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Brief report

From a magnetoacoustic system to a J-T black hole11: A little trip down memory lane

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Abstract: We assign a Riemannian metric to a system of nonlinear equations that describe the one-dimensional propagation of long magnetoacoustic waves (also called magnetosonic waves) in a cold plasma under the inference of a transverse magnetic field. The metric, which in general is expressed in terms of the density of the plasma and its speed across the magnetic field, when specialized to a particular solution of the nonlinear system (the Gurevich-Krylov (G-K) solution) is mapped explicitly to a Jackiw-Teitelboim (J-T) black hole metric, which is the main result. Dilaton fields, constructed from data involved in the G-K solution, are presented - which with the plasma metric provide for elliptic function solutions of the J-T equations of motion in 2d dilaton gravity. A correspondence between solutions of the nonlinear plasma system (whose Galilean invariance is also established) and certain solutions of a resonant nonlinear Schrödinger equation is set up, along with some other general background material to render an expository tone in the presentation.

Keywords: magnetoacoustic system; Lagrangian density; cold plasma; elliptic functions; resonant nonlinear Schrödinger equation; plasma metric; continuous Heisenberg model; Ricci scalar curvature; Jackiw-Teitelboim black hole; cosmological constant; dilaton field
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1. Introduction

We consider a direct connection of black holes in the Jackiw-Teitelboim model of two-dimensional dilaton gravity to the dynamics of two-component, cold collisionless plasma in the presence of an external transverse magnetic field. The propagation of long magnetoacoustic waves in the cold plasma (under a uni-axial propagation assumption) is described by the key nonlinear system of equations (2.5), where ρ is the plasma density, u is its speed across the magnetic field $\vec{B}$, and $\beta > 0$ is a parameter that arises in a particular power series expansion of the magnetic field strength $B(x,t)$ when a shallow water approximation is imposed. Such dynamics are discussed more generally in [1–3], for example, where a general set of equations is involved. These include the Maxwell equations $\nabla \cdot \vec{B} = 0$ and $\frac{\partial \vec{B}}{\partial t} = -\text{curl}\ \vec{E}$ (Gauss and Faraday laws), where the electric field $\vec{E}$ is eliminated from the discussion by way of a few simplifying assumptions, including the above uni-axial propagation assumption - namely that $\vec{B} = B(x,t)\vec{e}_z$ and $\vec{u} = u(x,t)\vec{e}_x$, for the velocity vector $\vec{u}$ of the plasma. Later, u is also referred to as the "velocity" of the plasma.Getting back to the parameter $\beta > 0$ in the second equation of (2.5) below, and the statement about the power series expansion of B, one can say that the latter expansion has the form

$$B = \rho + \beta^2 \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x}\right) + O(\beta^4). \tag{1.1}$$

The first equation in (2.5) is a continuity equation. The system of two equations in (2.5) is the reduction of an initial system of seven equations that are next reduced to a system of three equations, and then to the two equations by way of a shallow water approximation [1, 4]. Actually, we show that the magnetoacoustic system (2.5), which is of fundamental interest here, can be derived directly and quickly from the equations of motion of a suitable Lagrangian density $\mathscr{L}$, which is given in definition (2.1).

In [1], the system (2.5) is reduced to a resonant nonlinear Schrödinger equation (RNLSE) and particular solutions are generated that are descriptive of the interaction of solitonic magnetoacoustive waves. One can also obtain, conversely, solutions of the system (2.5) from solutions of the RNLSE, a matter that is discussed in section 3. The RNLSE is obtained from the nonlinear Schrödinger equation (NLSE) by the addition of a diffraction type term $|\psi|_{xx}/|\psi|$ called the de Broglie potential (see equation (3.1)), and it (like the NLSE) also admits both bright and dark soliton solutions. Equation (3.30) is an example of a dark soliton solution. The NLSE is known as an indispensable tool that facilitates the description of a multiplicity of nonlinear phenomena. The latter ranges, for example, from plasma physics and hydrodynamics to molecular biology (the nonlinear dynamics of DNA) and the propagation of light in optical fibers. There are applications to the modeling of extreme deep water rogue waves. The beautiful connection of the RNLSE to cold plasma physics is discussed, for example, in the papers [1, 4–7], and in the new book [8]. The work in [6] was, and continues to be, the direct inspiration afforded by the author in [1, 4, 5, 9].

Some other topics or ideas associated with the RNLSE - cold plasma physics connection are, for example, a reaction-diffusion system (RDS), a Madelung fluid system, and the continuous Heisenberg model. Of particular importance is the gauge equivalence (by way of the construction of suitable Lax pairs) of a particular RDS and a Heisenberg model. In the present paper, a direct connection of the cold plasma system (2.5) to a Jackiw-Teitelboim (J-T) black hole is emphasized. As is discussed in section 4, we use the continuous Heisenberg model to assign to the cold plasma system (2.5) a Riemannian metric g_{plasma}. Then, in section 5, an explicit transformation of variables is presented by which the metric g_{plasma} is mapped exactly to a J-T black hole metric g_{bh}. g_{plasma} and g_{bh} are given in equations (4.19) and (5.1) respectively, where the black hole mass M in (5.1) is given by equation (5.7). m there is also given by equation (5.7), so that, in fact, the cosmological constant Λ in the J-T theory is given by $\Lambda = -m^2$, which is negative. Section 5 contains the main result, where also some dilaton fields $\Phi^{(j)}_{\text{plasma}}$, $j = 1, 2, 3$, are presented for which the pairs $\left(g_{\text{plasma}}, \Phi^{(j)}_{\text{plasma}}\right)$ provide for elliptic function solutions of the J-T gravitational field equations.

2. The basic magnetoacoustic system

Of central interest for the discussions that follow is a system of two nonlinear partial differential equations that describe the propagation of one-dimensional long magneto-acoustic waves in a cold plasma of density $\rho(x,t) > 0$ with a velocity $u(x,t)$ across a magnetic field. See equations (2.5) below. The references [1–3], for example, provide background material and further details. Here, we are contented to start with two functions $S(x,t)$ and $\rho(x,t) > 0$, and to consider the Lagrangian density given by

$$\mathscr{L} = \mathscr{L}(x,t,S,S_x,S_t,\rho,\rho_x,\rho_t) \overset{\text{def.}}{=} \rho\left(S_t - (S_x)^2\right) - \frac{\rho^2}{4} + \frac{\beta^2}{4}\frac{(\rho_x)^2}{\rho} \tag{2.1}$$

where $\beta > 0$ is a fixed real number, and where the subscripts x, t, xx denote partial differentiation as usual. The corresponding equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial f_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\mathscr{L}}{\partial f_x}\right) = \frac{\partial\mathscr{L}}{\partial f},\ f = S,\rho, \tag{2.2}$$

respectively, are directly computed:

$$\begin{aligned} &\rho_t - 2(\rho S_x)_x = 0, \\ &-S_t + (S_x)^2 + \tfrac{\rho}{2} + \tfrac{\beta^2}{2}\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right] = 0, \end{aligned} \tag{2.3}$$

the first equation here being a continuity equation and the second one being a Hamilton-Jacobi equation. Next, we differentiate the second equation in (2.3) with respect to x, assuming that $S_x = -u/2$ for a third function $u(x,t)$. Using the equality of mixed partial derivatives, we get $S_{tx} = S_{xt} = -u_t/2$, which with the equations

$$\frac{\partial}{\partial x}(S_x)^2 = 2S_xS_{xx} = (-u)(-u_x/2) \tag{2.4}$$

and (2.3) lead to the system

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &u_t + uu_x + \rho_x + \beta^2\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right]_x = 0. \end{aligned} \tag{2.5}$$

This is the magnetoacoustic system (MAS) of interest for what follows.

We note that this system is *Galilean invariant*. That is, for $c_0 \in \mathbb{R}$ the field of real numbers, c_0 fixed, define the *Galilean transforms* $\hat{u}, \hat{\rho}$ of u, ρ by

$$\hat{u}(x,t) = c_0 + u(x - c_0t, t), \hat{\rho}(x,t) = \rho(x - c_0t, t). \tag{2.6}$$

Then, by the chain rule,

$$\begin{aligned} &(\hat{\rho}_t + (\hat{\rho}\hat{u})_x)(x,t) = (\rho_t + (\rho u)_x)(x - c_0t, t), \\ &(\hat{u}_t + \hat{u}\hat{u}_x + \hat{\rho}_x)(x,t) = (u_t + uu_x + \rho_x)(x - c_0t, t), \\ &\left(\tfrac{\hat{\rho}_{xx}}{\hat{\rho}} - \tfrac{1}{2}\left(\tfrac{\hat{\rho}_x}{\hat{\rho}}\right)^2\right)(x,t) = \left(\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right)(x - c_0t, t), \\ &\left[\tfrac{\hat{\rho}_{xx}}{\hat{\rho}} - \tfrac{1}{2}\left(\tfrac{\hat{\rho}_x}{\hat{\rho}}\right)^2\right]_x(x,t) = \left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right]_x(x,t). \end{aligned} \tag{2.7}$$

Thus if (u, ρ) is a solution of the MAS (2.5) then the pair $(\hat{u}, \hat{\rho})$ is also a solution.

