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Research article

Normal forms, invariant manifolds and Lyapunov theorems

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Abstract: We present an approach to Lyapunov theorems about a center for germs of analytic vector fields based on the Poincaré–Dulac and Birkhoff normal forms. Besides new proofs of three Lyapunov theorems, we prove their generalization: if the Poincaré–Dulac normal form indicates the existence of a family of periodic solutions, then such a family really exists. We also present new proofs of Weinstein and Moser theorems about lower bounds for the number of families of periodic solutions; here, besides the normal forms, some topological tools are used, i.e., the Poincaré–Hopf formula and the Lusternik–Schnirelmann category on weighted projective spaces.

Keywords: Lyapunov theorems; Poincaré–Dulac normal form; invariant manifolds
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1. Introduction

In his memoir [21], Lyapunov stated several theorems about analytic families of periodic solutions of some analytic autonomous vector fields near equilibrium point. Below, we distinguish three of them.

First, consider an analytic planar vector field with a singular point having a pair of pure imaginary eigenvalues. Thus, we can assume an autonomous differential system of the form

$$\dot{x} = -\omega y + \ldots, \ \dot{y} = \omega x + \ldots. \tag{1.1}$$

with real analytic right-hand sides.

Here, one looks for a first integral in the form of a formal power series:

$$F = \frac{\omega}{2}\left(x^2 + y^2\right) + \ldots. \tag{1.2}$$

In general, such a formal first integral does not exist; one finds

$$\dot{F} = c_{2k+1}\left(x^2 + y^2\right)^{k+1} + \ldots, \tag{1.3}$$

where $c_{2k+1} \neq 0$ is the so-called **Poincaré–Lyapunov focus quantity** and we have a weak focus of order $2k-1$ (see Section 2 below). Otherwise, we have the following result attributed to Lyapunov [21, Section 39] and Poincaré [27, Chapter 11].

Theorem 1. *(Lyapunov–Poincaré) If all focus quantities vanish, then the formal integral (1.2) is, in fact, analytic, and all solutions near* $x = y = 0$ *are periodic.*

In [21, Section 40] the latter situation is generalized to the following system:

$$\dot{x} = Ax + \dots, \tag{1.4}$$

$x \in (\mathbb{R}^n, 0)$, where the matrix A has a pair $\lambda_{1,2} = \pm i\omega$ of pure imaginary eigenvalues and all of its other eigenvalues lie in the half-plane $\{\mathrm{Re}\lambda < 0\}$. Again, one looks for a formal first integral of the form

$$F = \frac{\omega}{2}\left(x_1^2 + x_2^2\right) + \dots, \tag{1.5}$$

where $x_{1,2}$ are the variables associated with $\lambda_{1,2}$, and there are again obstacles to such an integral occurring.

Theorem 2. *(Lyapunov) If all of these obstacles vanish, then the first integral is analytic and system (1.4) has an analytic 1–parameter family of periodic solutions.*

The last situation considered in [21, Section 42]* is the case when the matrix A in system (1.4) has the pure imaginary eigenvalues

$$\pm i\omega_1, \dots, \pm i\omega_m, \quad \omega_j > 0,$$

and one of the frequencies, say, ω_1, is such that none of the other frequencies is an integer multiple of it; thus,

$$\omega_j/\omega_1 \notin \mathbb{Z}, \quad j \geq 1. \tag{1.6}$$

Theorem 3. *(Lyapunov) In this case, there exists a family of periodic solutions*

$$x = \phi(t; c), \quad c \in (\mathbb{R}_+, 0),$$

of period $T(c) \approx 2\pi/\omega_1$*, depending analytically on* c *and such that* $\phi(t; 0) \equiv 0$.

The latter theorem has attracted the attention of specialists in Hamiltonian dynamics.†

Recall that an autonomous analytic Hamiltonian system with $m = \frac{n}{2}$ degrees of freedom takes the form

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, \dots, n; \tag{1.7}$$

*In fact, Lyapunov considered a more general situation where, besides $\pm i\omega_j$, there are also eigenvalues in the left half-plane.

