalities and Lemma 2.1 we discover

$$\begin{aligned}|\psi(t)| \le& \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2\\ \le& \frac{S_2^2}{2}\|u(t)\|_{*2}^2 + \frac{1}{2}\|u_t(t)\|^2.\end{aligned}$$

Thus, there exists a constant $K > 0$ such that

$$|\psi(t)| \le K\left(\|u(t)\|_{*2}^2 + \|u_t(t)\|^2\right). \tag{4.6}$$

In view of (4.3) in Lemma 4.3, we can get

$$\|u(t)\|_{*2}^2 + \|u_t(t)\|^2 \le \frac{1}{M_1}E(t) + \frac{M_2}{M_1}\left(\|g\|^2 + 2kl\right).$$

Inserting this inequality into (4.6), we obtain

$$|\psi(t)| \le \frac{K}{M_1}E(t) + \frac{KM_2}{M_1}\left(\|g\|^2 + 2kl\right).$$

Hence, we deduce from (4.4) that

$$\left(1 - \varepsilon\frac{K}{M_1}\right)E(t) - \varepsilon\frac{KM_2}{M_1}\left(\|g\|^2 + 2kl\right) \le \Psi(t) \le \left(1 + \varepsilon\frac{K}{M_1}\right)E(t) + \varepsilon\frac{KM_2}{M_1}\left(\|g\|^2 + 2kl\right).$$

Thus, assertion (4.5) is proved, and $\gamma_1 > 0$ will be ensured by the selection of ε later.

Next, we claim that

$$\Psi'(t) \le -\varepsilon E(t) + 2\varepsilon\varrho kl. \tag{4.7}$$

To confirm this, we note that

$$\psi'(t) = \|u_t(t)\|^2 + \langle u_{tt}(t), u(t)\rangle.$$

By Remark 2.6, we get

$$\psi'(t) = \|u_t(t)\|^2 - \|u(t)\|_{*2}^2 - (au(t), u(t)) - \mu(u_t(t), u(t)) - (f(u(t)), u(t)) + (g, u(t)).$$

Hence, from (4.4) and (4.2) in Lemma 4.3, we have

$$\begin{aligned}\Psi'(t) =& -(\mu - \varepsilon)\|u_t(t)\|^2 - \varepsilon\|u(t)\|_{*2}^2 - \varepsilon(au(t), u(t)) \\ & - \varepsilon\mu(u_t(t), u(t)) - \varepsilon(f(u(t)), u(t)) + \varepsilon(g, u(t)).\end{aligned}$$

By virtue of (4.1), we further get

$$\begin{aligned}\Psi'(t) =& -\varepsilon E(t) - \left(\mu - \frac{3\varepsilon}{2}\right)\|u_t(t)\|^2 - \frac{\varepsilon}{2}\|u(t)\|_{*2}^2 - \frac{\varepsilon}{2}(au(t), u(t)) \\ & - \varepsilon\mu(u_t(t), u(t)) + \varepsilon\left(\int_\Omega F(u(t))\,\mathrm{d}x\mathrm{d}y - (f(u(t)), u(t))\right).\end{aligned} \tag{4.8}$$

For the fifth term on the right-hand side of (4.8), we deduce from Schwarz's inequality, Lemma 2.1 and Cauchy's inequalities with $\epsilon > 0$ that

$$\begin{aligned}-\mu(u_t(t), u(t)) \leq& \mu\|u(t)\|\|u_t(t)\| \\ \leq& \epsilon S_2^2\|u(t)\|_{*2}^2 + \frac{\mu^2}{4\epsilon}\|u_t(t)\|^2.\end{aligned} \tag{4.9}$$

For the last term on the right-hand side of (4.8), it follows from (1.6) and Lemma 2.1 that

$$\begin{aligned}\varepsilon\left(\int_\Omega F(u(t))\,\mathrm{d}x\mathrm{d}y - (f(u(t)), u(t))\right) \leq& \varepsilon\left(\frac{\eta}{2}\|u\|^2 + 2\varrho kl\right) \\ \leq& \varepsilon\left(\frac{\eta S_2^2}{2}\|u\|_{*2}^2 + 2\varrho kl\right).\end{aligned} \tag{4.10}$$

Consequently, by substituting (4.9) and (4.10) into (4.8) we obtain

$$\begin{aligned}\Psi'(t) \leq& -\varepsilon E(t) - \left(\mu - \varepsilon\left(\frac{3}{2} + \frac{\mu^2}{4\epsilon}\right)\right)\|u_t(t)\|^2 \\ & - \varepsilon\left(\frac{1}{2} - \frac{\eta S_2^2}{2} - \epsilon S_2^2\right)\|u(t)\|_{*2}^2 - \frac{\varepsilon}{2}(au(t), u(t)) + 2\varepsilon\varrho kl.\end{aligned} \tag{4.11}$$