The standard Jacobi elliptic functions $sn(x,\kappa)$, $cn(x,\kappa)$, $dn(x,\kappa)$ with elliptic modulus κ will be needed. Their basic properties are the following [8,10]:

$$\begin{aligned} & sn^2(x,\kappa) + cn^2(x,\kappa) = 1,\ dn^2(x,\kappa) + \kappa^2 sn^2(x,\kappa) = 1, \\ & sn(x,1) = \tanh x,\ cn(x,1) = dn(x,1) = \operatorname{sech} x, \\ & \tfrac{d}{dx} sn(x,\kappa) = cn(x,\kappa) dn(x,\kappa),\ \tfrac{d}{dx} cn(x,\kappa) = -sn(x,\kappa) dn(x,\kappa) \\ & \tfrac{d}{dx} dn(x,\kappa) = -\kappa^2 sn(x,\kappa) cn(x,\kappa). \end{aligned} \tag{2.8}$$

In particular, some attention will be given to the following traveling wave solution of the system (2.5), due to A.Gurevich and A.Krylov [11], expressed in terms of the elliptic function $dn(x,\kappa)$. For $u_0 \in \mathbb{R}$, $u_0 > 0$ fixed, and for choices $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_3 > \alpha_2 \geqslant \alpha_1 \geqslant 0$

$$\begin{aligned} & \rho(x,t) = \alpha_1 + (\alpha_3 - \alpha_1)\, dn^2\left(\tfrac{(\alpha_3-\alpha_1)^{\frac{1}{2}}}{2\beta}\,(x - u_0 t)\,, \kappa\right) > 0, \\ & u(x,t) = u_0 + \tfrac{C}{\rho(x,t)},\ \kappa \overset{\text{def.}}{=} +\left(\tfrac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}\right)^{\frac{1}{2}},\ C \overset{\text{def.}}{=} +(\alpha_1\alpha_2\alpha_3)^{\frac{1}{2}}\,. \end{aligned} \tag{2.9}$$

A convenient way to express $\rho(x,t)$ and C is

$$\begin{aligned} & \rho(x,t) = \alpha_1 + 4a^2\beta^2 dn^2(a(x-\beta v t),\kappa), \\ & C = \alpha_1^{\frac{1}{2}}\left[4a^2\beta^2\left(1-\kappa^2\right)+\alpha_1\right]^{\frac{1}{2}}\left[4a^2\beta^2+\alpha_1\right]^{\frac{1}{2}} \end{aligned} \tag{2.10}$$

for

$$a \overset{\text{def.}}{=} +\frac{(\alpha_3-\alpha_1)^{\frac{1}{2}}}{2\beta} > 0,\ v \overset{\text{def.}}{=} \frac{u_0}{\beta} > 0, \tag{2.11}$$

since

$$\begin{aligned} & 1-\kappa^2 \overset{\text{def.}}{=} 1 - \tfrac{(\alpha_3-\alpha_2)}{(\alpha_3-\alpha_1)} = \tfrac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}, \\ & 4a^2\beta^2 \overset{\text{def.}}{=} \alpha_3 - \alpha_1,\ \left(4a^2\beta^2+\alpha_1\right)^{\frac{1}{2}} = \alpha_3^{\frac{1}{2}}, \\ & 4a^2\beta^2\left(1-\kappa^2\right) = (\alpha_3-\alpha_1)\left(\tfrac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}\right) = \alpha_2 - \alpha_1 \Rightarrow \\ & \left[4a^2\beta^2\left(1-\kappa^2\right)+\alpha_1\right]^{\frac{1}{2}} = \alpha_2^{\frac{1}{2}}. \end{aligned} \tag{2.12}$$

For plasma physics, convenient choices for α_1 and α_2 are the values $\alpha_2 = \alpha_1 = 1$ since then $\kappa = 1$, and by (2.8), (2.10) the solution

$$\rho(x,t) = 1 + 4a^2\beta^2 \operatorname{sech}^2 a(x - \beta v t) \tag{2.13}$$

for the plasma density does achieve the convenient value of 1 as $|x| \to \infty$. Also by (2.8), (2.10) note that

$$\begin{aligned} & \rho_x(x,t) = -8a^3\beta^2\kappa^2(sncndn)(a(x-\beta v t),\kappa), \\ & \rho_t(x,t) = 8a^3\beta^3\kappa^2 v(sncndn)(a(x-\beta v t),\kappa) \Rightarrow \\ & \rho_t + u_0\rho_x \overset{\text{def.}}{=} \rho_t + v\beta\rho_x = 0. \end{aligned} \tag{2.14}$$

For the choice $\alpha_1 = 0$, $C = 0$ and $u = u_0$ by (2.9). Also one can check that

$$\begin{aligned}&\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right](x,t) = \tfrac{2}{\sqrt{\rho(x,t)}}\tfrac{\partial^2}{\partial x^2}\left(\sqrt{\rho(x,t)}\right) \overset{\text{for } \alpha_1=0}{=} \\ &\tfrac{2}{2a\beta dn(a(x-\beta vt),\kappa)}\tfrac{\partial^2}{\partial x^2}\left(2a\beta dn(a(x-\beta vt),\kappa\right) = \\ &-2a^2\kappa^2\left[-sn^2 + cn^2\right](a(x-\beta vt),\kappa) = a^2\left[-2dn^2(a(x-\beta vt),\kappa) + 2 - \kappa^2\right]2\end{aligned} \tag{2.15}$$

by (2.8), which with (2.10) gives (again for $\alpha_1 = 0$)

$$\left(\frac{\rho}{2} + \frac{\beta^2}{2}\left[\frac{\rho_{xx}}{\rho} - \frac{1}{2}\left(\frac{\rho_x}{\rho}\right)^2\right]\right)(x,t) = \beta^2 a^2\left(2-\kappa^2\right). \tag{2.16}$$

As $S_x = -\frac{u}{2} = -\frac{u_0}{2} = \frac{-v\beta}{2}$

$$\rho_t - 2\left(\rho S_x\right)_x = \rho_t + u_0\rho_x = 0 \tag{2.17}$$

(by (2.14)), which is the first equation in (2.3). By (2.16), the second equation in (2.3) is

$$\begin{aligned}&-S_t + \tfrac{v^2\beta^2}{4} + \beta^2a^2\left(2-\kappa^2\right) = 0 \Rightarrow \\ &S(x,t) = \left[\tfrac{v^2\beta^2}{4} + \beta^2a^2\left(2-\kappa^2\right)\right]t + f(x),\end{aligned} \tag{2.18}$$

for some function of integration $f(x)$.

$$\begin{aligned}&-\tfrac{v\beta}{2} = S_x(x,t) \doteq f'(x) \Rightarrow f(x) = -\tfrac{v\beta}{2}x + b, b \in \mathbb{R}, \\ &\Rightarrow S(x,t) = \left[\tfrac{v^2\beta^2}{4} + \beta^2a^2\left(2-\kappa^2\right)\right]t - \tfrac{v\beta x}{2} + b.\end{aligned} \tag{2.19}$$

Thus, in summary, for the choice $\alpha_1 = 0$ in (2.9), so that $C = 0$ and $u(x,t) = u_0 = v\beta$ (by (2.11)), $S(x,t)$ given by (2.19), for $b \in \mathbb{R}$, which satisfies $S_x = -u/2$, is a solution of the equations of motion given by the system (2.3) for the Gurevich-Krylov (G-K) solution $\rho(x,t)$ given in (2.9), or in (2.10), of the MAS (2.5).