†I have somehow overlooked this result and have learned about it only recently on the occasion of reviewing the Ph.D. thesis of D. Strzelecki in the Nicolas Copernicus University in Toruń.

the corresponding vector field is usually denoted as

$$X_H = \sum H'_{p_i}\partial/\partial_{q_i} - \sum H'_{p_i}\partial/\partial_{p_i}.$$

Assume that it has equilibrium point $q = p = 0$ with the eigenvalues $\pm i\omega_1, \dots, \pm i\omega_m$, $\omega_j > 0$. Assuming $H(0) = 0$, the leading part of the Taylor expansion of the Hamiltonian function is

$$H_2 = \sum \frac{1}{2}\epsilon_j\omega_j\left(q_j^2 + p_j^2\right), \tag{1.8}$$

where q_j, p_j are suitable canonical variables, i.e., with the Poisson brackets $\left\{p_i, q_j\right\} = \delta_{ij}$, and $\epsilon_j = \pm 1$ are well-defined signs. [‡]

Schmidt [28] studied 1–parameter families of periodic solutions for such systems with two degrees of freedom in the cases of resonant frequencies ω_1 and ω_2. His analysis was mainly focused on definite Hamiltonians, i.e., when $\epsilon_1 = \epsilon_2$; but, he also considered the situation near the Lagrangian libration point in the restricted three-body problem, where the Hamiltonian function is indefinite. He did not refer to the third Lyapunov theorem. Anyway, his results agree with what is presented below.

Weinstein [35] has applied the Lusternik–Schnirelmann category to prove the following.

Theorem 4. *(Weinstein) Assume that H_2 is positive definite, i.e., all $\epsilon_j = 1$ in Eq. (1.8). Then, system (1.7) has at least $m = \frac{n}{2}$ 1–parameter families of periodic solutions.*

Next, Moser tried in [23, Theorem 4] to specify Weinstein's result by assuming that H_2 is positive on a linear subspace E associated with one frequency ω, i.e., all solutions in E of the linear system have the period $2\pi/\omega$ and these are the only such solutions. He claimed that there exists at least $\frac{1}{2}\dim E$ of periodic solution of the period $\approx 2\pi/\omega$ of the full Hamiltonian system. But his own example [23, Example 2] contradicts his statement; see also Example 4 below. Note also that the change $H \longmapsto -H$ means the reversion of the time, and it does not influence the periodic property of solutions.

We have the following specification of Theorem 4. Again assume an analytic Hamiltonian system (1.6) with the eigenvalues $\pm i\omega_j$, $\omega_j > 0$.

Following [35, Proof of Theorem 2.1] we consider the following equivalence relation on the set $\{\omega_1, \dots, \omega_m\}$ of frequencies

$$\omega_i \sim \omega_j \text{ iff } \omega_i/\omega_j \in \mathbb{Q}. \tag{1.9}$$

For each equivalence class C_ν we have a linear subspace $\mathcal{E}_\nu$ that is invariant for the linear part of the system.

Choose some equivalence class C_1 and let $\mathcal{E}_1$ be the corresponding linear subspace. Let us order the frequencies from C_1 as follows:

$$\begin{aligned} \tilde{\omega}_1 \quad &: \ = \omega_1 = \ldots = \omega_{k_1} > \tilde{\omega}_2 := \omega_{k_1+1} = \ldots = \omega_{k_1+k_2} \\ &> \ \ldots > \tilde{\omega}_r := \omega_{k_1+\ldots+k_{r-1}+1} = \ldots = \omega_{k_1+\ldots+k_r}. \end{aligned} \tag{1.10}$$

[‡]We have $\dot{f} = \{f, H\}$ in the case of a general Hamiltonian system with the symplectic structure defined by a Poisson bracket. In particular, for $z_j = q_j + ip_j$ and $v_j = q_j - ip_j$, we have $\left\{z_j, z_k\right\} = \left\{v_j, v_k\right\} = 0$ and $\left\{z_j, v_k\right\} = 2i\delta_{jk}$. Thus $\dot{z}_j = \left\{z_j, v_k\right\}\partial H/\partial v_k = 2i\cdot\partial H/\partial v_k$. Also, the resonant monomials $g = z^k v^l$ that form the Birkhoff theorem satisfy $\{g, H_2\} = 0$.

In the case of (1.8) the corresponding linear system is diagonalizable in the complex variables. The non-diagonalizable case is somewhat special and we consider it only in Example 7 in Section 5.