From the condition $a_1 \geq 0$ and Lemma 2.2, it is apparent that $A_1 = 1$. We are now in a position to choose sufficiently small ϵ such that

$$\frac{1}{2} - \frac{\eta S_2^2}{2} - \epsilon S_2^2 > 0.$$

For fixed ϵ, we choose

$$\varepsilon < \min\left\{\frac{M_1}{K}, \frac{4\epsilon\mu}{6\epsilon + \mu^2}\right\}.$$

Thus, the middle three terms on the right-hand side of (4.11) are non-positive and could be neglected so assertion (4.7) is demonstrated. Here, $\varepsilon < M_1/K$ ensures $\gamma_1 > 0$ in assertion (4.5).

By assertion (4.7) and the second inequality in assertion (4.5), we have

$$\Psi'(t) \leq -\frac{\varepsilon}{\gamma_3}\Psi(t) + \varepsilon\frac{\gamma_4}{\gamma_3}\|g\|^2 + 2\varepsilon\left(\varrho + \frac{\gamma_4}{\gamma_3}\right)kl.$$

Hence,

$$\Psi(t) \leq \Psi(0)e^{-\frac{\varepsilon}{\gamma_3}t} + \gamma_4\|g\|^2 + 2\left(\gamma_3\varrho + \gamma_4\right)kl. \tag{4.12}$$

By the second inequality in assertion (4.5), we have

$$\Psi(0) \leq \gamma_3 E(0) + \gamma_4\left(\|g\|^2 + 2kl\right)$$

which together with (4.12) and the first inequality in assertion (4.5) yields

$$E(t) \leq \left(\frac{\gamma_3}{\gamma_1}E(0) + \frac{\gamma_4}{\gamma_1}\left(\|g\|^2 + 2kl\right)\right)e^{-\frac{\varepsilon}{\gamma_3}t} + \frac{\gamma_4 + \gamma_2}{\gamma_1}\|g\|^2 + \frac{2(\gamma_3\varrho + \gamma_4 + \gamma_2)}{\gamma_1}kl.$$

Accordingly, we deduce from (4.3) that

$$\|S(t)z\|_Z^2 \leq \frac{\gamma_3}{\gamma_1 M_1}E(0)e^{-\frac{\varepsilon}{\gamma_3}t} + \frac{\gamma_2 + 2\gamma_4 + \gamma_1 M_2}{\gamma_1 M_1}\|g\|^2 + \frac{2(\gamma_3\varrho + \gamma_2 + 2\gamma_4 + \gamma_1 M_2)}{\gamma_1 M_1}kl.$$

This shows that any closed ball $\overline{B}_Z(0, R)$ with the radius

$$R > \sqrt{\frac{\gamma_2 + 2\gamma_4 + \gamma_1 M_2}{\gamma_1 M_1}\|g\|^2 + \frac{2(\gamma_3\varrho + \gamma_2 + 2\gamma_4 + \gamma_1 M_2)}{\gamma_1 M_1}kl}$$

is a bounded absorbing set for $S(t)$.

In order to show that the dynamical system $(Z, S(t))$ corresponding to problem (1.1)–(1.3) is asymptotically smooth as in the proof of Lemma 4.4, we use the perturbed energy method to establish a stabilizability estimate.

Lemma 4.5 (Stabilizability estimate)**.** *Under the conditions of Theorem 1.2, for a given bounded set $B \subset Z$ there exist constants $\alpha, \beta > 0$ and $\sigma := \sigma(B) > 0$ (depending on B) such that*

$$\|S(t)\bar{z} - S(t)z\|_Z^2 \leq \alpha e^{-\beta t}\|\bar{z} - z\|_Z^2 + \sigma\int_0^t e^{-\beta(t-\tau)}\|\bar{u}(\tau) - u(\tau)\|_{*1}^2\,\mathrm{d}\tau$$

for every $z, \bar{z} \in B$ and $t > 0$ where $S(t)\bar{z} = (\bar{u}(t), \bar{u}_t(t))$.

Proof. Set $\tilde{u} := \bar{u} - u$ and

$$\widetilde{\Psi}(t) := \widetilde{E}(t) + \varepsilon\widetilde{\psi}(t) \tag{4.13}$$

where

$$\widetilde{E}(t) := \|\tilde{u}(t)\|_{*2}^2 + \|\tilde{u}_t(t)\|^2, \tag{4.14}$$

$$\widetilde{\psi}(t) := (\tilde{u}(t), \tilde{u}_t(t))$$

and $\varepsilon > 0$ is a constant to be determined later.