Given a triple of functions (S, ρ, u), we set up the Lagrangian density $\mathscr{L}$ in (2.1) for the pair (S, ρ) and we differentiated with respect to x the second corresponding equation of motion in (2.3) to derive the MAS of (2.5) for the pair (ρ, u) assuming that S was a velocity potential for u - namely that $S_x = -u/2$. Conversely, given a pair (ρ, u) that solves the system (2.5), there always exists a velocity potential S for u that solves the system (2.3). To see this, start with any velocity potential S^0 of u whatsoever: $S^0_x = -u/2$. Then, again, $S^0_{tx} = S^0_{xt} = -u_t/2$ and $\left(\left(S^0_x\right)^2\right)_x = uu_x/2$ (as in (2.4)) so that for

$$F \overset{\text{def.}}{=} -2S^0_t + 2\left(S^0_x\right)^2 + \rho + \beta^2\left[\frac{\rho_{xx}}{\rho} - \frac{1}{2}\left(\frac{\rho_x}{\rho}\right)^2\right], \tag{2.20}$$

we get by (2.5) that

$$F_x = -2\left(-u_t/2\right) + \rho_x + \beta^2\left[\frac{\rho_{xx}}{\rho} - \frac{1}{2}\left(\frac{\rho_x}{\rho}\right)^2\right]_x = 0, \tag{2.21}$$

or that $(F/2)_x = 0$, which says that

$$F(x,t)/2 = \phi(t) \tag{2.22}$$

for some function $\phi(t)$ of integration. Now choose any function $h(t)$ such that $h'(t) = \phi(t)$. Then

$$\begin{aligned}
&S(x,t) \overset{\text{def.}}{=} S^0(x,t) + h(t) \Rightarrow \\
&S_x = S^0_x \overset{\text{def.}}{=} -u/2,\ S_t = S^0_t + h' \overset{\text{def.}}{=} S^0_t + \phi \overset{\text{def.}}{=} S^0_t + F/2 \\
&\Rightarrow -S_t + (S_x)^2 + \tfrac{\rho}{2} + \tfrac{\beta^2}{2}\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right] = \\
&-S^0_t - \tfrac{F}{2} + \left(S^0_x\right)^2 + \tfrac{\rho}{2} + \tfrac{\beta^2}{2}\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right] = 0
\end{aligned} \tag{2.23}$$

by the definition of F in (2.20). That is, S defined in (2.23) solves the second equation in (2.3), S is a velocity potential for u ($S_x = -u/2$ by (2.23)), and S solves the first equation in (2.3) since by (2.5), $0 = \rho_t + (\rho u)_x = \rho_t + \rho(-2S_x)_x$. Thus the converse assertion regarding the existence of S is established. For the G-K solution in (2.10), we know that for $\alpha_1 = 0$, S is given by (2.19). Note that the first equation in (2.3) can be written as

$$0 = \frac{\rho_t - 2\left(\rho S_{xx} + \rho_x S_x\right)}{2\rho} = \frac{\rho_t}{2\rho} - S_{xx} - \frac{\rho_x S_x}{\rho}. \tag{2.24}$$

3. The MAS $\longleftrightarrow$ RNLS equation correspondence

The purpose of this section is to set up an explicit correspondence between solutions $(u, \rho > 0)$ of the magnetoacoustic system (MAS) in (2.5) and certain solutions ψ of the *resonant nonlinear Schrödinger* (RNLS) *equation*

$$\begin{aligned}
&i\psi_t + \psi_{xx} + \gamma|\psi|^2\psi = \delta\tfrac{|\psi|_{xx}\psi}{|\psi|}, \\
&\gamma, \delta \in \mathbb{R},
\end{aligned} \tag{3.1}$$

with de Broglie quantum potential $|\psi|_{xx}/|\psi|$. The choice for δ will be $\delta \overset{\text{def.}}{=} 1 + \beta^2$, for β in (2.5). Besides its occurrence in plasma physics, the RNLS equation occurs in quite many other studies as well. It occurs in the study of nonlinear fiber optics, for example. It's stability and dynamic properties, which are not discussed here, are considered in [12], for example. The Galilean invariance established for the MAS (2.5) is considered for equation (3.1).

Given a pair of real-valued functions $(S, \rho > 0)$ of (x, t) and $c > 0$, $c \in \mathbb{R}$, set

$$\psi = \sqrt{c\rho}\, e^{-iS} \tag{3.2}$$

for $i^2 = -1$. Then

$$\begin{aligned}
&\psi_t = \psi\left(-iS_t + \tfrac{\rho_t}{2\rho}\right),\ \psi_x = \psi\left(-iS_x + \tfrac{\rho_x}{2\rho}\right), \\
&\psi_{xx} = \psi\left(\tfrac{1}{2}\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right] - (S_x)^2 + i\left[-S_{xx} - \tfrac{S_x\rho_x}{\rho}\right]\right) \Rightarrow \\
&i\psi_t + \psi_{xx} = \\
&\psi\left(\tfrac{1}{2}\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right] - (S_x)^2 + S_t + i\left[-S_{xx} - \tfrac{S_x\rho_x}{\rho} + \tfrac{\rho_t}{2\rho}\right]\right).
\end{aligned} \tag{3.3}$$

Also (as noted in (2.15))

$$
\begin{aligned}
&\tfrac{2}{\sqrt{\rho}}\tfrac{\partial^2}{\partial x^2}(\sqrt{\rho}) = \tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2, \ |\psi|^2 = c\rho \Rightarrow \\
&\tfrac{\rho}{2} = \tfrac{|\psi|^2}{2c}; \ \tfrac{1}{\sqrt{\rho}}\tfrac{\partial^2}{\partial x^2}(\sqrt{\rho}) = \tfrac{\sqrt{c}}{|\psi|}\tfrac{\partial^2}{\partial x^2}\left(\tfrac{|\psi|}{\sqrt{c}}\right) = \tfrac{|\psi|_{xx}}{|\psi|} \Rightarrow \\
&\tfrac{|\psi|_{xx}}{|\psi|} = \tfrac{1}{2}\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right] \Rightarrow i\psi_t + \psi_{xx} + \tfrac{|\psi|^2\psi}{-2c} = \\
&\psi\left(\tfrac{|\psi|_{xx}}{|\psi|} + S_t - (S_x)^2 - \tfrac{\rho}{2} + i\left[-S_{xx} - \tfrac{S_x\rho_x}{\rho} + \tfrac{\rho_t}{2\rho}\right]\right)
\end{aligned}
\tag{3.4}
$$

by (3.3).

Now suppose $(u, \rho > 0)$ solves the MAS (2.5). Then from section 2 we know that S with $S_x = -u/2$ can be chosen to solve the system (2.3); see (2.23). The first equation in (2.3) is the equation

$$
\frac{\rho_t}{2\rho} - S_{xx} - \frac{\rho_x S_x}{\rho} = 0 \tag{3.5}
$$

by (2.24). Then by (3.4) and the second equation in (2.3)

$$
\begin{aligned}
&i\psi_t + \psi_{xx} + \tfrac{|\psi|^2\psi}{-2c} = \psi\left(\tfrac{|\psi|_{xx}}{|\psi|} + S_t - (S_x)^2 - \tfrac{\rho}{2}\right) = \\
&\psi\left(\tfrac{|\psi|_{xx}}{|\psi|} + \tfrac{\beta^2}{2}\left[\tfrac{\rho_{xx}}{\rho} - \tfrac{1}{2}\left(\tfrac{\rho_x}{\rho}\right)^2\right]\right) = \\
&\psi\left(\tfrac{|\psi|_{xx}}{|\psi|} + \beta^2\tfrac{|\psi|_{xx}}{|\psi|}\right) = \psi\left(1 + \beta^2\right)\tfrac{|\psi|_{xx}}{|\psi|},
\end{aligned}
\tag{3.6}
$$

which is the RNLS equation (3.1) for $\delta = 1 + \beta^2 > 1$ and $\gamma = -1/2c < 0$,

Conversely, suppose we are given a solution ψ of the RNLS equation

$$
\begin{aligned}
&i\psi_t + \psi_{xx} + \tfrac{|\psi|^2\psi}{-2c} = \left(1 + \beta^2\right)\tfrac{|\psi|_{xx}\psi}{|\psi|}, \\
&c \in \mathbb{R}, \ c > 0,
\end{aligned}
\tag{3.7}
$$

where ψ is of the form

$$
\psi = e^{R - iS} \tag{3.8}
$$

for real-valued functions R, S of (x, t). Define

$$
u = -2S_x, \ \rho = \frac{e^{2R}}{c} > 0. \tag{3.9}
$$

Then we claim that the pair (u, ρ) solves the MAS (2.5). $\sqrt{c\rho} = e^R \Rightarrow \psi = \sqrt{c\rho}e^{-iS}$, which means that the formulas in (3.4) apply. By (3.4) and (3.7)

$$
\begin{aligned}
&\psi\left(\tfrac{|\psi|_{xx}}{|\psi|} + S_t - (S_x)^2 - \tfrac{\rho}{2} + i\left[-S_{xx} - \tfrac{S_x\rho_x}{\rho} + \tfrac{\rho_t}{2\rho}\right]\right) = \\
&i\psi_t + \psi_{xx} + \tfrac{|\psi|^2\psi}{-2c} \overset{\therefore}{=} \psi\left(1 + \beta^2\right)\tfrac{|\psi|_{xx}}{|\psi|} \Rightarrow \\
&\tfrac{|\psi|_{xx}}{|\psi|} + S_t - (S_x)^2 - \tfrac{\rho}{2} + i\left[-S_{xx} - \tfrac{S_x\rho_x}{\rho} + \tfrac{\rho_t}{2\rho}\right] = \\
&\left(1 + \beta^2\right)\tfrac{|\psi|_{xx}}{|\psi|} = \tfrac{|\psi|_{xx}}{|\psi|} + \beta^2\tfrac{|\psi|_{xx}}{|\psi|} \Rightarrow \\
&S_t - (S_x)^2 - \tfrac{\rho}{2} = \tfrac{\beta^2|\psi|_{xx}}{|\psi|}, \ -S_{xx} - \tfrac{S_x\rho_x}{\rho} + \tfrac{\rho_t}{2\rho} = 0,
\end{aligned}
\tag{3.10}
$$

where we equate real and imaginary parts. Again by (2.24), the last equation in (3.10) is the first equation in (2.3). Also by (3.4) again, the next to the last equation in (3.10) is the equation

$$S_t - (S_x)^2 - \frac{\rho}{2} = \frac{\beta^2}{2}\left[\frac{\rho_{xx}}{\rho} - \frac{1}{2}\left(\frac{\rho_x}{2}\right)^2\right]. \tag{3.11}$$

In other words, the last two equations in (3.10) are precisely the two equations in the system (2.3) - which are the equations of motion for the Lagrangian density $\mathscr{L}$ in (2.1). Moreover, since $S_x = -u/2$ (by definition (3.9)), we have already shown that differentiation of equation (3.11) with respect to x leads exactly to the second equation in (2.5). Thus, conversely, (u, ρ) solves (2.5).