Theorem 5. *Let H be analytic and such that $H_2|_{\mathcal{E}_1}$ is definite (say, positive), i.e., $\epsilon_1 = \ldots = \epsilon_{k_1+\ldots+k_r} = 1$ in Eq. (1.8); then, there exist the following:*

— at least k_1 1–parameter families of periodic solutions to system (1.7) with periods $\approx 2\pi/\tilde{\omega}_1$,

— at least $k_1 + k_2$ 1–parameter families of periodic solutions to system (1.7) with periods $\approx 2\pi/\tilde{\omega}_1$ or $\approx 2\pi/\tilde{\omega}_2, \ldots$

— at least $k_1 + \ldots + k_r$ 1–parameter families of periodic solutions to system (1.7) with periods $\approx 2\pi/\tilde{\omega}_1$, or $\approx 2\pi/\tilde{\omega}_2, \ldots,$ or $\approx 2\pi/\tilde{\omega}_r$.

Next, this subject was raised up by specialists in the nonlinear functional analysis, including Szulkin [32] and several other groups [10, 15, 25, 26]. The principal result here is that we assume the ordering (1.10) of all frequencies (not only from C_1).

Theorem 6. *(Szulkin) If, for a given l,*

(i) $\tilde{\omega}_j/\tilde{\omega}_l \notin \mathbb{N}$ for all $j \neq l$ in Eq. (1.10), and

(ii) $\epsilon_{k_1+\ldots+k_{l-1}+1} + \ldots + \epsilon_{k_1+\ldots+k_l} \neq 0$ for ϵ_i's from Eq. (1.8), then there exists a sequence $\{\gamma_n(t)\}$ of periodic solutions to the equation $\dot{x} = X_H(x)$ of non-constant periodic solutions tending to $\gamma(t) \equiv 0$ of periods tending to $2\pi/\tilde{\omega}_l$.

We see that the first assumption is like in the third Lyapunov theorem. But, the assumption (ii) is new; it replaces the definiteness assumption of H_2 from Theorems 4 and 5.

Finally, we recall the following non-Hamiltonian version of the third Lyapunov theorem due to Moser [23, Theorem 2].

Theorem 7. *(Moser) Assume that an analytic system (1.4) with only purely imaginary eigenvalues has an analytic first integral $F(x)$ such that $D^2F(0)$ is positive definite. Then, this system has at least one 1–parameter family of periodic solutions.*

The aim of this paper is to take a somehow different look at this center problem. Our approach is rather qualitative and less topological, like in [1] and [18]. It turns out that much can be deduced from the Poincaré–Dulac normal form and its Hamiltonian version, i.e., the Birkhoff normal form (Schmidt also used the Birkhoff normal form). Important is the use of some versions of the standard Poincaré return map combined with the analyticity assumption of the vector fields. We will reprove most of above theorems and we shall prove a new result.

Assume that the matrix A in system (1.4) has pure imaginary eigenvalues: $\pm i\omega_1, \ldots, \pm i\omega_m,\ \omega_j > 0$. We can reduce it, via a formal change of variables, to the Poincaré–Dulac normal form (see Theorem 9 in the next section). It may happen that the latter formal system admits a formal 1–parameter family of periodic solutions with the period $\approx 2\pi/\omega_l$.

More precisely, one can consider the situation with only one equivalence class for the frequencies, i.e., $\omega_j/\omega_k = p_j/p_k$, with relatively prime positive integers p_i, and that $\omega_1 \geq \ldots \geq \omega_l$. Let Z_j be the formal complex variables associated with the Poincaré–Dulac normal form. For the new variables $R \in \mathbb{R}$, $W_j \in \mathbb{C}$ and $\Theta \in \mathbb{S}^1$, defined by $Z_l = Re^{i\Theta}$ and $Z_j = W_j e^{ip_j\Theta/p_l}$, one obtains a formal differential system:

$$\frac{dR}{d\Theta} = \Pi\left(R, W, \bar{W}\right), \ \frac{dW_j}{d\Theta} = \Lambda_j\left(R, W, \bar{W}\right)$$

(called the return system). The analogue of the center conditions in this case means that the latter system has a formal curve Γ of non-isolated equilibrium points. All of these notions are made precise in Section 7 (Eqs. (7.1)–(7.3), Lemma 3 and Definition 4).

