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Research article

Boundedness of square functions related with fractional Schrödinger semigroups on stratified Lie groups

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Abstract: In this paper, we consider a Schrödinger operator $L = -\Delta_{\mathbb{H}} + V$ on the stratified Lie group $\mathbb{H}$. First, we establish fractional heat kernel estimates related to L^{β}, $\beta \in (0, 1)$. By utilizing kernel estimations and the fractional Carleson measure, we are able to derive a characterization of the Campanato type space $BMO^{\nu}_{L}(\mathbb{H})$. Second, we demonstrate that both Littlewood-Paley **g**-functions and area functions are bounded on $BMO^{\nu}_{L}(\mathbb{H})$. Finally, we also obtain that the above square functions are bounded on the Morrey space $L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})$ and the weak Morrey space $WL^{\gamma,\theta}_{1,\kappa}(\mathbb{H})$, respectively.

Keywords: Schrödinger operators; Morrey spaces; Carleson measures; fractional heat semigroups; Campanato spaces
Mathematics Subject Classification: 22E30, 42B35

1. Introduction

Over the past few years, the Schrödinger operator $L := -\Delta + V$ has attracted considerable attention, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and V is a nonnegative potential. In 1995, Shen [1] investigated the estimate of the fundamental solution of L assuming that the nonnegative potential V belongs to a reverse Hölder class B_s for some $s > n/2$. Since then, there have been a lot of researches which focus on the related problems on L, see [2–11]. During the same period, function spaces related to L were also studied extensively. Similarly to the classical Hardy spaces, one can define the Hardy space $H^1_L(\mathbb{R}^n)$, related to L, as the set of all L^1-functions f that satisfy the property $M_L(f) \in L^1(\mathbb{R}^n)$, where $M_L(f)(x) := \sup_{t>0} |e^{-tL} f(x)|$ and $\{e^{-tL}\}_{t>0}$ denotes the heat semigroup related with L. In [12], Dziubański and Zienkiewicz utilized the local Hardy space to explore the atomic characterization and Riesz transform characterization of $H^1_L(\mathbb{R}^n)$, see also [13] for the formulation of $H^p_L(\mathbb{R}^n)$, $p \in (0, 1)$. As the dual space of $H^1_L(\mathbb{R}^n)$, the BMO type space $BMO_L(\mathbb{R}^n)$ was introduced by Dziubański et al. in [14]. As an analogue of the case of

Euclidean spaces, Li in [15] investigated the fundamental solution of L and related singular integral operators in the context of nilpotent groups. Similarly to the idea of [12] and [14], the Hardy space and BMO type space related to L were investigated by Lin Liu and Liu. [16] and Lin and Liu [17] on the Heisenberg group, respectively. For more information, we recommend referring to [18–24] and the references therein.

One of objectives of this paper is to study the fractional heat kernel related to $L^{\beta}, \beta \in (0,1)$, within the context of stratified Lie groups. In stratified Lie groups $\mathbb{H}$, the Schrödinger operator is defined as $L = -\Delta_{\mathbb{H}} + V$, where $\Delta_{\mathbb{H}}$ is the sub-Laplacian on $\mathbb{H}$.

Definition 1.1. A nonnegative potential V is said to belong to B_s ($\infty > s > 1$) if there exists $C > 0$ such that, for every ball B,

$$\Big(\frac{1}{|B|}\int_B V^s(g)dg\Big)^{1/s} \leq \frac{C}{|B|}\int_B V(g)dg$$

holds.

In the sequel, we always assume that the nonnegative potential $0 \not\equiv V \in B_s$ for some s, where d is the homogeneous dimension of $\mathbb{H}$ and $d/2 < s < d$. Let $\delta_0 = 2 - d/s \in (0,1)$ and $\delta < \min\{2\beta, \delta_0\}$. In the whole paper, we maintain the assumption and definitions of δ_0, δ and s.

Let $-\Delta$ be the Laplace operator on $\mathbb{R}^n$. The fractional heat semigroup related to $(-\Delta)^{\beta}, \beta \in (0,1)$, can be defined as

$$\widehat{e^{-t(-\Delta)^{\beta}}u}(x) := e^{-t|x|^{2\beta}}\widehat{u}(x), \quad \beta \in (0,1), \tag{1.1}$$

where $\widehat{f}$ denotes the Fourier transform of f.

The fractional heat semigroup $\{e^{-t(-\Delta)^{\beta}}\}_{t>0}$ has found extensive application in the fields of partial differential equations and mathematical physics, owing to the background of quantum mechanics. It is well-known that $\{e^{-t(-\Delta)^{\beta}}\}_{t>0}$ can aid in constructing the linear component of solutions to fluid equations in mathematical physics, including the generalized Navier-Stokes equation and the MHD equation. We refer readers to [25–28] and the references therein. In [27], Miao Yuan and Zhang proved that the kernel of $\{e^{-t(-\Delta)^{\beta}}\}_{t>0}$ has the following regularity estimate in the Euclidean space:

$$e^{-t(-\Delta)^{\beta}}(x) \leq \frac{Ct}{(t^{1/2\beta} + |x|)^{n+2\beta}} \quad \forall (x,t) \in \mathbb{R}^{n+1}_{+}.$$

In [22], Wang et al. got the estimate of $t^m\partial_t^m e^{-t(-\Delta_{\mathbb{G}})}(\cdot)$, $m \in \mathbb{Z}_+$, on the Heisenberg group $\mathbb{G}$. In this paper, we extend the estimate of $t^m\partial_t^m e^{-t(-\Delta_{\mathbb{G}})}(\cdot)$ to the fractional case on the stratified Lie group, which is defined as

$$t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^{\beta}}(\cdot). \tag{1.2}$$

By the aid of the estimate of $t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})}(\cdot)$, we can obtain the estimate of (1.2). In particular, when $\beta = 1$, the estimate of (1.2) returns to the estimate of $t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})}(\cdot)$. See Lemma 3.2.

For $L^{\beta} = (-\Delta_{\mathbb{H}} + V)^{\beta}$, $\beta \in (0,1)$, the fractional heat semigroup $\{e^{-tL^{\beta}}\}_{t>0}$ not be defined by the Fourier transform as in (1.1). Therefore, the formula in [27] cannot be used for the kernel estimate of $e^{-tL^{\beta}}$.

In Section 3, we give some regularity estimates of $\{e^{-tL^{\beta}}\}_{t>0}$. Unlike $\{e^{-t(-\Delta)^{\beta}}\}_{t>0}$, we utilize the following subordinative formula to represent $e^{-tL^{\beta}}$ (cf. [25]), that is,

$$e^{-tL^{\beta}}(g,h) = \int_0^{\infty} \eta_t^{\beta}(s)e^{-sL}(g,h)ds \ \ \forall\, g,h \in \mathbb{H}, \tag{1.3}$$

where $\eta_t^\beta(\cdot)$ is a continuous function on $(0,\infty)$ satisfying

$$\begin{cases} \eta_t^\beta(s) = 1/t^{1/\beta}\eta_1^\beta(s/t^{1/\beta}); \\ \eta_t^\beta(s) \le t/s^{1+\beta}\ \forall s,t>0; \\ \int_0^\infty s^{-r}\eta_1^\beta(s)ds < \infty, \quad r>0; \\ \eta_t^\beta(s) \simeq t/s^{1+\beta} \quad \forall s \ge t^{1/\beta} > 0. \end{cases}$$

So, the identity (1.3) and the estimate of $e^{-tL}(\cdot,\cdot)$ can be used to estimate $e^{-tL^\beta}(\cdot,\cdot)$.

In Section 4, applying Carleson measures generated by $\{e^{-tL^\beta}\}_{t>0}$, we can characterize the Campanato type space $BMO_L^\nu(\mathbb{H})$, which is defined in Definition 1.2. Define

$$\gamma(g) = \sup\Big\{r>0 : \frac{1}{r^{d-2}}\int_B V(h)dh \le 1\Big\} \quad \forall\, g\in\mathbb{H}, \quad B=B(g,r).$$

Let f_B denote the mean of f on B, that is, $f_B = |B|^{-1}\int_B f(h)dh$ and

$$f(B,V) = \begin{cases} f_B, & r<\gamma(g); \\ 0, & r\ge\gamma(g). \end{cases}$$

Definition 1.2. Let $0 \le \nu \le \delta/d$. If a locally integrable function f satisfies

$$\|f\|_{BMO_L^\nu} = \sup_{B\subset\mathbb{H}}\Big\{|B|^{-\nu}\Big(\int_B |f(g)-f(B,V)|^2\frac{dg}{|B|}\Big)^{1/2}\Big\} < \infty,$$

we say that it belongs to the Campanato type space $BMO_L^\nu(\mathbb{H})$.

Section 4.2 is devoted to the $BMO_L^\nu(\mathbb{H})$-boundedness of the following square functions:

$$\begin{cases} S_{m,\beta}^L(f)(g) := \Big(\int_0^\infty \int_{|z^{-1}g|<t} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}f(z)|^2\frac{dzdt}{t^{d+1}}\Big)^{1/2}, \quad g\in\mathbb{H}; \\ G_{m,\beta}^L(f)(g) := \Big(\int_0^\infty |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}f(g)|^2\frac{dt}{t}\Big)^{1/2}, \quad g\in\mathbb{H}. \end{cases}$$

We also prove that square functions $S_{m,\beta}^L$ and $G_{m,\beta}^L$ are bounded from $L^\infty(\mathbb{H})$ to $BMO_L^\nu(\mathbb{H})$. See Theorem 4.6.

Remark 1.3. When $\nu=0$, we can see that $BMO_L^0(\mathbb{H}) = BMO_L(\mathbb{H})$. Hence, if $\beta=1$ and $m=1$, Theorem 4.6 returns to [17, Theorem 6]. When $\nu\neq 0$, if $\beta=1$, Theorem 4.6 returns to [22, Theorem 4.12].

Finally, inspired by [17, 29], using Theorem 4.6, Lemma 4.2 and an interpolation argument, we show that square functions $S_{m,\beta}^L$ and $G_{m,\beta}^L$ are bounded on $L^p(\mathbb{H})$ for $1<p<\infty$. See Theorem 4.7. As applications of Theorem 4.7, we also obtain the boundednesses of $S_{m,\beta}^L$ and $G_{m,\beta}^L$ on the Morrey space $L_{p,\kappa}^{\gamma,\theta}(\mathbb{H})$ and the weak Morrey space $WL_{1,\kappa}^{\gamma,\theta}(\mathbb{H})$. See Theorems 5.4 & 5.5.

Remark 1.4. In this paper, if we assume that the nonnegative potential $0\not\equiv V\in B_s$ for some $s>d/2$, where d is the homogeneous dimension of $\mathbb{H}$, then $\delta<\min\{2\beta,\delta_0,1\}$.

Throughout this article, we will use the following notation. Let $\mathbb{Z}_+=\{0,1,2,\cdots\}$ and $\mathbb{N}=\{1,2,\cdots\}$. We shall write c and C for various positive constants that are independent of the main variables involved and may be different at each occurrence. The notation $B_1 \lesssim B_2$ means that the inequality $B_1\le CB_2$ holds. The notation $B_1\sim B_2$ means that there exists a constant $C>1$ such that $C^{-1}\le B_1/B_2\le C$.