In summary, the following has been established. Given a solution$(u, \rho > 0)$ of the MAS (2.5), we can always choose a velocity potential S for u (ie. $S_x = -u/2$) such that the pair (S, ρ) solves the system (2.3). Given $\gamma \in \mathbb{R}$, $\gamma < 0$, set

$$\psi \overset{\text{def.}}{=} \sqrt{c\rho}e^{-iS},\ c = \frac{1}{-2\gamma} > 0. \tag{3.12}$$

Then ψ is a solution of the RNLS equation (3.1) for $\delta \overset{\text{def.}}{=} 1+\beta^2 > 1$. Conversely, suppose for real-valued functions R, S that

$$\psi \overset{\text{def.}}{=} e^{R-iS} \tag{3.13}$$

is a solution of equation (3.1), with $\delta = 1+\beta^2$, $\gamma < 0$. Define

$$u \overset{\text{def.}}{=} -2S_x, \quad \rho \overset{\text{def.}}{=} \frac{e^{2R}}{c} > 0, \quad c = \frac{1}{-2\gamma} > 0. \tag{3.14}$$

$\psi = \sqrt{c\rho}\, e^{-iS}$, the pair (S, ρ) solves the system (2.3) and the pair (u, ρ) solves the MAS (2.5).

For the Gurevich-Krylov (G-K) solution (2.10) with the choice $\alpha_1 = 0$, we know that S is given by (2.19). Therefore by (3.12) one obtains the solution

$$\psi(x,t) = \frac{2a\beta}{\sqrt{-2\gamma}}\,\text{dn}(a(x-\beta vt), k)e^{-i\left(\beta^2\left[\frac{v^2}{4}+a^2\left(2-\kappa^2\right)\right]t-\frac{v\beta x}{2}+b\right)} \tag{3.15}$$

of (3.1), again for $\delta = 1+\beta^2 > 1$, $b \in \mathbb{R}$, $\alpha_1 = 0$, $b \in \mathbb{R}$. In particular for the choice $\kappa = 1$, this solution reduces to the 1-soliton solution

$$\psi(x,t) = \frac{2a\beta}{\sqrt{-2\gamma}}\ \text{sech}(a(x-\beta vt))e^{-i\left(\beta^2\left[\frac{v^2}{4}+a^2\right]t-\frac{v\beta x}{2}+b\right)} \tag{3.16}$$

of (3.1), by (2.8).

The resonant nonlinear Schrödinger equation (3.1) is the equation of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\overline{\psi}_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\mathscr{L}}{\partial\overline{\psi}_x}\right) = \frac{\partial\mathscr{L}}{\partial\overline{\psi}} \tag{3.17}$$

for the Lagrangian density $\mathscr{L}$ given by

$$\mathscr{L} \overset{\text{def.}}{=} \frac{i}{2}\left(\overline{\psi}\psi_t - \overline{\psi}_t\psi\right) - \overline{\psi}_x\psi_x + \delta\left(|\psi|_x\right)^2 + \frac{\gamma}{2}|\psi|^4. \tag{3.18}$$

The main point here is the key formula

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial \psi_x}\left(|\psi|_x\right)^2\right] = \frac{|\psi|_{xx}\overline{\psi}}{|\psi|} + \frac{\partial}{\partial \psi}\left(|\psi|_x\right)^2 . \tag{3.19}$$

Namely, one has that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \overline{\psi}_t}\right) = \frac{\partial}{\partial t}\left(-\frac{i\psi}{2}\right) = -\frac{i}{2}\psi_t,\ \frac{\partial \mathscr{L}}{\partial \overline{\psi}_x} = -\psi_x + \delta\frac{\partial}{\partial \overline{\psi}_x}\left(|\psi|_x\right)^2 \Rightarrow \\
&\frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial \overline{\psi}_x}\right) = -\psi_{xx} + \delta\frac{\partial}{\partial x}\left[\frac{\partial}{\partial \overline{\psi}_x}\left(|\psi|_x\right)^2\right] = -\psi_{xx} + \delta\frac{|\psi|_{xx}\psi}{|\psi|} + \delta\frac{\partial}{\partial \overline{\psi}}\left(|\psi|_x\right)^2 ,
\end{aligned} \tag{3.20}$$

by conjugation of the key formula (3.19), Also

$$\frac{\partial \mathscr{L}}{\partial \overline{\psi}} = \frac{i}{2}\psi_t + \delta\frac{\partial}{\partial \overline{\psi}}\left(|\psi|_x\right)^2 + \gamma\overline{\psi}\psi^2 \tag{3.21}$$

since $\frac{\gamma}{2}|\psi|^4 = \frac{\gamma}{2}\bar{\psi}^2\psi^2$. Thus equation (3.17) is the equation

$$-\frac{i}{2}\psi_t - \psi_{xx} + \delta\frac{|\psi|_{xx}\psi}{|\psi|} + \delta\frac{\partial}{\partial \overline{\psi}}\left(|\psi|_x\right)^2 = \frac{i}{2}\psi_t + \delta\frac{\partial}{\partial \overline{\psi}}\left(|\psi|_x\right)^2 + \gamma\overline{\psi}\psi\psi. \tag{3.22}$$

That is,

$$\delta\frac{|\psi|_{xx}\psi}{|\psi|} = i\psi_t + \psi_{xx} + \gamma|\psi|^2\psi \tag{3.23}$$

which is exactly the RNLS equation (3.1).

For $c_0 \in \mathbb{R}$ fixed, the Galilean transforms $\hat{u}, \hat{\rho}$ of u, ρ were defined in (2.6). Now $S_x = -u/2$, and we can define the Galilean transform $\hat{S}$ of S by

$$\hat{S}(x,t) \overset{\text{def.}}{=} -\frac{c_0}{2}x + \frac{c_0^2}{4}t + S\left(x - c_0 t, t\right), \tag{3.24}$$

since then

$$\hat{S}(x,t) = -\frac{c_0}{2} + S_x\left(x - c_0 t, t\right) = -\frac{c_0}{2} - \frac{u}{2}\left(x - c_0 t, t\right) = -\frac{\hat{u}(x,t)}{2}, \tag{3.25}$$

by (2.6). For $\psi = e^{R-iS}$, with R, S = real functions, we saw (following (3.9)) that $\psi = \sqrt{\rho}e^{-iS}$ for $\rho = e^{-2R}$. This suggests that we set

$$\hat{\psi}(x,t) = \sqrt{\hat{\rho}(x,t)}e^{-i\hat{S}(x,t)}; \tag{3.26}$$

that is, by (2.6), (3.24),

$$\hat{\psi}(x,t) = \sqrt{\rho\left(x - c_0 t, t\right)}e^{-i[-c_0 x/2 + c_0^2 t/4 + S(x - c_0 t, t)]} = e^{i[c_0 x/2 - c_0^2 t/4]}\psi\left(x - c_0 t, t\right). \tag{3.27}$$

The point is that indeed for any solution ψ of the RNLS equation (3.1), with ψ of the form e^{R-iS} for real functions R, S, its Galilean transform $\hat{\psi}$ defined by the last equation in (3.27) will also be a solution of equation (3.1). Throughout, one may regard c_0 as a velocity parameter.

As a simple example, for $\rho_0, \gamma \in \mathbb{R}, \rho_0 > 0$

$$\psi_0(x,t) \overset{\text{def.}}{=} \sqrt{\rho_0} e^{i\gamma\rho_0 t} \tag{3.28}$$

is a solution of (3.1) - a ground state on condensate solution. It's Galilean transform

$$\hat{\psi}_0(x,t) = \sqrt{\rho_0} e^{i\left[\frac{c_0}{2}x + \left(\gamma\rho_0 - c_0^2/4\right)t\right]}, \ c_0 \in \mathbb{R} \tag{3.29}$$

therefore is also a solution.