We claim that there exist two constants $\kappa_1, \kappa_2 > 0$ depending on ε, such that

$$\kappa_1 \widetilde{E}(t) \leq \widetilde{\Psi}(t) \leq \kappa_2 \widetilde{E}(t). \tag{4.15}$$

By the arguments similar to the proof of the (4.6), we have $|\widetilde{\psi}(t)| \leq K\widetilde{E}(t)$. Hence, we deduce from (4.13) that

$$(1 - \varepsilon K)\,\widetilde{E}(t) \leq \widetilde{\Psi}(t) \leq (1 + \varepsilon K)\,\widetilde{E}(t).$$

Thus, assertion (4.15) is demonstrated and $\kappa_1 > 0$ will be guaranteed by the selection of ε later.

Next, we claim that there exists a constant $\kappa_3 := \kappa_3(B) > 0$ such that

$$\widetilde{\Psi}'(t) \leq -\varepsilon\widetilde{E}(t) + \kappa_3\|\tilde{u}(t)\|_{*1}^2. \tag{4.16}$$

To see this, by the arguments similar to the proof of (4.2) in Lemma 4.3, we have

$$\widetilde{E}'(t) = -2(a\tilde{u}(t), \tilde{u}_t(t)) - 2\mu\|\tilde{u}_t(t)\|^2 - 2(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)). \tag{4.17}$$

Concerning the first term on the right-hand side of (4.17), we deduce from Schwarz's inequality, Corollary 2.4 and Cauchy's inequality with $\epsilon_1 > 0$ that

$$\begin{aligned} -2(a\tilde{u}(t), \tilde{u}_t(t)) \leq& 2a_2\|\tilde{u}(t)\|\|\tilde{u}_t(t)\| \\ \leq& \frac{a_2^2\mathfrak{C}^2}{2\epsilon_1}\|\tilde{u}(t)\|_{*1}^2 + 2\epsilon_1\|\tilde{u}_t(t)\|^2. \end{aligned} \tag{4.18}$$

For the last term on the right-hand side of (4.17), it follows from the arguments similar to the proof of (3.20) and Corollary 2.4 that

$$\begin{aligned} -2(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)) \leq& C(B)\|\tilde{u}(t)\|_{*1}\|\tilde{u}_t(t)\| \\ \leq& C(B, \epsilon_2)\|\tilde{u}(t)\|_{*1}^2 + \epsilon_2\|\tilde{u}_t(t)\|^2. \end{aligned} \tag{4.19}$$

Hence, by substituting (4.18) and (4.19) into (4.17) we obtain

$$\widetilde{E}'(t) \leq C(B, \epsilon_1, \epsilon_2)\|\tilde{u}(t)\|_{*1}^2 - (2\mu - 2\epsilon_1 - \epsilon_2)\|\tilde{u}_t(t)\|^2. \tag{4.20}$$

We are now in a position to choose sufficiently small ϵ_1 and ϵ_2 such that

$$\theta := 2\mu - 2\epsilon_1 - \epsilon_2 > 0.$$

Thus, (4.20) can be rewritten as

$$\widetilde{E}'(t) \leq C(B)\|\tilde{u}(t)\|_{*1}^2 - \theta\|\tilde{u}_t(t)\|^2. \tag{4.21}$$

Since

$$\widetilde{\psi}'(t) = \|\tilde{u}_t(t)\|^2 + \langle \tilde{u}_{tt}(t), \tilde{u}_t(t)\rangle,$$

we conclude from the analogous arguments in Remark 2.6 that

$$\widetilde{\psi}'(t) = \|\tilde{u}_t(t)\|^2 - \|\tilde{u}(t)\|_{*2}^2 - (a\tilde{u}(t), \tilde{u}(t)) - \mu(\tilde{u}_t(t), \tilde{u}(t)) - (f(\bar{u}(t)) - f(u(t)), \tilde{u}(t)).$$

On account of (4.14), we can get

$$\begin{aligned}\widetilde{\psi}'(t) \leq &- \widetilde{E}(t) + 2\|\tilde{u}_t(t)\|^2 - (a\tilde{u}(t), \tilde{u}(t)) - \mu(\tilde{u}_t(t), \tilde{u}(t))\\ &- (f(\bar{u}(t)) - f(u(t)), \tilde{u}(t)).\end{aligned} \tag{4.22}$$

For the third term on the right-hand side of (4.22), it follows from Corollary 2.4 that

$$-(a\tilde{u}(t), \tilde{u}(t)) \leq a_2\|\tilde{u}(t)\|^2 \leq a_2\mathfrak{C}^2\|\tilde{u}(t)\|_{*1}^2. \tag{4.23}$$