The next statement can be regarded as a fourth Lyapunov theorem.

Theorem 8. *In this case, there exists a continuous family of periodic solutions to system (1.4) with the period* $\approx 2\pi/\omega_l$.

The paper is organized as follows. In Section 2, we define the Poincaré–Dulac and Birkhoff normal forms, with their principal properties. In Section 3, we present and discuss theorems about invariant manifolds. Section 4 is devoted to various proofs of the three Lyapunov theorems; they are included in this paper (which is treated as a sort of review) because some of the Lyapunov theorems are not widely known, and because their proofs are (we hope) illuminating. In Section 5, we present some examples of Hamiltonian systems with explicit families of periodic solutions. In Section 6,we prove the existence of some additional invariant submanifolds following from the normal forms. In Section 7, we prove Theorem 8. Section 8 is devoted to the discussion of situations with a first integral; there, we give proofs of Theorems 4, 5, 7 and an introduction to the functional analytic method. In Section 9, we present our approach, based only on the Birkhoff normal form, of two examples from the celestial mechanics: geostationary orbits of satellites and libration points in the restricted four-body problem. The last section contains appendices about the topology of weighted projective spaces and the Lusternik–Schnirelmann category.

2. Poincaré–Dulac and Birkhoff normal forms

Consider the analytic system (1.4), but in $(\mathbb{C}^n, 0)$ (not in $(\mathbb{R}^n, 0)$. Thus, we can assume that the matrix A is in the Jordan form with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Definition 1. *We say that the latter system of eigenvalues satisfies the resonant relation of type* $(j;k)$, $j = 1, \ldots, n$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$, *if*

$$\lambda_j = (k, \lambda) = k_1\lambda_1 + \ldots + k_n\lambda_n. \tag{2.1}$$

Theorem 9. *(Poincaré–Dulac [13, 27]) There exists a change* $x \to X$, *defined by formal power series, which transforms system (1.4) to the following Poincaré–Dulac normal form:*

$$\dot{X}_j = \lambda_j X_j + \sum_k a_{j;k} X^k, \ \ j = 1, \ldots, n, \tag{2.2}$$

where the summation runs over multi-indices k such that the resonant relations of type $(j;k)$ *hold and* $X^k = X_1^{k_1} \cdots X_n^{k_n}$.

The proof can be found in [3, 17, 37].

Remark 1. *In the case when there are multiple eigenvalues λ_j and the corresponding Jordan cell is not diagonal,we have the resonant relations $(j;k)$, where $k = (0,\ldots,0,1,0,\ldots 0)$ with 1 in the $(j+1)^{\text{th}}$ entry. They imply non-diagonal entries in A.*

Remark 2. *In the so-called Poincaré domain, i.e., when the convex hull of the set $\{\lambda_1,\ldots,\lambda_n\} \subset \mathbb{C}$ is separated from $0 \in \mathbb{C}$, the Poincaré–Dulac normal form is polynomial and the normalizing change is analytic. We refer the reader to [30] for a short proof.*

Remark 3. *In the general, situation the Poincaré–Dulac normal form is divergent. Indeed, in the example*

$$\dot{x}_1 = x_1^2,\ \ \dot{x}_2 = x_2 - x_1^2, \tag{2.3}$$

due to Euler (see [37]), the Poincaré–Dulac normal form is

$$\dot{X}_1 = X_1^2 + a_3X_1^3 + \ldots,\ \ \dot{X}_2 = X_2\left(1 + b_1X_1 + b_2X_1^2 + \ldots\right).$$

But, the so-called center manifold W^c, corresponding to the zero eigenvalue and formally defined by $X_2 = 0$, in the original system (2.3) takes the form

$$W^c = \left\{x_2 = \sum (m-1)!x_1^m\right\}.$$

Consider now an autonomous analytic Hamiltonian system (1.7) with $m = \frac{n}{2}$ degrees of freedom near an equilibrium point $q = p = 0$ with the eigenvalues $\pm \mathrm{i}\omega_1,\ldots,\pm\mathrm{i}\omega_m$, $\omega_j > 0$. Assuming $H(0) = 0$, the leading part of the Taylor expansion of the Hamiltonian function is like in Eq. (1.8), i.e., $H_2 = \sum \frac{1}{2}\epsilon_j\omega_j\left(q_j^2 + p_j^2\right)$, where q_j, p_j are suitable canonical variables, i.e., with the Poisson brackets $\left\{p_i, q_j\right\} = \delta_{ij}$, and $\epsilon_j = \pm 1$ are well-defined signs.