We have connected equations (2.5) and (3.1) by the choice $\delta = 1 + \beta^2 > 1$. For $\delta < 1$, there are also important solutions of (3.1) of independent interest. For example, in [12] for $\delta < 1$ and $\mu < 0$ the dark (or topological) soliton solution

$$\psi(x,t) \overset{\text{def.}}{=} e^{i\mu t} \sqrt{-\mu} \ \tanh \sqrt{\frac{-\mu}{2(1-\delta)}} x \tag{3.30}$$

is considered, in case $\gamma = -1$ in (3.1)

4. The continuous Heisenberg model and a cold plasma metric

The ultimate goal of our discussion is to establish a connection between the MAS (2.5) and a J-T black hole. For this, an initial key point is the assignment of a suitable Riemannian metric g_{plasma} to the system (2.5). Fortunately, this can be done by way of the *classical continuous* (hyperbolic) *Heisenberg model*, to which we turn our attention. It is the genius and beauty of mathematics that often enough there exist startling, connective threads between seemingly disparate and unrelated topics or ideas. This happens in the present case here.

$< , >$ will denote the Minkowski inner product on $\mathbb{R}^3$ given by

$$< X, Y > \overset{\text{def.}}{=} -x_1 y_1 + x_2 y_2 - x_3 y_3 \tag{4.1}$$

for $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The Heisenberg model of interest is given by real-valued functions $S_1(x,t), S_2(x,t), S_3(x,t)$ for which

$$H : (x,t) \to H(x,t) \overset{\text{def.}}{=} (S_1(x,t), \ S_2(x,t), \ S_3(x,t)) \in \mathbb{R}^3 \tag{4.2}$$

satisfies certain equations of motion(see (4.6)), and

$$\begin{gathered} < H(x,t), \ H(x,t) >= -1 : \\ -S_1^2(x,t) + S_2^2(x,t) - S_3^2(x,t) = -1, \end{gathered} \tag{4.3}$$

by (4,1). Thus the points $H(x,t)$ lie on a single-sheeted hyperboloid. The function $H : \mathbb{R}^2 \to \mathbb{R}^3$ provides for a natural induced metric (or fundamental form) g_H on the model:

$$g_H = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \overset{\text{def.}}{=} \begin{bmatrix} \langle H_x, H_x \rangle & \langle H_x, H_t \rangle \\ \langle H_x, H_t \rangle & \langle H_t, H_t \rangle \end{bmatrix}. \tag{4.4}$$

In other words, by (4.1), (4.2)

$$\begin{aligned} g_{11} &= -\left(\frac{\partial S_1}{\partial x}\right)^2 + \left(\frac{\partial S_2}{\partial x}\right)^2 - \left(\frac{\partial S_3}{\partial x}\right)^2 \\ g_{21} = g_{12} &= -\frac{\partial S_1}{\partial x}\frac{\partial S_1}{\partial t} + \frac{\partial S_2}{\partial x}\frac{\partial S_2}{\partial t} - \frac{\partial S_3}{\partial x}\frac{\partial S_3}{\partial t} \\ g_{22} &= -\left(\frac{\partial S_1}{\partial t}\right)^2 + \left(\frac{\partial S_2}{\partial t}\right)^2 - \left(\frac{\partial S_3}{\partial t}\right)^2. \end{aligned} \tag{4.5}$$

It was mentioned that certain equations of motion are satisfied by the model. They are

$$\begin{aligned} S_t &= \frac{1}{2i}[S, S_{xx}] \overset{\text{def.}}{=} \frac{1}{2i}(SS_{xx} - S_{xx}S) \\ \text{for } S &\overset{\text{def.}}{=} i\begin{bmatrix} S_3 & S_1 - S_2 \\ S_1 + S_2 & -S_3 \end{bmatrix}. \end{aligned} \tag{4.6}$$

One can write

$$\begin{aligned} S_1 &= \frac{S^+ + S^-}{2},\ S_2 = \frac{S^+ - S^-}{2}, \\ \text{for } S^+ &\overset{\text{def.}}{=} S_1 + S_2,\ S^- \overset{\text{def.}}{=} S_1 - S_2. \end{aligned} \tag{4.7}$$

Then the metric g_H and the equations of motion assume the form

$$\begin{aligned} g_{11} &= -\frac{\partial S^+}{\partial x}\frac{\partial S^-}{\partial x} - \left(\frac{\partial S_3}{\partial x}\right)^2 \\ g_{21} = g_{12} &= \frac{-\frac{\partial S^+}{\partial x}\frac{\partial S^-}{\partial t} - \frac{\partial S^+}{\partial t}\frac{\partial S^-}{\partial x}}{2} - \frac{\partial S_3}{\partial x}\frac{\partial S_3}{\partial t} \\ g_{22} &= -\frac{\partial S^+}{\partial t}\frac{\partial S^-}{\partial t} - \left(\frac{\partial S_3}{\partial t}\right)^2, \\ \frac{\partial S_3}{\partial t} &= \frac{1}{2}\left(S^-\frac{\partial^2 S^+}{\partial x^2} - S^+\frac{\partial^2 S^-}{\partial x^2}\right),\quad \frac{\partial S^-}{\partial t} = S_3\frac{\partial^2 S^-}{\partial x^2} - S^-\frac{\partial^2 S_3}{\partial x^2} \\ \frac{\partial S^+}{\partial t} &= S^+\frac{\partial^2 S_3}{\partial x^2} - S_3\frac{\partial^2 S^+}{\partial x^2}. \end{aligned} \tag{4.8}$$

As an example [5], for $a, v \in \mathbb{R}$ with $a \neq 0$ and $4a^2 - v^2 \neq 0$, define

$$\begin{aligned} \phi(x,t) &= \left(a^2 + v^2/4\right)t - vx/2 \\ S^+(x,t) &\overset{\text{def.}}{=} \frac{8a^2}{4a^2 - v^2} e^{\phi(x,t)}(\operatorname{sech} a(x - vt))\left[\tanh a(x - vt) + \frac{v}{2a}\right] \\ S^-(x,t) &\overset{\text{def.}}{=} \frac{8a^2}{4a^2 - v^2} e^{-\phi(x,t)}(\operatorname{sech} a(x - vt))\left[\tanh a(x - vt) - \frac{v}{2a}\right] \\ S_3(x,t) &\overset{\text{def.}}{=} \frac{8a^2}{4a^2 - v^2}\operatorname{sech}^2 a(x - vt) - 1. \end{aligned} \tag{4.9}$$

Then

$$S_3(x,t)^2 + S^+(x,t)S^-(x,t) = 1. \tag{4.10}$$

That is, (4.3) holds since S_1, S_2 satisfy (4.7). Also the equations of motion for $S_3,\ S^+,\ S^-$ in (4.8) hold, which are equivalent to the equation of motion for $S_1,\ S_2,\ S_3$ in (4.6), as has been noted.

Of utmost practical importance is the fact that the metric g_H in (4.4) has *constant* Ricci scalar curvature. It is given by

$$R(g_H) = 2, \tag{4.11}$$

which can be verified by a Maple program(tensor), for example. In particular, for a positive multiple bg_H of g_H, $b > 0$, $b \in \mathbb{R}$, one has that

$$R\left(bg_H\right) = \frac{2}{b}. \tag{4.12}$$

We move now to the main point of this section that connects the MAS (2.5) and the continuous Heisenberg model - a connection explicated by way of the metric g_H in (4.4). Fix $\gamma < 0$, $\gamma \in \mathbb{R}$, γ as in (3.1) for example. Let $(u, \rho > 0)$ be a solution of the system (2.5). Then it is possible to construct a solution (r, s) of the *reaction-diffusion* system

$$r_t - r_{xx} + Br^2 s = 0,\ s_t + s_{xx} - Brs^2 = 0,\ B \overset{\text{def.}}{=} \frac{-\gamma}{\beta^2}. \tag{4.13}$$

Namely, for $S^0(x,t)$ with $S^0_x = -u/2$ (as usual)

$$\begin{aligned} r(x,t) &\overset{\text{def.}}{=} \left[\rho(x,t/\beta)/(-2\gamma)\right]^{\frac{1}{2}} e^{\phi^0(x,t)} \\ s(x,t) &\overset{\text{def.}}{=} -\left[\rho(x,t/\beta)/(-2\gamma)\right]^{\frac{1}{2}} e^{-\phi^0(x,t)} \\ \phi^0(x,t) &\overset{\text{def.}}{=} S^0(x,t/\beta)/\beta. \end{aligned} \tag{4.14}$$