2. Preliminaries

As in [30], a Lie group $\mathbb{H}$ is known as stratified if it is nilpotent, connected and simple connected, and its Lie algebra $\mathfrak{g}$ is equipped with a family of dilations: $\{\delta_r : r > 0\}$ and $\mathfrak{g}$ can be expressed as a direct sum $\oplus_{j=1}^{m} \mathfrak{g}_j$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset g_{i+j}$, $\mathfrak{g}_1$ generates $\mathfrak{g}$, and $\delta_r(\mathbf{X}) = r^j \mathbf{X}$ for $\mathbf{X} \in \mathfrak{g}_j$. $d = \sum_{j=1}^{m} jd_j$ is called the homogeneous dimension of $\mathbb{H}$, where $d_j = \dim \mathfrak{g}_j$. $\mathbb{H}$ is topologically identified with $\mathfrak{g}$ by the map $\exp : \mathfrak{g} \mapsto \mathbb{H}$. Moreover, we also view δ_r as an automorphism of $\mathbb{H}$. For a homogeneous norm of $\mathbb{H}$, it needs to satisfy

$$\begin{cases} |gh| \leq \gamma(|g| + |h|) & \forall g, h \in \mathbb{H},\ \gamma \geq 1; \\ ||gh| - |g|| \leq \gamma |h| & \forall g, h \in \mathbb{H}, |h| \leq |g|/2. \end{cases} \tag{2.1}$$

Denote by $B(g,r) = \{h \in \mathbb{H} : |g^{-1}h| < r\}$ a ball of radius r centered at g. The Haar measure on $\mathbb{H}$ is simply the Lebesgue measure on $\mathbb{R}^n$ under the identification of $\mathbb{H}$ with $\mathfrak{g}$ and the identification of $\mathfrak{g}$ with $\mathbb{R}^n$, where $n = \sum_j^m d_j$. The measure of $B(g,r)$ is $|B(g,r)| = b' r^d$, where b' is a constant.

We identify $\mathfrak{g}$ with $\mathfrak{g}_L$, the Lie algebra of left-invariant vector fields on $\mathbb{H}$. Let $\{\mathbf{X}_j : j = 1, ..., d_1\}$ be a basis of $\mathfrak{g}_1$. The sub-Laplacian $\Delta_{\mathbb{H}}$ is defined by $\Delta_{\mathbb{H}} := \sum_{j=1}^{d_1} \mathbf{X}_j^2$. Also, the gradient operator ∇ is denoted by $\nabla = (\mathbf{X}_1, ..., \mathbf{X}_{d_1})$.

The notation $\mathbb{S}$ denote the semidirect extension of $\mathbb{H}$ by the one parameter group of dilations.

$$(g,a)(h,b) = (g(\delta_a h), ab), g, h \in \mathbb{H}, a, b > 0$$

is the group law of $\mathbb{S}$. The Carleson box $\Omega(B) = \Omega(g,r)$ is defined as

$$\Omega(g,r) = \{(h,s) \in \mathbb{S} : |g^{-1}h| < r,\ 0 < s < r\}.$$

Lemma 2.1. (*[17, Lemmas 4 & 5]*) *There exist $C > 0$ and $N_0 \geq 1$ such that, for all $g, h \in \mathbb{H}$,*

$$C^{-1}\gamma(g)\Big(1 + |g^{-1}h|/\gamma(g)\Big)^{-N_0} \leq \gamma(h) \leq C\gamma(g)\Big(1 + |g^{-1}h|/\gamma(g)\Big)^{N_0/(1+N_0)}. \tag{2.2}$$

Specifically, $\gamma(h) \sim \gamma(g)$ if $|h^{-1}g| < C\gamma(g)$.

Using (2.2), we obtain that for each fixed $k \in \mathbb{N}$

$$\Big(1 + \frac{r}{\gamma(g)}\Big)^{-N_0/(1+N_0)}\Big(1 + \frac{2^k r}{\gamma(g)}\Big) \leq C\Big(1 + \frac{2^k r}{\gamma(h)}\Big) \tag{2.3}$$

holds for any $h \in B(g,r)$ with $g \in \mathbb{H}$ and $r \in (0,\infty)$, where the constant C is the same as in (2.2).

Below we state some basic facts about the Hardy type space $H_L^p(\mathbb{H})$, $d/(d+\delta) < p \leq 1$. Denote by $\mathcal{M}_L$ the maximal operator, that is, $\mathcal{M}_L(f)(g) := \sup_{t>0} |e^{-tL} f(g)|$, $g \in \mathbb{H}$. According to [22, Theorem 3.12], as a distribution in $(BMO_L^{1/p-1}(\mathbb{H}))^*$, it is known that $\mathcal{M}_L f$ is well defined. Then, for $d/(d+\delta) < p < 1$, we have the following definition

$$H_L^p(\mathbb{H}) := \Big\{f \in (BMO_L^{1/p-1}(\mathbb{H}))^* : \mathcal{M}_L f \in L^p(\mathbb{H})\Big\}$$

and its norm is defined as $\|f\|_{H_L^p} := \|\mathcal{M}_L f\|_{L^p}$.

Definition 2.2. Assume that $p \in (d/(d+\delta), 1]$, $q \in [1,\infty]$ and $p \neq q$. If a function a satisfies $\text{supp}\, a \subset B(g_0, r)$, $\|a\|_{L^q} \leq |B(g_0, r)|^{1/q-1/p}$ and $\int_{B(g_0,r)} a(h)dh = 0$, $r < \gamma(g_0)$, then we say that a is an $H_L^{p,q}$-atom related to a ball $B(g_0, r)$.

The atomic norm of $H_L^p(\mathbb{H})$ is defined by $\|f\|_{H_L^{p,q}-atom} := \inf\{(\sum |c_j|^p)^{1/p}\}$, where the infimum is taken over all decompositions $f = \Sigma c_j a_j$ with a_j being $H_L^{p,q}$-atoms and c_j being scalars.

Let $p \in (0,\infty)$ and $q \in [1,\infty]$. If a function φ on $\mathbb{S}$ satisfies the following two cases: for $q \in [1,\infty)$,

$$\Big(\int\int_{\{(h,t):\ |g^{-1}h|<t\}} |\varphi(h,t)|^q \frac{dhdt}{t^{d+1}}\Big)^{1/q} \in L^p(\mathbb{H})$$

and

$$\sup_{(h,t)\in\{(h,t):\ |g^{-1}h|<t\}} |\varphi(h,t)| \in L^p(\mathbb{H}), \quad q = \infty,$$

then we say that φ belongs to the tent space $T_q^p(\mathbb{S})$.

Let $T_2^{p,\infty}(\mathbb{S}) = \Big\{u(h,t):\ \text{measurable on } \mathbb{S} \text{ and } \|u\|_{T_2^{p,\infty}} < \infty\Big\}$, where

$$\|u\|_{T_2^{p,\infty}} := \sup_{B\subset\mathbb{H}} \frac{1}{|B|^{1/p-1/2}}\Big(\int_{\{(h,t):\ |g^{-1}h|<t\}} |u(h,t)|^2 \frac{dhdt}{t}\Big)^{1/2}.$$

3. Estimates of the kernels

Similarly to the proof of [31, Proposition 1], we have the following estimates of $Q_{t,m}(\cdot)$.

Lemma 3.1. *Let* $m \in \mathbb{Z}_+$.

(i) *There exist* $C, c > 0$ *such that*

$$|t^m \partial_t^m e^{-t(-\Delta_{\mathbb{H}})}(g)| \leq Ct^{-d/2} e^{-ct^{-1}|g|^2}.$$

(ii) *There exist* $C, c > 0$ *such that for* $|\omega| \leq |g|/2$,

$$|t^m \partial_t^m e^{-t(-\Delta_{\mathbb{H}})}(g\omega) - t^m \partial_t^m e^{-t(-\Delta_{\mathbb{H}})}(g)| \leq C|\omega| t^{-(d+1)/2} e^{-ct^{-1}|g|^2}.$$

Lemma 3.2. *Let* $0 < \beta < 1$.

(i) *There exists* $C > 0$ *such that*

$$|t^m \partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \leq \frac{Ct}{(t^{1/2\beta} + |g|)^{d+2\beta}}.$$

(ii) *There exists* $C > 0$ *such that for* $|\omega| \leq |g|/2$,

$$|t^m \partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g\omega) - t^m \partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \leq \frac{C|\omega|}{(t^{1/2\beta} + |g|)^{d+1}}.$$

Proof: By (1.3), we have

$$e^{-t(-\Delta_{\mathbb{H}})^\beta} = \int_0^\infty \eta_1^\beta(\tau) e^{-t^{1/\beta}\tau(-\Delta_{\mathbb{H}})} d\tau. \tag{3.1}$$

Then,

$$|t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \le \int_0^\infty \eta_1^\beta(\tau)|t^m\partial_t^m e^{-t^{1/\beta}\tau(-\Delta_{\mathbb{H}})}(g)|d\tau \lesssim \int_0^\infty \eta_1^\beta(\tau)(t^{1/\beta}\tau)^{-d/2}e^{-c|g|^2/t^{1/\beta}\tau}d\tau.$$

Note that $\eta_1^\beta(\tau) \le C/\tau^{1+\beta}$. We obtain

$$|t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \lesssim (t^{1/\beta})^{-d/2}\int_0^\infty \tau^{-1-\beta-d/2}e^{-c|g|^2/t^{1/\beta}\tau}d\tau.$$

Let $|g|^2/t^{1/\beta}\tau = v$. It holds that

$$|t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \lesssim t|g|^{-2\beta-d}\int_0^\infty v^{1+\beta+d/2-2}e^{-cv}dv \lesssim t/|g|^{2\beta+d}.$$

On the other hand,

$$|t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \lesssim \int_0^\infty \eta_1^\beta(\tau)(t^{1/\beta}\tau)^{-d/2}d\tau \lesssim t^{-d/2\beta}.$$

If $t^{1/2\beta} \le |g|$, then

$$|t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \lesssim \frac{t}{(t^{1/2\beta}+|g|)^{2\beta+d}}.$$

If $t^{1/2\beta} > |g|$, then

$$|t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)| \lesssim \frac{t}{t^{d/2\beta+1}} \le \frac{Ct}{(t^{1/2\beta}+|g|)^{d+2\beta}}.$$

Now we prove (ii). Applying (1.3) again, we can get

$$\begin{aligned}\left|t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g\omega) - t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g)\right| &\le \left|t^m\partial_t^m\Big(\int_0^\infty \eta_1^\beta(\tau)\Big(e^{-t^{1/\beta}\tau(-\Delta_{\mathbb{H}})}(g\omega) - e^{-t^{1/\beta}\tau(-\Delta_{\mathbb{H}})}(g)\Big)d\tau\Big)\right| \\ &\lesssim \int_0^\infty \eta_1^\beta(\tau)|\omega|(t^{1/\beta}\tau)^{-(d+1)/2}e^{-c|g|^2/t^{1/\beta}\tau}d\tau.\end{aligned}$$

Similarly to (i), we can verify that (ii) holds. So, the details are omitted.

Below, we investigate the kernel estimate of $e^{-tL^\beta}, \beta \in (0,1)$.

Lemma 3.3. (*[28, Proposition 3.3]*) *Let* $0 < \beta < 1$.

(i) *There exists* $C_M > 0$ *such that for any* $M > 0$

$$\left|e^{-tL^\beta}(g,h)\right| \le \frac{C_M t}{(t^{1/2\beta}+|g^{-1}h|)^{d+2\beta}}\Big(1+\frac{t^{1/2\beta}}{\gamma(g)}+\frac{t^{1/2\beta}}{\gamma(h)}\Big)^{-M}.$$

(ii) *Let* $0 < \delta < \min\{2\beta, \delta_0\}$. *For any* $M > 0$, *there exists* $C_M > 0$ *such that for all* $|\omega| \le t^{1/2\beta}$,

$$\left|e^{-tL^\beta}(g\omega,h) - e^{-tL^\beta}(g,h)\right| \le C_M\Big(\frac{|\omega|}{t^{1/2\beta}}\Big)^\delta \frac{1}{(t^{1/2\beta}+|g^{-1}h|)^{d+2\beta}}\Big(1+\frac{t^{1/2\beta}}{\gamma(g)}+\frac{t^{1/2\beta}}{\gamma(h)}\Big)^{-M}.$$

Lemma 3.4. (*[28, Proposition 3.4]*) *Let* $0 < \beta < 1$ *and* $m \in \mathbb{Z}_+$.