It is natural to introduce the complex variables

$$z_j = q_j + \mathrm{i}p_j,\ \ v_j = q_j - \mathrm{i}p_j, \tag{2.4}$$

with the Poisson brackets $\left\{z_j, v_k\right\} = 2\mathrm{i}\delta_{jk}$. Then, we have

$$H_2 = \sum \frac{1}{2}\epsilon_j\omega_j z_j v_j \tag{2.5}$$

and

$$\dot{z}_j = \lambda_j z_j + \ldots,\ \ \dot{v}_j = -\lambda_j v_j + \ldots,\ \ \ \lambda_j = -\mathrm{i}\epsilon_j\omega_j. \tag{2.6}$$

Of course, in the real domain (where p_j and q_k are real), we have $v_j = \bar{z}_j$.

The following result from [8] is a Hamiltonian analogue of the Poincaré–Dulac theorem.

Theorem 10. *(Birkhoff) There exists a formal canonical change $(z,v) \longmapsto (Z,V)$ which reduces the Hamiltonian function to the following one:*

$$H_2(Z,V) + \sum a_{k;l}Z^kV^l,$$

where summation runs over the pairs $(k;l) = (k_1,\ldots,k_n;l_1,\ldots,l_n) \in \mathbb{Z}_{\geq 0}^n \times \mathbb{Z}_{\geq 0}^n$ such that the resonant relations

$$(k-l,\lambda) = \sum \left(k_j - l_j\right)\lambda_j = 0$$

hold and $|k| + |l| \geq 3$.

3. Invariant manifolds

Definition 2. *The singular point $x = 0$ of system (1.4) is hyperbolic if none of the eigenvalues of the matrix A is imaginary. Thus, the eigenvalues are divided into two groups:*

$$\left\{\lambda_1, \ldots, \lambda_p\right\} \text{ with } \mathrm{Re}\lambda_j < 0 \text{ and } \left\{\lambda_{p+1}, \ldots, \lambda_n\right\} \text{ with } \mathrm{Re}\lambda_j > 0; \tag{3.1}$$

i.e., separated by the imaginary axis.

The principal result about invariant manifolds is the Hadamard–Perron theorem for hyperbolic singular points. These manifolds are the local stable manifold (of dimension p)

$$W^s = \{x : g^t(x) \to 0 \text{ as } t \to +\infty\} \tag{3.2}$$

and the local unstable manifold (of dimension $q = n - p$)

$$W^u = \{x : g^t(x) \to 0 \text{ as } t \to -\infty\}, \tag{3.3}$$

where $\{g^t\}$ is the phase flow generated by the vector field (1.4); $g^t(x_0)$ is the solution $x(t)$ obeying the initial condition $x(0) = x_0$. In the case of a finitely smooth vector field,the proof of the Hadamard–Perron theorem is rather involved. But, in the analytic case, it is quite easy.

Theorem 11. *(Holomorphic Hadamard–Perron) Assume that the vector field (1.4) is analytic. Assume also that the set $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ of eigenvalues of the matrix A can be separated by a straight line ℓ through $\lambda = 0$ into two groups: $\left\{\lambda_1, \ldots, \lambda_p\right\}$ on one side of ℓ and $\left\{\lambda_{p+1}, \ldots, \lambda_n\right\}$ on the other side. Then, there exist local analytic invariant manifolds for (1.4) tangent to the corresponding linear subspaces invariant for A.*

It follows that, if the singular point $x = 0$ is hyperbolic, then the invariant manifolds W^s and W^u are real analytic.

Proof. We present a rather novel proof, following a method from [30]. Another proof can be found in [17]; note also that, in the case $n = 2$ and $p = 1$, this result was proved by Briot and Bouquet [9] (see also [37]).