From the solution (r, s) in (4.14), one can construct, moreover, a solution $H(x,t)$ in (4.3) of the equations of motion (4.6) (or (4.8)) that define the continuous Heisenberg model. The construction of $H(x,t)$ is a bit involved. It amounts to the construction of a suitable *Lax pair* to establish in fact that the reaction-diffusion system and Heisenberg model are *gauge equivalent*. Details are presented in [5, 8], for example. In the end, from a solution $(u, \rho > 0)$ of the MAS (2.5) one obtains a metric g_H (see (4.4)) such that the multiple

$$g_{\text{plasma}} \overset{\text{def.}}{=} \frac{g_H}{(-2\gamma/\beta^2)} \tag{4.15}$$

of g_H is given explicitly by the following formulas:

$$\begin{aligned} \left(g_{\text{plasma}}\right)_{11}(x,t) &= \rho(x,t/\beta)/2\gamma \\ \left(g_{\text{plasma}}\right)_{21}(x,t) &= \left(g_{\text{plasma}}\right)_{12}(x,t) = \tfrac{\rho(x,t/\beta)u(x,t/\beta)}{-4\gamma\beta} \\ \left(g_{\text{plasma}}\right)_{22}(x,t) &= \tfrac{\rho(x,t/\beta)}{-8\gamma}\left[\left(\tfrac{1}{\rho}\tfrac{\partial\rho}{\partial x}\right)^2 - \tfrac{u^2}{\beta^2}\right](x,t/\beta) \end{aligned} \tag{4.16}$$

Also by (4.12), g_{plasma} has *constant* Ricci scalar curvature given by

$$R\left(g_{\text{plasma}}\right) = -\frac{4\gamma}{\beta^2} > 0. \tag{4.17}$$

In particular, the formulas in (4.16) apply to the Gurevich-Krylov solution in (2.10), where $(g_{\text{plasma}})_{22}$ is a little messy to compute. In the end, one obtains the following formulas, where details appear in section 15 of [8]:

$$\begin{aligned} \left(g_{\text{plasma}}\right)_{11}(x,t) &= \tfrac{\alpha_1 + 4a^2\beta^2 dn^2(a(x-vt),\kappa)}{2\gamma} \\ \left(g_{plasma}\right)_{21}(x,t) &= \left(g_{plasma}\right)_{12}(x,t) = -\tfrac{v}{\gamma}a^2\beta^2 dn^2(a(x-vt),\kappa) - \tfrac{v\alpha_1}{4\gamma} - \tfrac{C}{4\gamma\beta} \\ \left(g_{\text{plasma}}\right)_{22}(x,t) &= 4a^2\beta^2\left[\tfrac{a^2\kappa^4}{-2\gamma}(sncn)^2(a(x-vt),\kappa) + \tfrac{v^2}{8\gamma}dn^2(a(x-vt),\kappa)\right] \\ &\quad + \frac{16\alpha_1 a^4\beta^2\kappa^4(sncn)^2(a(x-vt),\kappa) + \frac{C^2}{\beta^2}}{8\gamma\left[\alpha_1 + 4a^2\beta^2 dn^2(a(x-vt),\kappa)\right]} + \tfrac{v^2\alpha_1}{8\gamma} + \tfrac{vC}{4\gamma\beta}. \end{aligned} \tag{4.18}$$

Again C is given by (2.9), or by (2.10).

Finally, although (apriori) g_{plasma} is non-diagonal (ie.$\left(g_{\text{plasma}}\right)_{12} \neq 0$) there is a suitable change of variables $(x,t) \to (\tau,\delta)$ [6, 13] by which g_{plasma} assumes the diagonal form

$$g_{\text{plasma}} : ds^2 = A(\delta)d\tau^2 - \frac{4a^4\beta^4\kappa^4(sncndn)^2(\delta,\kappa)}{A(\delta)\gamma^2}d\delta^2, \tag{4.19}$$

where

$$\begin{aligned} A(\delta) =& 4a^2\beta^2\left[\frac{a^2\kappa^4}{-2\gamma}(sncn)^2(\delta,k) + \frac{v^2}{8\gamma}dn^2(\delta,\kappa)\right] + \\ & \frac{16\alpha_1 a^4\beta^2\kappa^4(sncn)^2(\delta,k) + \frac{C^2}{\beta^2}}{8\gamma\left[\alpha_1 + 4a^2\beta^2 dn^2(\delta,\kappa)\right]} + \frac{v^2\alpha_1}{8\gamma} - \frac{vC}{4\gamma\beta}, \end{aligned} \tag{4.20}$$

with conditions of course that $A(\delta)$ never vanishes. This is the case if v^2 is sufficiently large. More precisely, as is shown in [6], if $\alpha_1 \neq 0$, then $A(\delta)$ never vanishes if

$$v^2 > 4a^2\kappa^4 \text{ and } v^2 \geqslant \frac{4C^2}{\alpha_1^2\beta^2}. \tag{4.21}$$

If $\alpha_1 = 0$, then the single condition

$$v^2 > 4a^2\kappa^4 \tag{4.22}$$

suffices for the non-vanishing of $A(\delta)$. We point out that in [6], the notation $A(\rho)$ is used for the $A(\delta)$ here. The g_{11} in [6, 7, 13] corresponds to the notation g_{22} used here and vice versa. In [13], $\alpha_1 = 0$, $\gamma = -1/2$, and b^2 there is $4\beta^2$ here.

As in section 2, the various results assume a much simpler form for the choice $\alpha_1 = 0$, mainly as then $C = 0$ by definition (2.9). For example, (4.20) reduces to

$$A(\delta) = 4a^2\beta^2\left[\frac{a^2\kappa^4}{-2\gamma}(sncn)^2(\delta,\kappa) + \frac{v^2}{8\gamma}dn^2(\delta,\kappa)\right]. \tag{4.23}$$

Moreover, for the choice $\kappa = 1$ and for γ of the form $\gamma = \Lambda/4$ for a suitable cosmological constant Λ, $A(\delta)$ in (4.23) assumes the form

$$A(\delta) = -\frac{8a^2\beta^2}{\Lambda}\left(\operatorname{sech}^2\delta\right)\left[a^2\tanh^2\delta - \frac{v^2}{4}\right], \tag{4.24}$$

(by (2.8)), and consequently (4.19) reduces to

$$\begin{aligned} g_{\text{plasma}} : ds^2 =& -\frac{8a^2\beta^2}{\Lambda}\left(\operatorname{sech}^2\delta\right)\left[a^2\tanh^2\delta - \frac{v^2}{4}\right]d\tau^2 \\ & + \frac{8a^2\beta^2\tanh^2\delta\operatorname{sech}^2\delta}{\Lambda\left[a^2\tanh^2\delta - \frac{v^2}{4}\right]}d\delta^2 = \\ & -\frac{8a^2\beta^2}{\Lambda}\left(sech^2\delta\right)\left[\left(a^2\tanh^2\delta - \frac{v^2}{4}\right)d\tau^2 - \frac{\tanh^2\delta}{\left(a^2\tanh^2\delta - \frac{v^2}{4}\right)}d\delta^2\right], \end{aligned} \tag{4.25}$$

which, up to the factor β^2, is exactly the black hole metric ds^2 in equation (3.14) of [9], with a horizon singularity at

$$a^2 \tanh^2 \delta - \frac{v^2}{4} = 0 : \ \tanh \delta = \pm \frac{v}{2a}. \tag{4.26}$$

Note that by formula (4.24), $A(\delta) \neq 0$ indeed for $v^2 > 4a^2$, as asserted in (4.22).

Given a solution ψ of equation (3.1) of the form (3.13) (with $\delta = 1 + \beta^2$, $\gamma < 0$ in (3.1)), we saw that the prescription

$$u = -2S_x, \ \rho = \frac{e^{2R}}{c}, \ c = \frac{1}{-2\gamma} > 0 \tag{4.27}$$

in (3.14) provided for solutions $(S, \rho > 0)$, $(u, \rho > 0)$ of the systems (2.3), (2.5), respectively. By the equations in (4.16) we can express the components g_{ij} of the plasma metric $g = g_{\text{plasma}}$ in terms of ψ. We proceed as follows. Since $|\psi|^2 = c\rho$, the 1st formula in (4.16) gives

$$g_{11}(x;t) = -c\rho\,(x, t|\beta) = -\,|\psi(x, t/\beta)|^2\,. \tag{4.28}$$

Next by (3.3) and (4.27)

$$\begin{aligned} &\psi_x = \psi\left(-iS_x + \frac{\rho_x}{2\rho}\right) \Rightarrow (\bar{\psi})_x = \bar{\psi}\left(iS_x + \frac{\rho_x}{2\rho}\right) \Rightarrow \\ &\psi_x(\bar{\psi})_x = \psi\bar{\psi}\left[(S_x)^2 + \frac{1}{4}\left(\frac{\rho_x}{\rho}\right)^2\right] = c\rho\left[\frac{u^2}{4} + \frac{1}{4}\left(\frac{\rho_x}{\rho}\right)^2\right] \\ &= \frac{c\rho}{4}\left[u^2 + \left(\frac{\rho_x}{\rho}\right)^2\right], \ \psi(\bar{\psi})_x - \bar{\psi}\psi_x = |\psi|^2(2iS_x) = -ic\rho u. \end{aligned} \tag{4.29}$$

By the last equation here and the 2nd equation in (4.16)

$$\begin{aligned} g_{21}(x,t) = g_{12}(x,t) = &\frac{(\rho u)(x, t|\beta)}{-4\gamma\beta} = \frac{c}{2\beta}(\rho u)(x, t/\beta) = \\ &\frac{i}{2\beta}\left[\psi(\bar{\psi})_x - \bar{\psi}\psi_x\right](x, t/\beta). \end{aligned} \tag{4.30}$$