(i) *For any* $M > 0$*, there exists* $C_M > 0$ *such that*

$$|t^m\partial_t^m e^{-tL^\beta}(g,h)| \le \frac{C_M t}{(t^{1/2\beta}+|g^{-1}h|)^{d+2\beta}}\Big(1+\frac{t^{1/2\beta}}{\gamma(g)}+\frac{t^{1/2\beta}}{\gamma(h)}\Big)^{-M}.$$

(ii) *Let* $0<\delta<\min\{2\beta,\delta_0\}$ *. For any* $M>0$*, there exists* $C_M>0$ *such that for all* $|\omega|\le t^{1/2\beta}$*,*

$$|t^m\partial_t^m e^{-tL^\beta}(g\omega,h)-t^m\partial_t^m e^{-tL^\beta}(g,h)| \le \frac{C_M t}{(t^{1/2\beta}+|g^{-1}h|)^{d+2\beta}}\Big(\frac{|\omega|}{t^{1/2\beta}}\Big)^{\delta}\Big(1+\frac{t^{1/2\beta}}{\gamma(g)}+\frac{t^{1/2\beta}}{\gamma(h)}\Big)^{-M}.$$

(iii) *For any* $M>0$*, there exists* $C_M>0$ *such that*

$$\Big|\int_{\mathbb{H}} t^m\partial_t^m e^{-tL^\beta}(g,h)dh\Big| \le \frac{C_M(t^{1/2\beta}/\gamma(g))^{\delta}}{(1+t^{1/2\beta}/\gamma(g))^M}.$$

Let $F_{\beta,t}(\cdot,\cdot)=t^m\partial_t^m e^{-tL^\beta}(\cdot,\cdot)-t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(\cdot)$.

Lemma 3.5. *Let* $0<\beta<1$*. There exist* $C>0$ *such that*

$$\Big|F_{\beta,t}(g,z)\Big| \le \begin{cases} \dfrac{Ct}{(t^{1/2\beta}+|z^{-1}g|)^{d+2\beta}}\Big(\dfrac{t^{1/2\beta}}{\gamma(g)}\Big)^{\delta_0}, & t^{1/2\beta}>|z^{-1}g|;\\ \dfrac{Ct}{(t^{1/2\beta}+|z^{-1}g|)^{d+2\beta}}\Big(\dfrac{|z^{-1}g|}{\gamma(g)}\Big)^{\delta_0}, & t^{1/2\beta}\le|z^{-1}g|.\end{cases}$$

Proof: Since $e^{-tL^\beta}=\int_0^\infty \eta_1^\beta(\tau)e^{-t^{1/\beta}\tau L}d\tau$, we can get

$$\Big|t^m\partial_t^m e^{-tL^\beta}(g,z)-t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g,z)\Big| = \Big|t^m\int_0^\infty \eta_1^\beta(\tau)\partial_t^m\Big(e^{-t^{1/\beta}\tau L}(g,z)-e^{-t^{1/\beta}\tau(-\Delta_{\mathbb{H}})}(g,z)\Big)d\tau.$$

Then we first recall that the higher-order derivative formula of the composite function: if $y=f(u)$ and $u=\varphi(g)$, then

$$\frac{\partial^m y}{\partial g^m}=\sum_{i=1}^m\frac{\mathbf{p}_{m.i}(g)}{i!}f^{(i)}(u),$$

where $\mathbf{p}_{m,i}(g)=\sum_{k=0}^{i-1}(-1)^kC_i^ku^k\frac{\partial^m}{\partial g^m}u^{i-k}$. So, let $f(u)=e^{-uL}$ and $u=t^{1/\beta}\tau$, we obtain

$$\begin{aligned}&\Big|t^m\partial_t^m e^{-tL^\beta}(g,z)-t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g,z)\Big|\\ &\lesssim\Big|t^m\sum_{i=1}^m\sum_{k=0}^{i-1}\frac{(-1)^kC_i^k}{i!}\int_0^\infty\eta_1^\beta(\tau)t^{i/\beta-m}\tau^i\Big(\partial_s^ie^{-sL}(g,z)|_{s=t^{1/\beta}\tau}-\partial_s^ie^{-s(-\Delta_{\mathbb{H}})}(g,z)|_{s=t^{1/\beta}\tau}\Big)d\tau\Big|\\ &\lesssim\sum_{i=1}^m\int_0^\infty\eta_1^\beta(\tau)\Big|(t^{1/\beta}\tau)^i\partial_s^ie^{-sL}(g,z)|_{s=t^{1/\beta}\tau}-(t^{1/\beta}\tau)^i\partial_s^ie^{-s(-\Delta_{\mathbb{H}})}(g,z)|_{s=t^{1/\beta}\tau}\Big|d\tau.\end{aligned}$$

Note that $\Big|t^m\partial_t^m e^{-tL}(g,z)-t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})}(g,z)\Big|\lesssim t^{-d/2}e^{-c|z^{-1}g|^2/t}\Big(\frac{\sqrt{t}}{\gamma(g)}\Big)^{\delta_0}$ (cf. [22]). Then, we obtain

$$\Big|t^m\partial_t^m e^{-tL^\beta}(g,z)-t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g,z)\Big|\lesssim\int_0^\infty\eta_1^\beta(\tau)(t^{1/\beta}\tau)^{-d/2}e^{-c|z^{-1}g|^2/t^{1/\beta}\tau}\Big(\frac{t^{1/2\beta}\tau^{1/2}}{\gamma(g)}\Big)^{\delta_0}d\tau.$$

One the one hand, since $\eta_1^\beta(\tau) \le C/\tau^{\beta+1}$, we get

$$\left|t^m\partial_t^m e^{-tL^\beta}(g,z) - t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g,z)\right| \lesssim t^{-d/2\beta}\Big(\frac{t^{1/2\beta}}{\gamma(g)}\Big)^{\delta_0}\int_0^\infty \tau^{-d/2+\delta_0/2-1-\beta}e^{-c|z^{-1}g|^2/t^{1/\beta}\tau}d\tau.$$

Let $|z^{-1}g|^2/t^{1/\beta}\tau = v$. We obtain

$$\begin{aligned}\left|t^m\partial_t^m e^{-tL^\beta}(g,z) - t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g,z)\right| &\lesssim t^{-d/2\beta}\Big(\frac{t^{1/2\beta}}{\gamma(g)}\Big)^{\delta_0}\frac{(t^{1/\beta})^{d/2-\delta_0/2+\beta}}{|z^{-1}g|^{d-\delta_0+2\beta}}\int_0^\infty e^{-cv}v^{d/2-\delta_0/2+1+\beta}dv\\ &\lesssim \frac{t^{1-\delta_0/2\beta}}{|z^{-1}g|^{d+2\beta-\delta_0}}\Big(\frac{t^{1/2\beta}}{\gamma(g)}\Big)^{\delta_0}. \end{aligned}\tag{3.2}$$

On the other hand,

$$\begin{aligned}\left|t^m\partial_t^m e^{-tL^\beta}(g,z) - t^m\partial_t^m e^{-t(-\Delta_{\mathbb{H}})^\beta}(g,z)\right| &\lesssim \int_0^\infty \eta_1^\beta(\tau)(t^{1/\beta}\tau)^{-d/2}\Big(\frac{t^{1/2\beta}\tau^{1/2}}{\gamma(g)}\Big)^{\delta_0}d\tau\\ &\lesssim \Big(\frac{t^{1/2\beta}}{\gamma(g)}\Big)^{\delta_0}(t^{1/\beta})^{-d/2}\int_0^\infty \eta_1^\beta(\tau)\tau^{-d/2+\delta_0}d\tau\\ &\lesssim t^{-d/2\beta}\Big(\frac{t^{1/2\beta}}{\gamma(g)}\Big)^{\delta_0}. \end{aligned}\tag{3.3}$$

Hence, we can deduce that (3.2) and (3.3) satisfy

$$\left|F_{\beta,t}(g,z)\right| \le \begin{cases}\dfrac{Ct}{(t^{1/2\beta}+|z^{-1}g|)^{d+2\beta}}\Big(\dfrac{t^{1/2\beta}}{\gamma(g)}\Big)^{\delta_0}, & t^{1/2\beta} > |z^{-1}g|;\\ \dfrac{Ct}{(t^{1/2\beta}+|z^{-1}g|)^{d+2\beta}}\Big(\dfrac{|z^{-1}g|}{\gamma(g)}\Big)^{\delta_0}, & t^{1/2\beta} \le |z^{-1}g|.\end{cases}$$

4. Square functions and characterizations of $BMO_L^\nu(\mathbb{H})$

4.1. Carleson measure characterization of $BMO_L^\nu(\mathbb{H})$

Similarly to [22], we can also get the following lemmas.

Lemma 4.1. *Let $m \in \mathbb{Z}_+$ and $0 < \beta < 1$. The operators $G_{m,\beta}^L$ and $S_{m,\beta}^L$ are bounded on $L^2(\mathbb{H})$.*

Lemma 4.2. *The operators $G_{m,\beta}^L$ and $S_{m,\beta}^L$ are bounded from $L^1(\mathbb{H})$ to $L^{1,\infty}(\mathbb{H})$.*

Proof: Since the proofs of $G_{m,\beta}^L$ and $S_{m,\beta}^L$ are similar, we only give the proof for $S_{m,\beta}^L$. Using the Calderón-Zygmund decomposition (cf. [32, Chapter 1, §4]), given $f \in L^1(\mathbb{H})$ and $\alpha > 0$, there holds the decomposition $f = f_1 + f_2$, with $f_2 = \sum_j b_j$, such that

(i) $|f_1(g)| \le C\alpha$, *a.e.* $g \in \mathbb{H}$;

(ii) each b_j is supported on a ball B_j,

$$\int_{B_j}|b_j(g)|dg \le C\alpha|B_j| \quad \textit{and} \quad \int_{B_j} b_j(g)dg = 0;$$

(iii) $\{B_j\}$ has finite overlaps property and $\sum_j |B_j| \lesssim \frac{1}{\alpha}\|f\|_{L^1}$.

It is easy to see that

$$\Big|\{g \in \mathbb{H} : S^L_{m,\beta} f_1(g) > \alpha/2\}\Big| \lesssim \frac{1}{\alpha^2}\|f\|^2_{L^2} \lesssim \frac{1}{\alpha}\|f\|_{L^1}.$$

Let $B_j = B(g_0, r_j)$ and $E = \bigcup_j B(g_j, 4r_j)$. Then,

$$|E| \lesssim \sum_j |B_j| \lesssim \frac{1}{\alpha}\|f\|_{L^1}.$$

By the same arguments as [22], we have

$$\int_{|g^{-1}g_j|\geq 4r_j} S^L_{m,\beta} b_j(g)dg \lesssim \int_{B_j} |b_j(g)|dg \lesssim \alpha|B_j|,$$

which implies

$$\begin{aligned}\Big|\{g \notin E : S^L_{m,\beta} f_2(g) > \alpha/2\}\Big| &\lesssim \frac{1}{\alpha}\int_{E^c} S^L_{m,\beta} f_2(g)dg \\ &\lesssim \frac{1}{\alpha}\sum_j \int_{|g^{-1}g_j|\geq 4r_j} S^L_{m,\beta} b_j(g)dg \\ &\lesssim \frac{1}{\alpha}\|f\|_{L^1}.\end{aligned}$$

The above discussion gives

$$\Big|\{g \in \mathbb{H} : S^L_{m,\beta} f(g) > \alpha\}\Big| \lesssim \frac{1}{\alpha}\|f\|_{L^1}.$$

This proves that $S^L_{m,\beta}$ is bounded from $L^1(\mathbb{H})$ to $L^{1,\infty}(\mathbb{H})$.

To establish the (H^1_L, L^1) boundedness, we need the following lemma.