Let $E^s \simeq \mathbb{C}^p$ and $E^u \simeq \mathbb{C}^q$ be the linear subspaces associated with the division of the eigenvalues set, and let (x, y) be the coordinates associated with the splitting $\mathbb{C}^n = E^s \oplus E^u$. We can assume that the matrix A has the form $A^s \oplus A^u$, where $A^s = \mathrm{diag}\left(\mu_1, \ldots, \mu_p\right) + A_1^s$ and $A^u = \mathrm{diag}\left(\nu_1, \ldots, \nu_q\right) + A_1^u$, where $A_1^{s,u}$ are off-diagonal with small entries. We have

$$\dot{x} = A^s x + \phi(x, y), \quad \dot{y} = A^u y + \psi(x, y).$$

We look for the manifold W^s as a graph of a map $F : E^s \longmapsto E^u$, $W^s = \{(x, F(x))\}$; the case of W^u is treated analogously. The invariance condition, i.e.,

$$\dot{y} - DF \cdot \dot{x}|_{y=F(x)} \equiv 0,$$

leads to the equation

$$A^u F(x) - DF(x) \cdot A^s x = DF(x) \cdot \phi(x, F(x)) - \psi(x, F(x)). \tag{3.4}$$

This equation takes the form

$$\boldsymbol{L}F = \boldsymbol{T}(F),$$

where $\boldsymbol{L}$ and $\boldsymbol{T}$ are operators, linear and nonlinear. We treat these operators as acting on the Banach space $\mathcal{F}$ consisting of power series $F = \sum_{j,k} f_{j,k} x^k e_j$ with the norm

$$\|F\| = \sum \left| f_{j;k} \right| \rho^{|k|},$$

where ρ is the 'radius of convergence' (suitably chosen). We have $\boldsymbol{L} = \boldsymbol{L}_0 + \boldsymbol{L}_1$, where

$$\boldsymbol{L}_0 x^k e_j = \left[\nu_j - (k, \mu) \right] x^k e_j$$

and $\boldsymbol{L}_1$ is small with respect to $\boldsymbol{L}_0$. Because ν_j and μ_i are separated by the line ℓ, the operator $\boldsymbol{L}$ is invertible.

Moreover, if $\mathcal{F}_d = \left\{ F = \sum_{j,|k|=d} f_{j;k} x^k e_j \right\}$ are the subspaces of homogeneous maps of degree $d \geq 2$, then

$$\left\| \boldsymbol{L}^{-1}|_{\mathcal{F}_d} \right\| < C_1/d$$

for some constant C_1. On the other hand, the operators $\boldsymbol{T}|_{\mathcal{F}_d} : \mathcal{F}_d \longmapsto \mathcal{F}_{>d}$ have the Lipschitz constants bounded by $C_2 d\rho$, where the factor d arises from DF and C_2 is some constant. It follows that the operator

$$\boldsymbol{P} = \boldsymbol{L}^{-1}\boldsymbol{T}$$

is Lipschitz-continuous with a small Lipschitz constant. Therefore, the fixed-point problem $F = \boldsymbol{P}(F)$ has a unique solution. □

Assume now the non-hyperbolic situation. Thus, we have the splittings $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$ and $A = A^s \oplus A^u \oplus A^c$, where the eigenvalues of A^s (respectively, A^u) lie in the left (respectively, right) half-plane and the eigenvalues of A^c lie on the imaginary axis. The next theorem is usually cited without proof (which is quite technical, see [3, 17]).

Theorem 12. *(Center Manifold) There exist smooth invariant manifolds W^s, W^u and W^c tangent to the subspaces E^s, E^u and E^c, respectively. The manifolds W^s and W^u are analytic; W^c is only infinitely smooth in general, but its Taylor series is defined uniquely.*

Remark 4. *The Euler example from Remark 3 demonstrates that the center manifold can be non-analytic. If $W^c = \{y = F(x)\}$, then the analogue of Eq. (3.4) is*

$$\boldsymbol{L}F = x^2 \frac{\mathrm{d}}{\mathrm{d}x} F + x^2,$$

where $\boldsymbol{L} = Id$. Here $\boldsymbol{L}$, although invertible, is not dominating in comparison to $x^2 \frac{\mathrm{d}}{\mathrm{d}x}$; we have $\left\| x^2 \frac{\mathrm{d}}{\mathrm{d}x} x^k \right\| / \left\| x^k \right\| = k\rho$.