For the fourth term on the right-hand side of (4.22), we deduce from Schwarz's and Cauchy's inequalities and Corollary 2.4 that

$$\begin{aligned}-\mu(\tilde{u}_t(t), \tilde{u}(t)) \leq &\frac{\mu}{2}\left(\|\tilde{u}(t)\|^2 + \|\tilde{u}_t(t)\|^2\right)\\ \leq &\frac{\mu}{2}\left(\mathfrak{C}^2\|\tilde{u}(t)\|_{*1}^2 + \|\tilde{u}_t(t)\|^2\right).\end{aligned} \tag{4.24}$$

For the last term on the right-hand side of (4.22), we deduce from the arguments similar to the proof of (3.20) and Corollary 2.4 that

$$-(f(\bar{u}(t)) - f(u(t)), \tilde{u}(t)) \leq C(B)\|\tilde{u}(t)\|_{*1}^2. \tag{4.25}$$

Hence, by inserting (4.23)–(4.25) into (4.22) we derive

$$\widetilde{\psi}'(t) \leq -\widetilde{E}(t) + C(B)\|\tilde{u}(t)\|_{*1}^2 + \left(2 + \frac{\mu}{2}\right)\|\tilde{u}_t(t)\|^2. \tag{4.26}$$

From (4.13), (4.21) and (4.26), we conclude that

$$\widetilde{\Psi}'(t) \leq -\varepsilon\widetilde{E}(t) + (C(B) + \varepsilon C(B))\|\tilde{u}(t)\|_{*1}^2 - \left(\theta - \varepsilon\left(2 + \frac{\mu}{2}\right)\right)\|\tilde{u}_t(t)\|^2. \tag{4.27}$$

We can choose

$$\varepsilon < \min\left\{\frac{1}{K}, \frac{2\theta}{4+\mu}\right\}$$

such that the last term on the right-hand side of (4.27) are non-positive and could be neglected. Thus, assertion (4.16) is proved.

By assertion (4.16) and the second inequality in assertion (4.15), we can derive

$$\widetilde{\Psi}'(t) \leq -\frac{\varepsilon}{\kappa_2}\widetilde{\Psi}(t) + \kappa_3\|\tilde{u}(t)\|_{*1}^2.$$

Hence,

$$\widetilde{\Psi}(t) \leq \widetilde{\Psi}(0)e^{-\beta t} + \kappa_3\int_0^t e^{-\beta(t-\tau)}\|\tilde{u}(\tau)\|_{*1}^2\,\mathrm{d}\tau$$

where $\beta = \varepsilon/\kappa_2$. This combined with assertion (4.15) gives

$$\widetilde{E}(t) \leq \alpha\widetilde{E}(0)e^{-\beta t} + \sigma\int_0^t e^{-\beta(t-\tau)}\|\tilde{u}(\tau)\|_{*1}^2\,\mathrm{d}\tau$$

where $\alpha = \kappa_2/\kappa_1$ and $\sigma = \kappa_3/\kappa_1$. Thus, the proof of this lemma is finished.

Proof of Theorem 1.2. In terms of Lemma 4.5, we learn that for any bounded positively invariant set $B \subset Z$ and $\varsigma > 0$ there exists $T := T(B, \varsigma)$ such that

$$\|S(T)\bar{z} - S(T)z\|_Z^2 \leq \varsigma + \Phi_T(z, \bar{z})$$

where

$$\Phi_T(z, \bar{z}) = \sigma \int_0^T \|\bar{u}(\tau) - u(\tau)\|_{*1}^2 \, \mathrm{d}\tau.$$

We conclude from Theorem 1.1 that for every $\{z_n\} = \{(u_{0n}, u_{1n})\} \subset B$,

$$\{(u_n, u_{nt})\} \text{ is bounded in } C([0, T]; Z).$$

Since the embedding $H_*^2(\Omega) \hookrightarrow H_*^1(\Omega)$ is compact, we conclude that up to a subsequence

$$\{u_n\} \text{ converges stongly in } C\left([0, T]; H_*^1(\Omega)\right).$$

Therefore,

$$\lim_{m\to\infty} \lim_{n\to\infty} \Phi_T(z_n, z_m) = \sigma \lim_{m\to\infty} \lim_{n\to\infty} \int_0^T \|u_m - u_n\|_{*1}^2 \, \mathrm{d}\tau = 0.$$

Thus, in light of Proposition 4.2, $(Z, S(t))$ is asymptotically smooth. According to Theorem 4.1 and Lemma 4.4, $(Z, S(t))$ possesses a compact global attractor.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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