Lemma 4.3. *([16, Lemma 18]) If T is a bounded sublinear operator from $L^1(\mathbb{H})$ to $L^{1,\infty}(\mathbb{H})$ and satisfies $\|Ta\|_{L^1} \leq C$ for any $H^{1,\infty}_L$-atom a, then T is bounded from $H^1_L(\mathbb{H})$ to $L^1(\mathbb{H})$.*

When $v = 1$, from [28, Lemma 4.2] and Lemmas 4.2 & 4.3, we obtain

Lemma 4.4. *$G^L_{m,\beta}$ and $S^L_{m,\beta}$ are bounded from $H^1_L(\mathbb{H})$ to $L^1(\mathbb{H})$.*

Theorem 4.5. *Let $0 \leq v < \delta/d$. Then, we have*

(i)

$$\sup_{B\subset\mathbb{H}} \frac{1}{|B|^{v+1/2}}\Big(\int_{\hat{B}} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta} f(g)|^2 \frac{dgdt}{t}\Big)^{1/2} \leq C\|f\|_{BMO^v_L(\mathbb{H})},$$

where $f \in BMO^v_L(\mathbb{H})$ and $\hat{B}$ denote the tent based on $B = B(g_0, r)$, i.e., $\hat{B} = \{(g,t) : |g^{-1}g_0| \leq r - t\}$.

(ii) *Suppose f belongs to $L^1((1 + |g|)^{-(d+v+\varepsilon)}dg)$ for some $\varepsilon > 0$ and*

$$\sup_{B\subset\mathbb{H}} \frac{1}{|B|^{v+1/2}}\Big(\int_{\hat{B}} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta} f(g)|^2 \frac{dgdt}{t}\Big)^{1/2} < \infty.$$

Then, $f \in BMO^v_L(\mathbb{H})$ and

$$\|f\|_{BMO^v_L(\mathbb{H})} \leq C \sup_{B\subset\mathbb{H}} \frac{1}{|B|^{v+1/2}}\Big(\int_{\hat{B}} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta} f(g)|^2 \frac{dgdt}{t}\Big)^{1/2}.$$

Proof: Let $f \in BMO_L^{\nu}(\mathbb{H})$. Then, $f \in L^1((1+|g|)^{-(d+\nu+\varepsilon)}dg)$. By Lemma 3.4, we can see that

$$t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}f(g) = \int_{\mathbb{H}} t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}(g,h)f(h)dh$$

is absolutely convergent. In order to prove (i) , we only need to show that for any ball $B = B(g_0, r)$,

$$\frac{1}{|B|^{2\nu+1}}\int_{\hat{B}} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}f(g)|^2\frac{dgdt}{t} \lesssim \|f\|^2_{BMO_L^{\nu}}.$$

Denote by B_k the ball $B(g_0, 2^k r)$. Let

$$f = (f - f_{B_1})_{\chi_{B_1}} + (f - f_{B_1})_{\chi_{B_1^c}} + f_{B_1} = \widetilde{f_1} + \widetilde{f_2} + f_{B_1}.$$

Using Lemma 4.1, we have

$$\begin{aligned}
\int_{\hat{B}} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}\widetilde{f_1}(g)|^2\frac{dgdt}{t} &\leq \int_{B} |G^L_{m,\beta}\widetilde{f_1}(g)|^2dg\\
&\lesssim \|\widetilde{f_1}\|^2_{L^2} \lesssim \int_{B_1} |f(g) - f_{B_1}|^2dg\\
&\lesssim |B|^{2\nu+1}\|f\|^2_{BMO_L^{\nu}}.
\end{aligned}$$

Note that $|f_{B_2} - f_{B_1}| \leq 2^d|B_2|^{\nu}\|f\|_{BMO_L^{\nu}}$. Therefore,

$$|f_{B(g_0,2^{k+1}r)} - f_{B(g_0,2r)}| \lesssim k|B(g_0, 2^{k+1}r)|^{\nu}\|f\|_{BMO_L^{\nu}}. \tag{4.1}$$

For $g \in B(g_0, r)$, by Lemma 3.4 (i) and (4.1), we have

$$\begin{aligned}
|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}\widetilde{f_2}(g)| &\lesssim \int_{(B_1)^c} \frac{t^{2\beta}}{|(g_0^{-1}h|^{d+2\beta}}|f(h) - f_{B_1}|dh\\
&\lesssim \sum_{k=1}^{\infty}\frac{t^{2\beta}}{(2^k r)^{d+2\beta}}\Big(\int_{B_{k+1}\backslash B_k} |f(h) - f_{B_k}|dh + (2^k r)^d|f_{B_k} - f_{B_1}|\Big)\\
&\lesssim \frac{t^{2\beta}}{r^{2\beta-d\nu}}\sum_{k=1}^{\infty}2^{k(d\nu-2\beta)}(1+k)\|f\|^2_{BMO_L^{\nu}}\\
&\lesssim \frac{t^{2\beta}}{r^{2\beta-d\nu}}\|f\|^2_{BMO_L^{\nu}},
\end{aligned}$$

where we have used the fact $0 \leq \nu < \delta/d$ to get $d\nu - 2\beta < 0$ in the last step.

Thus, we have

$$\frac{1}{|B|^{2\nu+1}}\int_{\hat{B}} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}\widetilde{f_2}(g)|^2\frac{dgdt}{t} \lesssim \|f\|^2_{BMO_L^{\nu}}.$$

If $r < \gamma(g_0)$, taking k_0 such that $2^{k_0}r < \gamma(g_0) \leq 2^{k_0+1}r$, we have $|f_{B_1}| \lesssim (1 + \log_2\frac{\gamma(g_0)}{r})|B_{k_0+1}|^{\nu}\|f\|_{BMO_L^{\nu}}$. Note that $\gamma(g) \sim \gamma(g_0) > r$ for any $g \in B(g_0, r)$, by Lemma 3.4 (iii), we obtain

$$\frac{1}{|B|^{2\nu+1}}\int_{\hat{B}} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}(f_{B_1}1)(g)|^2dg\frac{dt}{t} \lesssim \frac{|f_{B_1}|^2}{|B|^{2\nu+1}}\int_{\hat{B}}\Big(\frac{t}{\gamma(g_0)}\Big)^{2\delta}dg\frac{dt}{t}$$

$$
\begin{aligned}
&\lesssim \frac{|B_{k_0+1}|^{2\nu+1}}{|B|^{2\nu+1}}\Big(1+\log_2\frac{\gamma(g_0)}{r}\Big)^2\Big(\frac{r}{\gamma(g_0)}\Big)^{2\delta}\|f\|^2_{BMO^\nu_L}\\
&\lesssim \|f\|^2_{BMO^\nu_L},
\end{aligned}
$$

where we have used the fact $dv < \delta$ in the last step. For $r \geq \gamma(g_0)$, we obtain $|f_{B(g_0,2r)}| \lesssim |B(g_0,2r)|^\nu\|f\|_{BMO^\nu_L(\mathbb{H})}$.

Noticing that $\gamma(g) \lesssim r$ for any $g \in B(g_0,r)$, by (iii) of Lemma 3.4 again, we have

$$
\begin{aligned}
&\frac{1}{|B|^{2\nu+1}}\int_{\hat{B}}|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}(f_{B_1}1)(g)|^2dg\frac{dt}{t}\\
&\lesssim \frac{|f_{B_1}|^2}{|B|^{2\nu+1}}\int_B\int_0^\infty\Big|\int_{\mathbb{H}}t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}(g,h)dh\Big|^2dg\frac{dt}{t}\\
&\lesssim \frac{|f_{B_1}|^2}{|B|^{2\nu+1}}\Big(\int_B\int_0^{\gamma(g)}\Big(\frac{t}{\gamma(g)}\Big)^{2\delta}\frac{dt}{t}dg+\int_B\int_{\gamma(g)}^\infty\Big(\frac{t}{\gamma(g)}\Big)^{-2}\frac{dt}{t}dg\\
&\lesssim \|f\|^2_{BMO^\nu_L}.
\end{aligned}
$$

Suppose $f \in L^1((1+|g|)^{-(d+\nu+\varepsilon)}dg)$ and $t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}f(g) \in T_2^{1/(\nu+1),\infty}(\mathbb{S})$. We want to prove that $f \in BMO^\nu_L(\mathbb{H})$. Using [22, Theorem 3.13], we can see that for every $u \in H_L^{1/(\nu+1)}(\mathbb{H})$

$$u \mapsto L_f(u) := \int_{\mathbb{H}} f(g)u(g)dg$$

satisfies

$$|L_f(u)| \lesssim \|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}f\|_{T_2^{1/(\nu+1),\infty}}\|u\|_{H_L^{1/(\nu+1)}}.$$

Then, we obtain

$$
\begin{aligned}
|L_f(u)| &= \Big|\int_{\mathbb{H}} f(g)u(g)dg\Big| = C\Big|\int_{\mathbb{S}} t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}f(g)t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}u(g)\frac{dgdt}{t}\Big|\\
&\lesssim \|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}f\|_{T_2^{1/(\nu+1),\infty}}\|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}u\|_{T_2^{1/(\nu+1)}}\\
&\lesssim \|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^\beta}f\|_{T_2^{1/(\nu+1),\infty}}\|u\|_{H_L^{1/(\nu+1)}}.
\end{aligned}
$$

This proves Theorem 4.5.

4.2. The BMO^ν_L-boundedness of square functions

In this section, we will show that $S^L_{m,\beta}$ and $G^L_{m,\beta}$ are bounded on $BMO^\nu_L(\mathbb{H})$.

Theorem 4.6. *Let $m \in \mathbb{Z}_+$, $0 < \beta \leq 1$ and and $0 \leq \nu < \delta/d$.*

(i) *The square function $G^L_{m,\beta}$ is bounded on $BMO^\nu_L(\mathbb{H})$.*

(ii) *The square function $S^L_{m,\beta}$ is bounded on $BMO^\nu_L(\mathbb{H})$.*

Proof: Let $\varphi \in BMO^\nu_L(\mathbb{H})$, $B = B(g_o, r)$. Denote by $2B$ the ball $B(g_0, 2r)$. We consider the following two cases.

Case 1: $r > \gamma(g_0)$. Let $f = f\chi_{2B} + f\chi_{(2B)^c} = f_1 + f_2$. By the Hölder inequality, we get

$$\frac{1}{|B|^{2\nu+1}}\int_B|S^L_{m,\beta}f_1(g)|^2dg \lesssim \frac{1}{|2B|^{2\nu+1}}\int_{2B}|f_1(g)|^2dg \lesssim \|f\|^2_{BMO^\nu_L}. \tag{4.2}$$

By Lemma 3.4, if $|g^{-1}z| < t$, we obtain

$$|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}(z,w)| \lesssim t^{-d}(\frac{|g^{-1}w|}{t})^{-(d+\beta)}(1+\frac{t}{\gamma(g)})^{-M}. \tag{4.3}$$

Since $g \in B(g_0, r)$ and (4.3), we deduce that $\gamma(g) \le r$ and

$$\begin{aligned}(S^L_{m,\beta}f_2(g))^2 &\lesssim \Big\{\sum_{k=1}^{\infty} 2^{-\beta k}(2^k r)^{-d}\int_{|g_0^{-1}w|\sim 2^k r}|f(w)|dw\Big\}^2\\ &\lesssim |B|^{2\nu}\|f\|^2_{BMO^\nu_L},\end{aligned}$$

which, together with (4.2), implies that

$$\int_B |S^L_{m,\beta}f(g)|^2 dg \lesssim |B|^{2\nu+1}\|f\|^2_{BMO^\nu_L}.$$

Similarly, we also have

$$\int_B |G^L_{m,\beta}f(g)|^2 dg \lesssim |B|^{2\nu+1}\|f\|^2_{BMO^\nu_L}.$$

Case 2: $r \le \gamma(g_0)$. We begin by giving an estimate of $G^L_{m,\beta}$.We set $f = \widetilde{f_1} + \widetilde{f_2} + f_{2B}$, where $\widetilde{f_1} := (f - f_{2B})\chi_{2B}$ and $\widetilde{f_2} := (f - f_{2B})\chi_{(2B)^c}$. Denote by $G_{m,\beta}$ the Littlewood-Paley **g**-function, that is,

$$G_{m,\beta}(f)(g_0) = \Big(\int_0^{\infty} |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}f(g_0)|^2\frac{dt}{t}\Big)^{1/2}.$$