In [17] and [37], one can find an explanation of the non-analyticity of W^c in terms of the so-called Stokes phenomena.

Next, the center manifold can be non-unique. Indeed, the real Euler example is the so-called saddle-node singularity; on the left from the unstable manifold $W^u = \{x = 0\}$, we have two hyperbolic sectors *separated by the left part of W^c (which is unique), but, on the right of W^u, we have a parabolic sector, and any of its phase curves can serve as the right part of W^c. This non-uniqueness is explained by the method of the proof of the existence of W^c; there, one extends the vector field to the whole $\mathbb{R}^n$ and W^c (which is a fixed point of some functional operator) depends on this extension.*

Below, we present other examples with non-analytic invariant manifolds.

Example 1. *Let $z = x_1 + \mathrm{i}x_2$, $v = x_1 - \mathrm{i}x_2$ and y be coordinates in $\mathbb{C}^3$. The system is as follows:*

$$\dot{z} = \mathrm{i}z + z^2 v,\ \dot{v} = -\mathrm{i}v + zv^2,\ \dot{y} = -y + zv. \tag{3.5}$$

One finds that

$$W^c = \left\{ y = \frac{1}{2} \sum (n-1)!\,(2zv)^n \right\}.$$

The non-analyticity of the center manifold is explained via the nonlinear Stokes phenomena as follows. In the variables $u = zv$ and y, we have

$$\dot{u} = 2u^2,\ \dot{y} = -y + u,$$

i.e., a saddle-node similar to the Euler example. In a sectorial domain around $\{(u, y) : \mathrm{Im}\,(u) = 0,\ \mathrm{Re}\,(u) > 0\}$, the phase portrait near the saddle, with a unique center manifold and unique stable manifold; but, in a sectorial domain near $\{(u, y) : \mathrm{Im}\,(u) = 0, \mathrm{Re}\,(u) < 0\}$, the phase portrait is like near the node with many center-ype manifolds.

Example 2. *Let $z_1 = x_1 + \mathrm{i}x_2$, $v_1 = x_1 - \mathrm{i}x_2$ and $z_2 = x_3 + \mathrm{i}x_4$, $v_2 = x_3 - \mathrm{i}x_4$ be coordinates in $\mathbb{C}^4$. The system is as follows:*

$$\dot{z}_1 = \mathrm{i}\omega_1 z_1 + z_1^2 v_1,\ \dot{v}_1 = -\mathrm{i}\omega_1 v_1 + z_1 v_1^2,\ \dot{z}_2 = \mathrm{i}\omega_2 z_2 + z_1 v_1,\ \dot{v}_2 = -\mathrm{i}\omega_2 v_2 + z_1 v_1. \tag{3.6}$$

It has two invariant planes:

$$W_1 = \{z_1 = v_1 = 0\}$$

(analytic) and

$$W_2 = \left\{ z_2 = -\frac{1}{2} \sum (n-1)! \left(\frac{2z_1 v_1}{\mathrm{i}\omega_2} \right)^n,\ v_2 = -\frac{1}{2} \sum (n-1)! \left(\frac{2\mathrm{i}z_1 v_1}{\omega_2} \right)^n \right\},$$

which is non-analytic.

We finish this section with another theorem about invariant manifolds.

Definition 3. *Let f be a smooth diffeomorphism of a manifold M, and let $L \subset M$ be a smooth invariant submanifold for f, $f(L) \subset L$. We say that f is normally hyperbolic at N if there is a splitting*

$$N_L = T_M / T_N \simeq E^s \oplus E^u$$

of the normal bundle to L into sub-bundles E^s and E^u that are invariant with respect to Df and the following estimates hold:

— the bundle E^s is contracted more strongly than the bundle TL, i.e.,

$$\sup_L \|Df|_{E^s}\| < \left\{\inf_L \left\|(Df|_{TL})^{-1}\right\|\right\}^{-1};$$

— the bundle E^u is expanded more strongly than the bundle TL, i.e.,

$$\left\{\inf_L \left\|(Df|_{E^u})^{-1}\right\|\right\}^{-1} > \sup_L \|Df|_{TL}\|.$$

We refer the reader to [16] for the proof of the following statement.