Finally, again as $|\psi|^2 = c\rho$, we see that $c\rho_x = 2|\psi||\psi|_x \Rightarrow c^2\,(\rho_x)^2 = 4c\rho(|\psi|_x)^2 \Rightarrow$

$$\frac{4}{c}(|\psi|_x)^2 = \frac{(\rho_x)^2}{\rho}, \tag{4.31}$$

and by the next to last equation in (4.29)

$$\begin{aligned} &\left(1 + \frac{1}{\beta^2}\right)\frac{4}{c}(|\,\psi|_x)^2 - \frac{4}{c\beta^2}\psi_x(\bar{\psi})_x = \left(1 + \frac{1}{\beta^2}\right)\frac{(\rho_x)^2}{\rho} \\ &- \frac{4}{c\beta^2}\frac{c\rho}{4}\left[u^2 + \left(\frac{\rho_x}{\rho}\right)^2\right] = \frac{(\rho_x)^2}{\rho} + \frac{1}{\beta^2}\frac{(\rho_x)^2}{\rho} - \frac{\rho u^2}{\beta^2} - \frac{(\rho_x)^2}{\beta^2\rho} \\ &= \frac{(\rho_x)^2}{\rho} - \rho\frac{u^2}{\beta^2} = \rho\left[\left(\frac{\rho_x}{\rho}\right)^2 - \frac{u^2}{\beta^2}\right] \Rightarrow \frac{c}{4}\rho\left[\left(\frac{\rho_x}{\rho}\right)^2 - \frac{u^2}{\beta^2}\right] \\ &= \left(1 + \frac{1}{\beta^2}\right)(|\psi|_x)^2 - \frac{1}{\beta^2}\psi_x(\bar{\psi})_x, \end{aligned} \tag{4.32}$$

where $c/4 = 1/-8\gamma$. In other words, by the last equation in (4.16)

$$g_{22}(x,t) = \left[\left(1+\frac{1}{\beta^2}\right)(|\psi|_x)^2 - \frac{1}{\beta^2}\psi_x(\bar{\psi})_x\right](x,t|\beta). \tag{4.33}$$

Formulas (4.28), (4.30) and (4.33) provide for the expression of the cold plasma metric components $g_{ij} = \left(g_{\text{plasma}}\right)_{ij}$ in (4.16) in terms of the solution ψ in (3.8) of the RNLS equation (3.1), for $\delta = 1+\beta^2$, $\gamma < 0$. Here for $(u, \rho > 0)$ in (4.16), we have the MAS $\leftrightarrow$ RNLS equation correspondence $(u, \rho > 0) \leftrightarrow \psi$ discussed in section 3.

5. A mapping of the cold plasma metric to a J-T black hole metric

Formula (4.25) provides for a realization of the cold plasma metric g_{plasma} as a black hole metric in case of the special choices $\alpha_1 = 0$ and $\kappa = 1$ in (2.9). For $\alpha_1 \geqslant 0$ arbitrary and for an arbitrary elliptic modulus κ, where g_{plasma} assumes the general form given in (4.19), in the variables τ, δ with $A(\delta)$ given by (4.20), we can in fact map g_{plasma}, more specifically, to a *Jackiw-Teitelboim* (J-T) black hole metric g_{bh} given by

$$g_{\text{bh}} = -\left(m^2r^2 - M\right)d\tau^2 + \frac{dr^2}{(m^2r^2 - M)} \tag{5.1}$$

in the variables τ, r. Here an explicit transformation of variables $(\tau,\delta) \to (\tau, r)$ is presented by which g_{plasma} in (4.19) is indeed mapped to g_{bh} in (5.1), which is the main result of this section, where the black hole parameters m, M are expressed in terms of the magnetoacoustic parameters in the G-K solution (2.9) (or (2.10)).

We begin first with some brief, contextual remarks regarding the J-T model of 2d gravity. Any metric g whatsoever on a two-dimensional space-time M^2 *automatically* solves the Einstein gravitational vacuum field equations with a zero matter tensor [14]. Given this well known fact, R. Jackiw and C. Teitlboim set out to construct a non-trivial theory of gravity for M^2 that involved in addition to g a scalar field Φ on M^2 [15, 16]. Φ is called a *dilaton* field. The action integral for the theory is given, up to some constant, by

$$S(g,\Phi) = \int_{M^2}\left(R(g) - \frac{2}{l^2}\right)\Phi\sqrt{|\det g|}d^2x, \tag{5.2}$$

where (as in section 4) $R(g)$ denotes the Ricci scalar curvature of g, and where $\Lambda = -1/l^2$ is a (negative) cosmological constant in the theory. The corresponding equations of motion for the pair (g,Φ) are given by

$$R(g) = \frac{2}{l^2} = -2\Lambda, \quad \nabla_i\nabla_j\Phi = \frac{g_{ij}\Phi}{l^2}, \quad 1 \le i, j \le 2, \tag{5.3}$$

where for local coordinates (x_1, x_2) on M^2, and for the Christoffel symbols (of the second kind) Γ^k_{ij} for g, the Hessian in (5.3) is given by

$$\nabla_i\nabla_j\Phi = \frac{\partial^2\Phi}{\partial x_i \partial x_j} - \sum_{k=1}^{2}\Gamma^k_{ij}\frac{\partial\Phi}{\partial x_k}. \tag{5.4}$$

Thus g has *constant* Ricci scalar curative, as does g_{plasma} in (4.17). We note that here, and throughout, our sign convention for scalar curvature is the negative of that employed by J-T in [15, 16], and possibly by others in the literature. A key solution of the theory, for example, is of course the J-T black hole solution g_{bh} given in the Lorentzian form (5.1) with coordinates $(x_1, x_2) = (\tau, r)$, where $m = 1/l$:

$$R(g) = 2m^2, \Phi(\tau, r) \stackrel{\text{def.}}{=} mr, \quad \Lambda = -m^2, \tag{5.5}$$

with M being a black hole mass parameter.

Returning to the remarks that followed equation (5.1), we now state the main result. Namely, the transformation of variables $(\tau, \delta) \to (\tau, r)$ by which the cold plasma metric g_{plasma} in (4.19) is mapped to the J-T black hole metric in (5.1) is given by

$$r = \Psi(\delta) \stackrel{\text{def.}}{=} \frac{a^2\beta^2 dn^2(\delta, \kappa)}{-\gamma} + \frac{a^2\beta^2\left(2-\kappa^2\right)}{2\gamma} - \frac{\left(\nu^2\beta^2 - \alpha_1\right)}{8\gamma}, \tag{5.6}$$

where m and M in (5.1) are given by

$$\begin{aligned} m &= +\tfrac{(-2\gamma)^{1/2}}{\beta} \\ M &= \tfrac{\beta^2}{-2\gamma}\left[\tfrac{\nu C}{2\beta^3} + A - \alpha_1\left(\tfrac{3\nu^2}{8\beta^2} + \tfrac{3\alpha_1}{16\beta^4} + \tfrac{a^2(2-\kappa^2)}{2\beta^2}\right)\right], \\ \text{for } A &\stackrel{\text{def.}}{=} -\tfrac{a^2\nu^2}{2}(2-\kappa^2) + a^4\kappa^4 + \tfrac{\nu^4}{16}. \end{aligned} \tag{5.7}$$

We keep the assumptions in (4.21) in order to have $A(\delta) \neq 0 \forall \delta$. Again see (2.9), (2.10), (2.11) for the notation in (5.7). Also see (4.15) where we chose $\gamma \in \mathbb{R}$, $\gamma < 0$ arbitrary. The result (5.6), (5.7) are proved in [6], but the more compact version of the black hole mass M in (5.7) is noted in [7]. In [6] it is shown that indeed $M > 0$ for

$$\nu^2 > 8a^2\left(2-\kappa^2\right) + \frac{6\alpha_1}{\beta^2}, \quad \frac{\nu C}{4\beta} - \frac{3\alpha_1^2}{32\beta^2} - \frac{\alpha_1 a^2}{2} \geqslant 0. \tag{5.8}$$

For $\alpha_1 = 0$, $C = 0$ (again by definition (2.9)) and so the second inequality in (5.8) is automatic. For $\alpha_1 > 0$ it is the statement

$$\nu \geqslant \frac{\alpha_1}{C}\left[\frac{3\alpha_1}{8\beta} + 2a^2\beta\right], \quad \alpha_1 \neq 0. \tag{5.9}$$

For $\alpha_1 = 0$, (5.7) and (5.8) reduce to

$$M = \frac{\beta^2}{-2\gamma}A, \quad \nu^2 > 8a^2(2-\kappa^2). \tag{5.10}$$

Now $2\kappa^2 + \kappa^4 \leqslant 2 + 1 < 4$(as $\kappa \leqslant 1$) $\Rightarrow \kappa^4 < 4 - 2\kappa^2 = 2\left(2-\kappa^2\right) \Rightarrow 4a^2\kappa^4 < 8a^2\left(2-\kappa^2\right) \Rightarrow \nu^2 - 4a^2\kappa^4 > \nu^2 - 8a^2\left(2-\kappa^2\right)$. That is, for $\alpha_1 = 0$ the inequality in (5.10) that implies that $M > 0$ also implies that $\nu^2 > 4a^2\kappa^4$, which is the condition in (4.22) for the non-vanishing of $A(\delta)$ in (4.23).