Set

$$A_2 := \Big(\int_0^{\gamma(g_0)} |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}\widetilde{f_2}(g_0)|^2\frac{dt}{t}\Big)^{1/2}.$$

Therefore, we know that $A_2 < \infty$ is a constant and $|G^L_{m,\beta}f(g) - A_2| \le L_1(g) + L_2(g) + L_3(g)$, where

$$\begin{cases} L_1(g) := \Big(\int_{\gamma(g_0)}^{\infty} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}f(g)|^2\frac{dt}{t}\Big)^{1/2};\\ L_2(g) := \Big|\Big(\int_0^{\gamma(g_0)} |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}f(g)|^2\frac{dt}{t}\Big)^{1/2} - A_2\Big|;\\ L_3(g) := \Big(\int_0^{\gamma(g_0)} |t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}f(g) - t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}f(g)|^2\frac{dt}{t}\Big)^{1/2}.\end{cases}$$

Using Lemma 2.1, we deduce that $\gamma(g) \sim \gamma(g_0)$ for any $g \in B(g_0, r)$. By Lemma 3.4 (i), we have, for $g \in B(g_0, r)$,

$$\begin{aligned}(L_1(g))^2 &\lesssim \Big[\gamma(g_0)^{-d}\int_{B(g,\gamma(g))}|f(z)|dz\Big]^2 + \Big[\sum_{k=1}^{\infty}\gamma(g_0)^{-d}2^{-k(d+\beta)}\int_{B(g,2^k\gamma(g))}|f(z)|dz\Big]^2\\ &\lesssim |B|^{2\nu}\|f\|^2_{BMO^\nu_L}.\end{aligned} \tag{4.4}$$

Since $t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}1=0$, we write $L_2(g)\le L_{2,1}(g)+L_{2,2}(g)$, where

$$\begin{cases} L_{2,1}(g):=\Big(\displaystyle\int_0^{\gamma(g_0)}|t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}\widetilde{f_1}(g)|^2\frac{dt}{t}\Big)^{1/2};\\ L_{2,2}(g):=\Big(\displaystyle\int_0^{\gamma(g_0)}|t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}\widetilde{f_2}(g)-t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}\widetilde{f_2}(g_0)|^2\frac{dt}{t}\Big)^{1/2}. \end{cases}$$

Similarly to the case of $G^L_{m,\beta}$, $G_{m,\beta}$ on $L^2(\mathbb{H})$ is also bounded. Therefore,

$$\frac{1}{|B|^{2\nu+1}}\int_B|L_{2,1}(g)|^2dg\lesssim\frac{1}{|2B|^{2\nu+1}}\int_{2B}|\widetilde{f_1}(g)|^2dg\lesssim\|f\|^2_{BMO^{\nu}_L}. \tag{4.5}$$

By Lemma 3.2,

$$\begin{aligned}(L_{2,2}(g))^2 &\lesssim \Big[\sum_{k=1}^{\infty}2^{-k}(2^kr)^{-d}\int_{|g_0^{-1}z|\sim 2^kr}|\widetilde{f_2}(z)|dz\Big]^2+\Big[\sum_{k=1}^{\infty}(2^kr)^{-d}\int_{|g_0^{-1}z|\sim 2^kr}|\widetilde{f_2}(z)|dz\Big]^2\\ &\lesssim |B|^{2\nu}\|f\|^2_{BMO^{\nu}_L}. \end{aligned}\tag{4.6}$$

It remains to estimate $L_3(g)$. Using the Lemma 2.1 and Lemma 3.5,

$$(L_3(g))^2\lesssim\frac{1}{|B|}\int_B|f(z)|^2dz\lesssim|B|^{2\nu}\|f\|^2_{BMO^{\nu}_L}. \tag{4.7}$$

It can be seen from (4.4)-(4.7) that

$$\frac{1}{|B|^{2\nu+1}}\int_B|G^L_{m,\beta}f(g)-A_2|^2dg\lesssim\|f\|^2_{BMO^{\nu}_L}.$$

Below we provide an estimate of $S^L_{m,\beta}$. Suppose $4B=B(g_0,4r)$, we write $f=\phi_1+\phi_2+f_{4B}$, where $\phi_1=(f-f_{4B})\chi_{4B}$ and $\phi_2=(f-f_{4B})\chi_{(4B)^c}$. Denote by $S_{m,\beta}$ the area function, that is,

$$S_{m,\beta}(f)(g):=\Big(\int_0^{\infty}\int_{|z^{-1}g|<t}|t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}f(z)|^2\frac{dzdt}{t^{d+1}}\Big)^{1/2}.$$

Set

$$A_3:=\Big(\int_{\Gamma^{\gamma(g_0)}(g_0)}|t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}f(z)|^2\frac{dzdt}{t^{d+1}}\Big)^{1/2},$$

where $\Gamma^{\gamma(g_0)}(g_0)=\{(z,t)\in\mathbb{S}:|z^{-1}g_0|<t,t<\gamma(g_0)\}$. We can see that A_3 is a constant and $|S^L_{m,\beta}f(g)-A_3|\le Z_1(g)+Z_2(g)+Z_3(g)$, where

$$\begin{cases} Z_1(g):=\Big(\displaystyle\int_{\gamma(g_0)}^{\infty}\int_{|z^{-1}g|<t}|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}f(z)|^2\frac{dzdt}{t^{d+1}}\Big)^{1/2};\\ Z_2(g):=\Big|\Big(\displaystyle\int_0^{\gamma(g_0)}\int_{|z^{-1}g|<t}|t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}f(z)|^2\frac{dzdt}{t^{d+1}}\Big)^{1/2}-A_3\Big|;\\ Z_3(g):=\Big(\displaystyle\int_0^{\gamma(g_0)}\int_{|z^{-1}g|<t}|t^{2\beta m}L^{m\beta}e^{-t^{2\beta}L^{\beta}}f(z)-t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta}e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}f(z)|^2\frac{dzdt}{t^{d+1}}\Big)^{1/2}. \end{cases}$$

Similarly to (4.4), we utilize (4.3) to obtain, for $g \in B(g_0, r)$,

$$(Z_1(g))^2 \lesssim \int_{\gamma(g_0)}^{\infty} \Big[\int_{\mathbb{H}} t^{-d} \Big(\frac{|w^{-1}g|}{t}\Big)^{-(d+\alpha\beta)} \Big(1+\frac{t}{\gamma(g)}\Big)^{-2\alpha\beta} |f(w)|dw \Big]^2 \frac{dt}{t} \lesssim |B|^{2\nu} \|f\|^2_{BMO^{\nu}_{L}}. \tag{4.8}$$

To estimate $Z_2(g)$, using the triangle inequality, we can see that $Z_2(g) \le Z_{2,1}(g) + Z_{2,2}(g)$, where

$$\begin{cases} Z_{2,1}(g) := \Big(\int_{\Gamma^{\gamma(g_0)}(g)} |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}} \phi_1(z)|^2 \frac{dzdt}{t^{d+1}} \Big)^{1/2}; \\ Z_{2,2}(g) := \Big(\int_{\Gamma^{\gamma(g_0)}(g) | \Gamma^{\gamma(g_0)}(g_0)} |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}} \phi_2(z)|^2 \frac{dzdt}{t^{d+1}} \Big)^{1/2}. \end{cases}$$

Here $O_1|O_2 = (O_1 \setminus O_2) \bigcup (O_2 \setminus O_1)$ denotes the symmetric difference of two sets O_1 and O_2. Using the functional calculus, similarly to [17, page 168] and [30, Chapter 7], we obtain $S_{m,\beta}$ is bounded on $L^2(\mathbb{H})$. Hence,

$$\frac{1}{|B|^{2\nu+1}} \int_B Z_{2,1}(g)^2 dg \lesssim \frac{1}{|B|^{2\nu+1}} \|S_{m,\beta}\phi_1\|^2_{L^2} \lesssim \frac{1}{|4B|^{2\nu+1}} \int_{4B} |\phi_1(g)|^2 dg \lesssim \|f\|^2_{BMO^{\nu}_{L}}.$$

For $Z_{2,2}$, it is easy to see that $Z_{2,2}(g)^2 \le Z_{2,2,1}(g) + Z_{2,2,2}(g)$, where

$$\begin{cases} Z_{2,2,1}(g) := \int_0^{2r} \int_{B(g,t)|B(g_0,t)} \Big(\int_{(4B)^c} |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}(w^{-1}z)||\phi_2(w)|dw \Big)^2 \frac{dzdt}{t^{d+1}}; \\ Z_{2,2,2}(g) := \int_{2r}^{\gamma(g_0)} \int_{B(g,t)|B(g_0,t)} \Big(\int_{(4B)^c} |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}(w^{-1}z)||\phi_2(w)|dw \Big)^2 \frac{dzdt}{t^{d+1}}. \end{cases}$$

If $2r > \gamma(g_0)$, then $Z_{2,2,2}(g) = 0$. For $g \in B(g_0, r)$, $w \in (4B)^c$ and $|z^{-1}g| < t$, we have

$$\begin{aligned} &|t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}(w^{-1}z)| \\ &\le |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}(w^{-1}z) - t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}(w^{-1}g)| + |t^{2\beta m}(-\Delta_{\mathbb{H}})^{m\beta} e^{-t^{2\beta}(-\Delta_{\mathbb{H}})^{\beta}}(w^{-1}g)| \\ &\lesssim t(t+|w^{-1}g_0|)^{-(d+1)}. \end{aligned}$$

Thus, for $g \in B(g_0, r)$,

$$\begin{aligned} Z_{2,2,1}(g) &\lesssim \int_0^{2r} \int_{B(g,t)|B(g_0,t)} \Big(\int_{(4B)^c} t^{-d} \Big(\frac{|w^{-1}g_0|}{t}\Big)^{-(d+1/2)} |\phi_2(w)|dw \Big)^2 \frac{dzdt}{t^{d+1}} \\ &\lesssim \int_0^{2r} \Big(\int_{(4B)^c} |w^{-1}g_0|^{-(d+1/2)} |\phi_2(w)|dw \Big)^2 dt \\ &\lesssim \Big[\sum_{k=2}^{\infty} r^{1/2} (2^k r)^{-(d+1/2)} \int_{|g_0^{-1}w| \sim 2^k r} |\phi_2(w)|dw \Big]^2 \\ &\lesssim |B|^{2\nu} \|f\|^2_{BMO^{\nu}_{L}}. \end{aligned}$$

Note that $|\{B(g,t)|B(g_0,t)\}| \le rt^{d-1}$ for $t \ge 2r > |g_0^{-1}g|$. Therefore, for $g \in B(g_0, t)$,

$$Z_{2,2,2}(g) \lesssim \int_{2r}^{\gamma(g_0)} \int_{B(g,t)|B(g_0,t)} \Big(\int_{(4B)^c} t^{-d} \Big(\frac{|w^{-1}g_0|}{t}\Big)^{-(d+1/4)} |\phi_2(w)|dw \Big)^2 \frac{dzdt}{t^{d+1}}$$

$$
\begin{aligned}
&\lesssim \int_{2r}^{\gamma(g_0)} r\Big(\int_{(4B)^c} |w^{-1}g_0|^{-(d+1/4)}|\phi_2(w)|dw\Big)^2 \frac{dt}{t^{3/2}}\\
&\lesssim \Big(\sum_{k=1}^{\infty} r^{1/4}(2^k r)^{-(d+1/4)} \int_{|g_0^{-1}w|\sim 2^k r} |\phi_2(w)|dw\Big)^2\\
&\lesssim |B|^{2\nu}\|f\|^2_{BMO_L^\nu}.
\end{aligned}
$$

This proves

$$
\frac{1}{|B|^{2\nu+1}}\int_B |Z_2(g)|^2 dg \lesssim \|f\|^2_{BMO_L^\nu}. \tag{4.9}
$$

Finally, we estimate $Z_3(g)$. Using Lemma 2.1, $t/\gamma(\eta) \leq (1 + |g^{-1}z|/\gamma(g))^{m_0} t/\gamma(g)$. For $g \in B(g_0, r)$ and $(z, t) \in \Gamma^{\gamma(g_0)}(g)$, we obtain $|g^{-1}z| < t < \gamma(g_0) \sim \gamma(g)$. By Lemma 3.5, if $|g^{-1}w| > t$, we can get

$$
\begin{aligned}
|F_{\beta,t^{2\beta}}(z,w)| &\leq |F_{\beta,t^{2\beta}}(z,w) - F_{\beta,t^{2\beta}}(g,w)| + |F_{\beta,t^{2\beta}}(g,w)|\\
&\lesssim \frac{Ct^{2\beta}}{(t+|w^{-1}g|)^{d+2\beta}}\Big(\frac{t}{\gamma(g)}\Big)^{\delta} + \frac{Ct^{2\beta}}{(t+|w^{-1}g|)^{d+2\beta}}\Big(\frac{|g^{-1}w|}{\gamma(g)}\Big)^{\delta}.
\end{aligned}
$$

Similarly to the estimate of G_3, we get, for $g \in B(g_0, r)$,

$$
\begin{aligned}
(Z_3(g))^2 &\leq \int_0^{\gamma(g_0)} \int_{|g^{-1}\eta|<t} \Big[\int_{\mathbb{H}} |F_{\beta,t^{2\beta}}(z,w)||f(w)|dw\Big]^2 \frac{dzdt}{t^{d+1}}\\
&\lesssim \int_0^{\gamma(g_0)} \Big[\int_{\mathbb{H}} \Big(\frac{Ct^{2\beta}}{(t+|w^{-1}g|)^{d+2\beta}}\Big(\frac{t}{\gamma(g)}\Big)^{\delta} + \frac{Ct^{2\beta}}{(t+|w^{-1}g|)^{d+2\beta}}\Big(\frac{|g^{-1}w|}{\gamma(g)}\Big)^{\delta}\Big)|f(w)|dw\Big]^2 \frac{dt}{t}\\
&\lesssim |B|^{2\nu}\|f\|^2_{BMO_L^\nu}.
\end{aligned} \tag{4.10}
$$

It follows from (4.8)-(4.10) that

$$
\frac{1}{|B|^{2\nu+1}}\int_B |S^L_{m,\beta}f(g) - A_3|^2 dg \lesssim \|f\|^2_{BMO_L^\nu},
$$

which gives the estimate for $S^L_{m,\beta}$.

Theorem 4.7. *The operators $S^L_{m,\beta}$ and $G^L_{m,\beta}$ are bounded from $H^1_L(\mathbb{H})$ to $L^1(\mathbb{H})$ and bounded from $L^1(\mathbb{H})$ to $L^{1,\infty}(\mathbb{H})$. Moreover, $S^L_{m,\beta}$ and $G^L_{m,\beta}$ are bounded on $L^p(\mathbb{H})$ for $1 < p < \infty$ with*

$$
\|S^L_{m,\beta}f\|_{L^p} \sim \|G^L_{m,\beta}f\|_{L^p} \sim \|f\|_{L^p}.
$$

Proof: When $\nu = 1$, it is easy to see that Theorem 4.6 implies that $S^L_{m,\beta}$ and $G^L_{m,\beta}$ are bounded from $L^\infty(\mathbb{H})$ to $BMO_L(\mathbb{H})$. Then, using Lemmas 4.2 & 4.4 and an interpolation argument, $S^L_{m,\beta}$ and $G^L_{m,\beta}$ are bounded on $L^p(\mathbb{H})$ for $1 < p < \infty$.

5. The $L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})$-boundedness of square functions

The classical Morrey space $M^{\lambda}_{p}(\mathbb{R}^n)$ has been studied intensively and extensively applied to the fields of analysis, mathematical physics and other related fields. As a promotion of Lebesgue spaces, Morrey [33] originally introduced the classical Morrey space to investigate the local behavior of solutions of second order elliptic partial differential equations. For the properties and applications of the classical Morrey space, please refer to [29, 34–36] and the references therein.

We first recall some related facts for the classical Morrey space $M^{\lambda}_{p}(\mathbb{R}^n)$, which consists of all p-locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M^{\lambda}_{p}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda/p}\|f\|_{L^p(B(x,r))} < \infty,$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$. Denote by $WM^{\lambda}_{1}(\mathbb{R}^n)$ the weak Morrey space, which consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{WM^{\lambda}_{1}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda}\|f\|_{WL^1(B(x,r))} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda} \sup_{\sigma>0} \sigma|\{y \in B(x,r) : |f(y)| > \sigma\}| < \infty.$$

Below, we introduce Morrey spaces related to L on the stratified Lie group (which also can be seen from [29]).

Definition 5.1. Assume that $1 \le p < \infty$, $0 \le \kappa < 1$ and $0 < \theta < \infty$. The Morrey space $L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})$ is defined as the set of all p-locally integrable functions f on $\mathbb{H}$ such that

$$\Big(\frac{1}{|B(g_0,r)|^{\kappa}}\int_{B(g_0,r)}|f(g)|^p dg\Big)^{1/p} \le C\Big(1+\frac{r}{\gamma(g_0)}\Big)^{\theta}$$

with norm

$$\|f\|_{L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})} := \sup_{B(g_0,r)} \Big(1+\frac{r}{\gamma(g_0)}\Big)^{-\theta}\Big(\frac{1}{|B(g_0,r)|^{\kappa}}\int_{B(g_0,r)}|f(g)|^p dg\Big)^{1/p} < \infty.$$

Define

$$L^{\gamma,\infty}_{p,\kappa}(\mathbb{H}) := \bigcup_{0<\theta<\infty} L^{\gamma,\theta}_{p,\kappa}(\mathbb{H}).$$

Definition 5.2. Assume that $p = 1$, $0 \le \kappa < 1$ and $0 < \theta < \infty$. The weak Morrey space $WL^{\gamma,\theta}_{1,\kappa}(\mathbb{H})$ is defined as the set of all measurable functions f on $\mathbb{H}$ such that

$$\frac{1}{|B(g_0,r)|^{\kappa}}\sup_{\lambda>0}\lambda\cdot|\{g\in B(g_0,r) : |f(g)|>\lambda\}| \le C\Big(1+\frac{r}{\gamma(g_0)}\Big)^{\theta}$$

with norm

$$\|f\|_{WL^{\gamma,\theta}_{1,\kappa}(\mathbb{H})} := \sup_{B(g_0,r)} \Big(1+\frac{r}{\gamma(g_0)}\Big)^{-\theta}\frac{1}{|B(g_0,r)|^{\kappa}}\sup_{\lambda>0}\lambda\cdot|\{g\in B(g_0,r) : |f(g)|>\lambda\}| < \infty.$$

Correspondingly, we define

$$WL^{\gamma,\infty}_{1,\kappa}(\mathbb{H}) := \bigcup_{0<\theta<\infty} WL^{\gamma,\theta}_{1,\kappa}(\mathbb{H}).$$

Remark 5.3. (i) If $\theta = 0$ or $V = 0$, the Morrey space $L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})$ (or weak Morrey space $WL^{\gamma,\theta}_{1,\kappa}(\mathbb{H})$) is just the Morrey space $L^{\kappa}_{p}(\mathbb{H})$ (or weak Morrey space $WL^{\kappa}_{1}(\mathbb{H})$), which is introduced by Guliyev et al. (see [37]).

(ii) From the definition, we can see

$$L^{\kappa}_{p}(\mathbb{H}) \subset L^{\gamma,\theta_1}_{p,\kappa}(\mathbb{H}) \subset L^{\gamma,\theta_2}_{p,\kappa}(\mathbb{H}); \tag{5.1}$$

$$WL^{\kappa}_{1}(\mathbb{H}) \subset WL^{\gamma,\theta_1}_{1,\kappa}(\mathbb{H}) \subset WL^{\gamma,\theta_2}_{1,\kappa}(\mathbb{H}), \tag{5.2}$$

whenever $0 < \theta_1 < \theta_2 < \infty$. Hence,

$$L^{\kappa}_{p}(\mathbb{H}) \subset L^{\gamma,\infty}_{p,\kappa}(\mathbb{H}) \ \ and \ \ WL^{\kappa}_{1}(\mathbb{H}) \subset WL^{\gamma,\infty}_{1,\kappa}(\mathbb{H})$$

for $(p, \kappa) \in [1, \infty) \times [0, 1)$.

(iii) Define a norm such that the space $L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})$ is a Banach space. In view of (5.1), for any given $f \in L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})$, let

$$\theta^* := \inf\{\theta > 0 : f \in L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})\}.$$

Define

$$\|f\|_{\star} = \|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})} := \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})},$$

which satisfies the axioms of a norm.

(iv) In view of (5.2), for any given $f \in WL^{\gamma,\infty}_{1,\kappa}(\mathbb{H})$, let

$$\theta^{**} := \inf\{\theta > 0 : f \in WL^{\gamma,\theta}_{1,\kappa}(\mathbb{H})\}.$$

Similarly, we can also define by

$$\|f\|_{\star\star} = \|f\|_{WL^{\gamma,\infty}_{1,\kappa}(\mathbb{H})} := \|f\|_{WL^{\gamma,\theta^{**}}_{1,\kappa}(\mathbb{H})}.$$

We can check that this functional $\|\cdot\|_{\star\star}$ satisfies the axioms of a (quasi-) norm. Hence $WL^{\gamma,\infty}_{1,\kappa}(\mathbb{H})$ is a (quasi-) normed linear space.

Since Morrey space $L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})$ (or weak Morrey space $WL^{\gamma,\theta}_{1,\kappa}(\mathbb{H})$) could be viewed as an extension of the Lebesgue (or the weak Lebesgue) space on $\mathbb{H}$, it is accordingly natural to investigate the boundedness properties of operators $S^{L}_{m,\beta}$ and $G^{L}_{m,\beta}$ in the framework of Morrey spaces. Therefore, we extend Theorem 4.7 to the Morrey spaces on $\mathbb{H}$.

Theorem 5.4. *Assume that* $0 < \kappa < 1$.

(i) *For* $1 < p < \infty$*, there exists a constant C such that for all functions* $f \in L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})$*,*

$$\|G^{L}_{m,\beta}f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})} \leq C\|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})}.$$

(ii) *Let* $p = 1$*. There exists a constant C such that for all functions* $f \in L^{\gamma,\infty}_{1,\kappa}(\mathbb{H})$

$$\|G^{L}_{m,\beta}f\|_{WL^{\gamma,\infty}_{1,\kappa}(\mathbb{H})} \leq C\|f\|_{L^{\gamma,\infty}_{1,\kappa}(\mathbb{H})}.$$

Proof: (i) For any given $f \in L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})$ with $1 \le p < \infty$ and $0 < \kappa < 1$, suppose that $f \in L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})$ for some $\theta^* > 0$, where

$$\theta^* = \inf\{\theta > 0 : f \in L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})\} \quad and \quad \|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})} = \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})}.$$

Then, we will prove that, for each fixed ball $B(g_0, r)$, there exist some $\nu > 0$ such that

$$\Big(\frac{1}{|B(g_0,r)|^\kappa}\int_{B(g_0,r)} |G^L_{m,\beta}(f)(g)|^p dg\Big)^{1/p} \lesssim \Big(1 + \frac{r}{\gamma(g_0)}\Big)^\nu \tag{5.3}$$

holds true for $f \in L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})$, $(p,\kappa) \in (1,\infty) \times (0,1)$.