Since the choice $\alpha_1 = 0$ is so practical and important as well, we provide a *direct* argument that $A > 0$ if $\nu^2 > 8a^2\left(2-\kappa^2\right)$ (as in (5.8) or (5.10)), and hence also $M > 0$ by (5.10), again as $\gamma < 0$. Note that by definition (5.7) we can write

$$\begin{aligned} A &= -a^2\nu^2 + \frac{a^2\kappa^2\nu^2}{2} + a^4\kappa^4 + \frac{\nu^4}{16} \\ &= \frac{2\gamma}{\beta^2}\left[-\frac{a^2\beta^2\nu^2}{2\gamma} + \frac{a^2\kappa^2\nu^2\beta^2}{4\gamma} + \frac{a^4\kappa^4\beta^2}{2\gamma} + \frac{\nu^4\beta^2}{32\gamma}\right]. \end{aligned} \tag{5.11}$$

Now if $v^2 > 8a^2\left(2-\kappa^2\right)$ then for $\gamma < 0$

$$\frac{v^4\beta^2}{32\gamma} = \left(\frac{v^2\beta^2}{32\gamma}\right)v^2 < \left(\frac{v^2\beta^2}{32\gamma}\right)8a^2\left(2-\kappa^2\right) = \left(\frac{v^2\beta^2}{32\gamma}\right)\left(16a^2 - 8a^2\kappa^2\right) = \\ \frac{v^2\beta^2a^2}{2\gamma} - \frac{v^2\beta^2a^2\kappa^2}{4\gamma} \Rightarrow \frac{v^4\beta^2}{32\gamma} - \frac{v^2\beta^2a^2}{2\gamma} + \frac{v^2\beta^2a^2\kappa^2}{4\gamma} < 0. \tag{5.12}$$

That is, for the bracket in (5.11), the sum of the 1st, 2nd and 4th terms is < 0. The 3rd term $a^4\kappa^4\beta^2/2\gamma$ in the bracket is also < 0 (since $\gamma < 0$) $\Rightarrow$ the bracket < 0, which multiplied by the negative number $2\gamma/\beta^2$ gives that $A > 0$ if $v^2 > 8a^2\left(2-\kappa^2\right)$, as claimed, in which case $M > 0$ also in (5.10).

By (5.5) and (5.7) the cosmological constant Λ is given by $\Lambda = 2\gamma/\beta^2 < 0$. Also by (5.5) and (5.6), for the dilaton field $\Phi^{(1)}_{\text{plasma}}$ given by

$$\Phi^{(1)}_{\text{plasma}}(\tau,\delta) \overset{\text{def.}}{=} m\Psi(\delta) = \\ m\left[\frac{a^2\beta^2 dn^2\left(\delta_1\kappa\right)}{-\gamma} + \frac{a^2\beta^2\left(2-\kappa^2\right)}{2\gamma} - \frac{\left(v^2\beta^2 - \alpha_1\right)}{8\gamma}\right], \tag{5.13}$$

we obtain a solution $(g_{\text{plasma}}, \Phi^{(1)}_{\text{plasma}})$ of the J-T field equations (5.3):

$$R\left(g_{\text{plasma}}\right) + 2\Lambda = 0, \quad \nabla_i\nabla_j\Phi^{(1)}_{\text{plasma}} + \Lambda g_{ij}\Phi^{(1)}_{\text{plasma}} = 0, \tag{5.14}$$

where the first equation here is equation (4.17), and the second set of equations are argued for in [6]. There are two more plasma dilaton fields $\Phi^{(j)}_{\text{plasma}}$, $j = 2, 3$, for g_{plasma} in (4,19) that are computed in [7] (not in [6]), and also in [13] by a different method. For the choice $\alpha_1 = 0$, equation (5.13) can be written as

$$\Phi^{(1)}_{\text{plasma}}(\tau,\delta) = \frac{m\beta^2}{-2\gamma}\left[2a^2dn^2(\delta,\kappa) - a^2\left(2-\kappa^2\right) + \frac{v^2}{4}\right], \tag{5.15}$$

and the other two dilaton fields are given by

$$\Phi^{(2)}_{\text{plasma}}(\tau,\delta) = \sqrt{\frac{2}{-\gamma}}a\beta dn(\delta,\kappa)\left[\frac{v^2}{4} - a^2\kappa^4\left(\frac{sncn}{dn}\right)^2(\delta,k)\right]^{\frac{1}{2}}\sinh(\sqrt{A}\tau) \\ \Phi^{(3)}_{\text{plasma}}(\tau,\delta) = \sqrt{\frac{2}{-\gamma}}a\beta dn(\delta,\kappa)\left[\frac{v^2}{4} - a^2\kappa^4\left(\frac{sncn}{dn}\right)^2(\delta,\kappa)\right]^{\frac{1}{2}}\cosh(\sqrt{A}\tau), \tag{5.16}$$

for A in (5.7), where we checked directly that $A > 0$, given the standing assumption that $v^2 > 8a^2\left(2-\kappa^2\right)$ in (5.8). Also, one has the inequality

$$\left(\frac{sncn}{dn}\right)^2(x,\kappa) \leqslant 1 \tag{5.17}$$

so that in (5.16) (where already $\gamma < 0$) we always have that

$$\left[\frac{v^2}{4} - a^2\kappa^4\left(\frac{sncn}{dn}\right)^2(\delta,\kappa)\right] \geqslant \frac{v^2}{4} - a^2\kappa^4 > 0. \tag{5.18}$$

Here as we have seen, in the argument that followed (5.10), the assumption $v^2 > 8a^2\left(2-\kappa^2\right) \Rightarrow v^2/4 - a^2\kappa^4 > 0$. Thus the scalar fields in (5.16) are real, as they should be, and they with the field in (5.15) and the cold plasma metric provide for new solutions in Jackiw-Teitelboim dilaton gravity. An additional reference for section 4 and for this section is [17].

6. Conclusions

We have maintained a special interest in the magnetoacoustic system (2.5) that describes the propagation of one-dimensional magnetoacoustic waves in a cold plasma of density ρ with a speed v across a transverse magnetic field. Using a positive multiple of the Riemannian metric g_H in (4.4), (4.5) (also see (4.8)) attached to the classical, continuous (hyperbolic) Heisenberg model, we assigned to the nonlinear system (2.5) the metric g_{plasma} defined in (4.15), where $\beta > 0$ is the parameter in the second equation of (2.5), which arises from the magnetic field strength expansion (1.1), and where γ is an arbitrary negative real number. Given the correspondence that was set up in section 3 between solutions of the system (2.5) and certain solutions of the resonant nonlinear Schrödinger equation in (3.1), a choice sometimes for γ is the coefficient of $|\psi|^2\psi$ in (3.1). g_{plasma} being a real number multiple of g_H also has constant Ricci scalar curvature given by $R\left(g_{\text{plasma}}\right) = -4\gamma/\beta^2$ in (4.17), and its components $(g_{\text{plasma}})_{ij}$, $1 \leqslant i, j \leqslant 2$, are given in terms of the cold plasma density and speed (ρ and u) by the concrete formulas in (4.18). If (ρ, u) is the traveling wave solution of the system (2.5) given by Gurevich and Krylov in (2.9) (or in (2.10)), then the plasma metric components are given by the concrete formulas in (4.18), As was shown in [6, 13], a change of variables (from (x, t) in (4.18) to new variables (τ, δ)) exists by which the plasma metric assumes the diagonal form (4.19), with $A(\delta)$ defined in (4.20). The main result is that by another change of variables $(\tau, \delta) \to (\tau, r)$, where r is given explicitly in terms of δ by (5.6), the cold plasma metric g_{plasma} diagonalized in (4.19) is mapped exactly to the Jackiw-Teitelboim (J-T) black hole metric g_{bh} given in (5.1), where m and M in (5.1) are given by the formulas in (5.7). M is the black hole mass, and $-m^2$ is the negative J-T cosmological constant Λ.

By way of the Gurevich-Krylov solution and the continuous Heisenberg model, we have established a direct connection of the magnetoacoustic system (2.5) for a cold plasma to a black hole solution g_{bh} in the J-T theory of 2d dilaton gravity. Dilaton fields $\Phi^{(j)}_{\text{plasma}}$, $1 \leqslant j \leqslant 3$, in terms of elliptic functions are presented in (5.15), (5.16) for which the pairs $(g_{\text{plasma}}, \Phi^{(j)}_{\text{plasma}})$ (again for g_{plasma} in (4.19)) are solutions of the J-T gravitational field equations (5.3).

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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