We split f as

$$\begin{cases} f = f_1 + f_2 \in L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H}); \\ f_1 = f \cdot \chi_{2B}; \\ f_2 = f \cdot \chi_{(2B)^c}. \end{cases}$$

Then, we write

$$\Big(\frac{1}{|B(g_0,r)|^\kappa}\int_{B(g_0,r)} |G^L_{m,\beta}(f)(g)|^p dg\Big)^{1/p} \le I_1 + I_2,$$

where

$$\begin{cases} I_1 := \Big(\frac{1}{|B(g_0,r)|^\kappa}\int_{B(g_0,r)} |G^L_{m,\beta}(f_1)(g)|^p dg\Big)^{1/p}; \\ I_2 := \Big(\frac{1}{|B(x_0,r)|^\kappa}\int_{B(x_0,r)} |G^L_{m,\beta}(f_2)(g)|^p dg\Big)^{1/p}. \end{cases}$$

For I_1, using Theorem 4.7, we obtain

$$I_1 \lesssim \frac{1}{|B|^{\kappa/p}}\Big(\int_{\mathbb{H}} |f_1(g)|^p dg\Big)^{1/p} \lesssim \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})} \frac{|2B|^{\kappa/p}}{|B|^{\kappa/p}}\Big(1 + \frac{2r}{\gamma(g_0)}\Big)^{\theta^*}.$$

Moreover, observe that for any fixed $\theta^* > 0$,

$$1 \le \Big(1 + \frac{2r}{\gamma(g_0)}\Big)^{\theta^*} \le 2^{\theta^*}\Big(1 + \frac{r}{\gamma(g_0)}\Big)^{\theta^*}, \tag{5.4}$$

which further implies that

$$I_1 \lesssim \|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})}\Big(1 + \frac{r}{\gamma(g_0)}\Big)^{\theta^*}.$$

Next we estimate the other term I_2. We can easily get

$$G^L_{m,\beta}(f_2)(g) \lesssim \int_{(2B)^c} \frac{1}{|h^{-1}g|^d}\Big(1 + \frac{|h^{-1}g|}{\gamma(g)}\Big)^{-N} |f(h)| dh. \tag{5.5}$$

In fact, from Lemma 3.4, it follows that

$$G^L_{m,\beta}(f_2)(g) \le G^{L,\infty}_{m,\beta}(f_2)(g) + G^{L,0}_{m,\beta}(f_2)(g),$$

where

$$\begin{cases} G^{L,\infty}_{m,\beta}(f_2)(g) := \Big(\int_{|g^{-1}h|^{2\beta}}^{\infty}\Big(\int_{\mathbb{H}}\frac{C_N s}{(s^{1/2\beta}+|g^{-1}h|)^{d+2\beta}}\Big(1+\frac{s^{1/2\beta}}{\gamma(g)}\Big)^{-N}|f_2(h)|dh\Big)^2\frac{ds}{s}\Big)^{1/2}; \\ G^{L,0}_{m,\beta}(f_2)(g) := \Big(\int_0^{|g^{-1}h|^{2\beta}}\Big(\int_{\mathbb{H}}\frac{C_N s}{(s^{1/2\beta}+|g^{-1}h|)^{d+2\beta}}\Big(1+\frac{s^{1/2\beta}}{\gamma(g)}\Big)^{-N}|f_2(h)|dh\Big)^2\frac{ds}{s}\Big)^{1/2}. \end{cases}$$

When $s > |g^{-1}h|^{2\beta}$, then $s^{1/2\beta} > |g^{-1}h|$. Hence,

$$\begin{aligned} G^{L,\infty}_{m,\beta}(f_2)(g) &\lesssim \Big(\int_{|g^{-1}h|^{2\beta}}^{\infty}\Big(\int_{\mathbb{H}}\frac{1}{s^{d/2\beta}}\Big[1+\frac{|g^{-1}h|}{\gamma(g)}\Big]^{-N}|f_2(h)|dh\Big)^2\frac{ds}{s}\Big)^{1/2} \\ &\lesssim \int_{\mathbb{H}}\Big[1+\frac{|g^{-1}h|}{\gamma(g)}\Big]^{-N}|f_2(h)|\Big(\int_{|g^{-1}h|^{2\beta}}^{\infty}\frac{ds}{s^{d/\beta+1}}\Big)^{1/2}dh \\ &\lesssim \int_{(2B)^c}\Big[1+\frac{|g^{-1}h|}{\gamma(g)}\Big]^{-N}\frac{1}{|g^{-1}h|^d}|f(h)|dh, \end{aligned}$$

where we have used the Minkowski inequality in the third step. On the other hand, since $0 < s \le |g^{-1}h|^{2\beta}$, a trivial computation leads to that

$$\int_{\mathbb{H}}\frac{C_N s}{|g^{-1}h|^{d+2\beta}}\Big(1+\frac{s^{1/2\beta}}{\gamma(g)}\Big)^{-N}|f_2(h)|dh \lesssim \int_{\mathbb{H}}\frac{1}{|g^{-1}h|^d}\Big(\frac{s^{1/2\beta}}{|g^{-1}h|}\Big)^{2\beta+N}\Big[1+\frac{|g^{-1}h|}{\gamma(g)}\Big]^{-N}|f_2(h)|dh.$$

Then, by the Minkowski's inequality for integrals, we can get

$$\begin{aligned} G^{L,0}_{m,\beta}(f_2)(g) &\lesssim \Big(\int_0^{|g^{-1}h|^{2\beta}}\Big(\int_{\mathbb{H}}\frac{1}{|g^{-1}h|^d}\Big(\frac{s^{1/2\beta}}{|g^{-1}h|}\Big)^{2\beta+N}\Big[1+\frac{|g^{-1}h|}{\gamma(g)}\Big]^{-N}|f_2(h)|dh\Big)^2\frac{ds}{s}\Big)^{1/2} \\ &\lesssim \int_{(2B)^c}\Big[1+\frac{|g^{-1}h|}{\gamma(g)}\Big]^{-N}\frac{1}{|g^{-1}h|^d}|f(h)|dh. \end{aligned}$$

Consequently, we have obtain the desired inequality (5.5) for any $g \in B(g_0, r)$.

Since $g \in B(g_0, r)$ and $h \in (2B)^c$, then $|g^{-1}h| \sim |g_0^{-1}h|$. Along with (5.5), we have for any $g \in B(g_0, r)$,

$$|G^L_{m,\beta}(f_2)(g)| \lesssim \sum_{k=1}^{\infty}\frac{1}{|B(g_0, 2^{k+1}r)|}\int_{|g_0^{-1}h|<2^{k+1}r}\Big(1+\frac{2^k r}{\gamma(g)}\Big)^{-N}|f(h)|dh.$$

In view of (2.3) and (5.4), we can deduce that

$$|G^L_{m,\beta}(f_2)(g)| \lesssim \sum_{k=1}^{\infty}\frac{1}{|B(g_0, 2^{k+1}r)|}\int_{B(g_0,2^{k+1}r)}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{-N}|f(h)|dh. \tag{5.6}$$

Using Hölder's inequality, we obtain

$$\begin{aligned} &\frac{1}{|B(g_0, 2^{k+1}r)|}\int_{B(g_0,2^{k+1}r)}|f(h)|dh \\ &\lesssim \frac{1}{|B(g_0, 2^{k+1}r)|}\Big(\int_{B(g_0,2^{k+1}r)}|f(h)|^p dh\Big)^{1/p}\Big(\int_{B(g_0,2^{k+1}r)}1dh\Big)^{1/p'} \end{aligned}$$

$$\lesssim \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})}\frac{|B(g_0,2^{k+1}r)|^{\kappa/p}}{|B(g_0,2^{k+1}r)|^{1/p}}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{\theta^*}.$$

Then,

$$\begin{aligned} I_2 &\lesssim \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})}\frac{|B(g_0,r)|^{1/p}}{|B(g_0,r)|^{\kappa/p}}\sum_{k=1}^{\infty}\frac{|B(g_0,2^{k+1}r)|^{\kappa/p}}{|B(g_0,2^{k+1}r)|^{1/p}}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{-N+\theta^*}\\ &\lesssim \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\sum_{k=1}^{\infty}\frac{|B(g_0,r)|^{(1-\kappa)/p}}{|B(g_0,2^{k+1}r)|^{(1-k)/p}}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{-N+\theta^*}. \end{aligned}$$

Consequently, Let N large enough such that $N \geq \theta^*$. we can see that the last series is convergent. Then, by the fact that $1-\kappa>0$, we have

$$\begin{aligned} I_2 &\lesssim \|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\sum_{k=1}^{\infty}\Big(\frac{|B(g_0,r)|}{|B(g_0,2^{k+1}r)|}\Big)^{(1-\kappa)/p}\\ &\lesssim \|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}. \end{aligned}$$

By adding the two estimates of I_1 and I_2 above, we get (5.3) by making $\nu=\max\{\theta^*, N\cdot\frac{N_0}{N_0+1}\}$, $N\geq\theta^*$.

(ii) It is only to prove that

$$\frac{1}{|B(g_0,r)|^{\kappa}}\sup_{\lambda>0}\lambda\cdot|\{g\in B(g_0,r):|G^L_{m,\beta}(f)(g)|>\lambda\}|\lesssim\Big(1+\frac{r}{\gamma(g_0)}\Big)^{\nu} \tag{5.7}$$

holds true for given $f\in L^{\gamma,\theta^*}_{1,\kappa}(\mathbb{H})$ with some $\theta^*>0$, $\nu>0$ and $0<\kappa<1$. We split f as

$$\begin{cases} f=f_1+f_2\in L^{\gamma,\theta^*}_{1,\kappa}(\mathbb{H});\\ f_1=f\cdot\chi_{2B};\\ f_2=f\cdot\chi_{(2B)^c}. \end{cases}$$

Then, for any given $\lambda>0$, we can write

$$\frac{1}{|B(g_0,r)|^{\kappa}}\lambda\cdot|\{g\in B(g_0,r):|G^L_{m,\beta}(f)(g)|>\lambda\}|\leq J_1+J_2,$$

where

$$\begin{cases} J_1:=\dfrac{1}{|B(g_0,r)|^{\kappa}}\lambda\cdot|\{g\in B(g_0,r):|G^L_{m,\beta}(f_1)(g)|>\lambda/2\}|;\\ J_2:=\dfrac{1}{|B(g_0,r)|^{\kappa}}\lambda\cdot|\{g\in B(g_0,r):|G^L_{m,\beta}(f_2)(g)|>\lambda/2\}|. \end{cases}$$

We first give the estimate for the term J_1. By the Lemma 4.2, we have

$$J_1\lesssim\frac{1}{|B|^{\kappa}}\Big(\int_{\mathbb{H}}|f_1(g)|dg\Big)\lesssim\|f\|_{L^{\gamma,\theta^*}_{1,\kappa}(\mathbb{H})}\frac{|2B|^{\kappa}}{|B|^{\kappa}}\Big(1+\frac{2r}{\gamma(g_0)}\Big)^{\theta^*}.$$

Therefore, in view of (5.4), we have

$$J_1 \lesssim \|f\|_{L^{\gamma,\infty}_{1,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{\theta^*}.$$

For J_2, by (5.6) and Chebyshev's inequality, we obtain

$$\begin{aligned} J_2 &\lesssim \frac{1}{|B(g_0,r)|^\kappa}\Big(\int_{B(g_0,r)}|G^L_{m,\beta}(f_2)(g)|dg\Big)\\ &\lesssim \frac{|B(g_0,r)|}{|B(g_0,r)|^\kappa}\sum_{k=1}^{\infty}\frac{1}{|B(g_0,2^{k+1}r)|}\int_{B(g_0,2^{k+1}r)}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{-N}|f(h)|dh. \end{aligned} \tag{5.8}$$

We consider the sum of (5.8) for each term, separately. We can see that

$$\frac{1}{|B(g_0,2^{k+1}r)|}\int_{B(g_0,2^{k+1}r)}|f(h)|dh \lesssim C\|f\|_{L^{\gamma,\theta^*}_{1,\kappa}(\mathbb{H})}\frac{|B(g_0,2^{k+1}r)|^\kappa}{|B(g_0,2^{k+1}r)|}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{\theta^*}.$$

Consequently,

$$J_2 \lesssim \|f\|_{L^{\gamma,\theta^*}_{1,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\sum_{k=1}^{\infty}\frac{|B(g_0,r)|}{|B(g_0,r)|^k}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{-N+\theta^*}\frac{|B(g_0,2^{k+1}r)|^\kappa}{|B(g_0,2^{k+1}r)|}.$$

Hence, let N large enough such that $N \geq \theta^*$. Then, we can get

$$\begin{aligned} J_2 &\lesssim \|f\|_{L^{\gamma,\infty}_{1,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\sum_{k=1}^{\infty}\Big(\frac{|B(g_0,r)|}{|B(g_0,2^{k+1}r)|}\Big)^{(1-\kappa)}\\ &\lesssim \|f\|_{L^{\gamma,\infty}_{1,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}. \end{aligned}$$

Let $\nu = \max\{\theta^*, N\cdot\frac{N_0}{N_0+1}\}$ with $N \geq \theta^*$. Summing up the above estimates for J_1 and J_2, and then taking the supremum over all $\lambda > 0$, we obtain the desired inequality (5.7).

Theorem 5.5. *Suppose* $0 < \kappa < 1$.

(i) *Let* $1 < p < \infty$. *There exists a constant C such that for all functions* $f \in L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})$,

$$\|S^L_{m,\beta}f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})} \leq C\|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})}.$$

(ii) *Let* $p = 1$. *There exists a constant C such that for all functions* $f \in L^{\gamma,\infty}_{1,\kappa}(\mathbb{H})$,

$$\|S^L_{m,\beta}f\|_{WL^{\gamma,\infty}_{1,\kappa}(\mathbb{H})} \leq C\|f\|_{L^{\gamma,\infty}_{1,\kappa}(\mathbb{H})}.$$

Proof: For any given $f \in L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})$ with $1 \leq p < \infty$ and $0 < \kappa < 1$, suppose that $f \in L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})$ for some $\theta^* > 0$, where

$$\theta^* = \inf\{\theta > 0 : f \in L^{\gamma,\theta}_{p,\kappa}(\mathbb{H})\} \quad \textit{and} \quad \|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})} = \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})}.$$

Below, we only need to show that

$$\Big(\frac{1}{|B(g_0,r)|^{\kappa}}\int_{B(g_0,r)}|S^{L}_{m,\beta}(f)(g)|^{p}dg\Big)^{1/p}\lesssim\Big(1+\frac{r}{\gamma(g_0)}\Big)^{\tilde{\theta}}$$

holds true for $f\in L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})$ with some $\tilde{\theta}>0$ and $(p,\kappa)\in(1,\infty)\times(0,1)$.

We split f as

$$\begin{cases} f=f_1+f_2\in L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H});\\ f_1=f\cdot\chi_{4B};\\ f_2=f\cdot\chi_{(4B)^c},\end{cases}$$

Then, we can write

$$\Big(\frac{1}{|B(g_0,r)|^{\kappa}}\int_{B(g_0,r)}|S^{L}_{m,\beta}(f)(g)|^{p}dg\Big)^{1/p}\leq I_1'+I_2',$$

where

$$\begin{cases} I_1':=\Big(\frac{1}{|B(g_0,r)|^{\kappa}}\int_{B(g_0,r)}|S^{L}_{m,\beta}(f_1)(g)|^{p}dg\Big)^{1/p};\\ I_2':=\Big(\frac{1}{|B(x_0,r)|^{\kappa}}\int_{B(x_0,r)}|S^{L}_{m,\beta}(f_2)(g)|^{p}dg\Big)^{1/p}.\end{cases}$$

For I_1, by Theorem 4.7 and (5.4), we can get

$$\begin{aligned} I_1' &\lesssim \frac{1}{|B|^{\kappa/p}}\Big(\int_{4B}|f(g)|^{p}dg\Big)^{1/p}\\ &\lesssim \|f\|_{L^{\gamma,\theta^*}_{p,\kappa}(\mathbb{H})}\frac{|4B|^{\kappa/p}}{|B|^{\kappa/p}}\Big(1+\frac{4r}{\gamma(g_0)}\Big)^{\theta^*}\\ &\lesssim \|f\|_{L^{\gamma,\infty}_{p,\kappa}(\mathbb{H})}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{\theta^*}.\end{aligned}$$

For I_2', we assert that the inequality

$$S^{L}_{m,\beta}(f_2)(g)\lesssim\int_{(4B)^c}\frac{1}{|z^{-1}g|^{d}}\Big(1+\frac{|z^{-1}g|}{\gamma(z)}\Big)^{-N}|f(z)|dz \tag{5.9}$$

is valid for any $g\in B(g_0,r)$.

Similarly to the proof of Theorem 5.4, we consider below two cases: $s>(\frac{|z^{-1}g|}{2})^{2\beta}$ and $0\leq s\leq(\frac{|z^{-1}g|}{2})^{2\beta}$. In fact, from Lemma 3.4, it follows that

$$S^{L}_{m,\beta}(f_2)(g)\leq S^{L,\infty}_{m,\beta}(f_2)(g)+S^{L,0}_{m,\beta}(f_2)(g),$$

where

$$\begin{cases} S^{L,\infty}_{m,\beta}(f_2)(g):=\Big(\int_{\frac{|z^{-1}g|^{2\beta}}{2^{2\beta}}}^{\infty}\int_{|h^{-1}g|<s^{1/2\beta}}\Big(\int_{\mathbb{H}}\frac{C_N s}{(s^{1/2\beta}+|z^{-1}h|)^{d+2\beta}}\Big(1+\frac{s^{1/2\beta}}{\gamma(z)}\Big)^{-N}|f_2(z)|dz\Big)^{2}\frac{dhds}{s^{d/2\beta+1}}\Big)^{1/2};\\ S^{L,0}_{m,\beta}(f_2)(g):=\Big(\int_0^{\frac{|z^{-1}g|^{2\beta}}{2^{2\beta}}}\int_{|h^{-1}g|<s^{1/2\beta}}\Big(\int_{\mathbb{H}}\frac{C_N s}{(s^{1/2\beta}+|z^{-1}h|)^{d+2\beta}}\Big(1+\frac{s^{1/2\beta}}{\gamma(z)}\Big)^{-N}|f_2(z)|dz\Big)^{2}\frac{dhds}{s^{d/2\beta+1}}\Big)^{1/2}.\end{cases}$$

When $s > |z^{-1}g|^{2\beta}/2^{2\beta}$, then $s^{1/2\beta} > |z^{-1}g|/2$. Hence,

$$\begin{aligned}
G^{L,\infty}_{m,\beta}(f_2)(g) &\lesssim \int_{\frac{|z^{-1}g|^{2\beta}}{4^\beta}}^{\infty}\int_{|h^{-1}g|<s^{1/2\beta}}\Big(\int_{\mathbb{H}}\frac{1}{s^{d/2\beta}}\Big[1+\frac{|z^{-1}g|}{\gamma(z)}\Big]^{-N}|f_2(z)|dz\Big)^2\frac{dhds}{s^{d/2\beta+1}}\Big)^{1/2}\\
&\lesssim \int_{\mathbb{H}}\Big[1+\frac{|z^{-1}g|}{\gamma(z)}\Big]^{-N}|f_2(z)|\Big(\int_{\frac{|z^{-1}g|^{2\beta}}{4^\beta}}^{\infty}\int_{|h^{-1}g|<s^{1/2\beta}}\frac{dhds}{s^{d/\beta+d/2\beta+1}}\Big)^{1/2}dzdh\\
&\lesssim \int_{(4B)^c}\Big[1+\frac{|z^{-1}g|}{\gamma(z)}\Big]^{-N}\frac{1}{|z^{-1}g|^d}|f(z)|dz,
\end{aligned}$$

where we have used the Minkowski inequality in the third step. On the other hand, it is easy to see that

$$\int_{\mathbb{H}}\frac{C_N s}{|z^{-1}h|^{d+2\beta}}\Big(1+\frac{s^{1/2\beta}}{\gamma(z)}\Big)^{-N}|f_2(z)|dz=\int_{\mathbb{H}}\frac{C_N}{|z^{-1}h|^d}\Big(\frac{s^{1/2\beta}}{|z^{-1}h|}\Big)^{2\beta}\Big(\frac{s^{1/2\beta}}{|z^{-1}h|}\Big)^{N}\Big[\frac{|z^{-1}h|}{s^{1/2\beta}}+\frac{|z^{-1}h|}{\gamma(z)}\Big]^{-N}|f_2(z)|dz.$$

Note that when $0 < s \le \frac{|z^{-1}g|^{2\beta}}{2^{2\beta}}$ and $|h^{-1}g| < s^{1/2\beta}$, then $|z^{-1}h| \ge |z^{-1}g|/2$. Hence, we have

$$\int_{\mathbb{H}}\frac{C_N s}{|z^{-1}h|^{d+2\beta}}\Big(1+\frac{s^{1/2\beta}}{\gamma(z)}\Big)^{-N}|f_2(z)|dz\lesssim\int_{\mathbb{H}}\frac{1}{|z^{-1}g|^d}\Big(\frac{s^{1/2\beta}}{|z^{-1}g|}\Big)^{2\beta+N}\Big[1+\frac{|z^{-1}g|}{\gamma(z)}\Big]^{-N}|f_2(z)|dz.$$

Hence, along with Minkowski's inequality, we obtain

$$\begin{aligned}
S^{L,0}_{m,\beta}(f_2)(g) &\lesssim \Big(\int_0^{\frac{|z^{-1}g|^{2\beta}}{4^\beta}}\int_{|h^{-1}g|<s^{1/2\beta}}\Big(\int_{\mathbb{H}}\frac{1}{|z^{-1}g|^d}\Big(\frac{s^{1/2\beta}}{|z^{-1}g|}\Big)^{2\beta+N}\Big[1+\frac{|z^{-1}g|}{\gamma(z)}\Big]^{-N}|f_2(z)|dz\Big)^2\frac{dhds}{s^{d/2\beta+1}}\Big)^{1/2}\\
&\lesssim \int_{\mathbb{H}}\frac{1}{|z^{-1}g|^d}\Big[1+\frac{|z^{-1}g|}{\gamma(z)}\Big]^{-N}|f_2(z)|\Big(\int_0^{\frac{|z^{-1}g|^{2\beta}}{4^\beta}}\int_{|h^{-1}g|<s^{1/2\beta}}\Big(\frac{s^{1/2\beta}}{|z^{-1}g|}\Big)^{4\beta+2N}\frac{dhds}{s^{d/2\beta+1}}\Big)^{1/2}dz\\
&\lesssim \int_{(4B)^c}\Big[1+\frac{|z^{-1}g|}{\gamma(z)}\Big]^{-N}\frac{1}{|z^{-1}g|^d}|f(z)|dz.
\end{aligned}$$

Taking the above two estimates together gives the required inequality (5.9) for any $g \in B(g_0, r)$. If $x \in B(g_0, r)$ and $z \in (4B)^c$, $|z^{-1}g| \sim |z^{-1}g_0|$. Then using (5.9) and (5.4), we have

$$|S^L_{m,\beta}(f_2)(g)| \lesssim \sum_{k=2}^{\infty}\frac{1}{|B(g_0,2^{k+1}r)|}\int_{|z^{-1}g_0|<2^{k+1}r}\Big(1+\frac{2^k r}{\gamma(z)}\Big)^{-N}|f(z)|dz.$$

Moreover, in view of (2.3), we can deduce that

$$|S^L_{m,\beta}(f_2)(g)| \lesssim \sum_{k=2}^{\infty}\frac{1}{|B(g_0,2^{k+1}r)|}\int_{B(g_0,2^{k+1}r)}\Big(1+\frac{r}{\gamma(g_0)}\Big)^{N\cdot\frac{N_0}{N_0+1}}\Big(1+\frac{2^{k+1}r}{\gamma(g_0)}\Big)^{-N}|f(z)|dz.$$

The rest of the proof is similar to the proof of Theorem 5.4, so it is omitted.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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