Theorem 13. *(Normal Hyperbolicity) If f is normally hyperbolic on L, then there exists a neighborhood $\mathcal{U}$ of f in the functional space of C^1–diffeomorphisms such that any $g \in \mathcal{U}$ has a unique invariant submanifold L_g close to L at which it is normally hyperbolic.*

4. The Lyapunov theorems

4.1. Poincaré–Lyapunov theorem

Eq. (1.1) with the variables $z = x + \mathrm{i}y$, $v = x - \mathrm{i}y$ takes the form

$$\dot{z} = \mathrm{i}\omega z + \dots, \ \dot{v} = -\mathrm{i}\omega v + \dots. \tag{4.1}$$

The Poincaré–Dulac normal form here is as follows:

$$\dot{Z} = Z\left\{\mathrm{i}\omega + \sum (a_k + \mathrm{i}b_k)(ZV)^k\right\}, \ \dot{V} = V\left\{-\mathrm{i}\omega + \sum (a_k - \mathrm{i}b_k)(ZV)^k\right\}. \tag{4.2}$$

It is easy to find that the first nonzero coefficient a_k is related to the first nonzero Poincaré–Lyapunov quantity:

$$c_{2k+1} = \omega a_k. \tag{4.3}$$

Next, after passing to the polar coordinates r, θ (in the real domain), i.e., with $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $v = \bar{z} = r\mathrm{e}^{-\mathrm{i}\theta}$, we get the equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = f(r, \theta),$$

where f is a convergent series in powers of r with trigonometric polynomials in θ as coefficients. We define the Poincaré return map

$$\mathcal{P}(r_0) = \phi(2\pi; r_0), \tag{4.4}$$

where $\phi(\theta; r_0)$ is the solution to the latter equation $dr/d\theta$ with the initial condition $\phi(0; r_0) = r_0$. The Poincaré map is analytic and has a Taylor expansion of the form

$$\mathcal{P}(r) = r + d_{2k+1} r^{2k+1} + \dots,$$

where

$$d_{2k+1} = (2\pi/\omega)\, a_k \tag{4.5}$$

provided the Poincaré–Lyapunov quantity $c_{2k+1} \neq 0$.

The assumption of Theorem 1 means formally that either

(i) the Poincaré–Dulac normal form equals

$$\dot{Z} = \mathrm{i}Z \cdot M(ZV), \; \dot{V} = -\mathrm{i}V \cdot M(ZV), \tag{4.6}$$

where $M = \omega + \sum b_k (ZV)^k$ is a formal integrating factor and $F = ZV$ is the formal first integral; or (equivalently),

(ii) formally,

$$\mathcal{P} = Id. \tag{4.7}$$

Moreover, the assumptions of the holomorphic Hadamard–Perron theorem for system (4.1) are satisfied. Therefore, we have two holomorphic invariant lines, W^+ and W^-, formally defined by $\{V = 0\}$ and $\{W = 0\}$, respectively.

Below, we present two geometrical proofs of Theorem 1.

First proof. (Note that this proof was not found in the literature). Since the Poincaré map is analytic and formally obeys Eq. (4.7), it is the identity. Therefore, the real phase portrait is of the center type and the equilibrium point is surrounded by a family of closed phase curves. Moreover, we have a first integral $\widetilde{G}(r,\theta)$ defined via the initial condition $\widetilde{G}(r,\theta) = r_0^2$ if $r = \phi(\theta; r_0)$.

But, this function is not correct because its value for negative r's should be compatible with the 'square root' of the Poincaré map $Q : (\mathbb{R}, 0) \longmapsto (\mathbb{R}, 0)$, $Q(r_0) = -r_0 + \ldots$, such that $\mathcal{P} = Q \circ Q$ on $(\mathbb{R}_+, 0)$. But, we can take a function defined initially on $(\mathbb{R}_+, 0)$ as $G(r_0) = \frac{1}{2}\left(r_0^2 + Q^2(r_0)\right)$. Thus,

$$G(r,\theta) = G(r_0) \tag{4.8}$$

if $r = \phi(\theta, r_0)$. This first integral is holomorphic near the circle $\{r = 0, 0 \leq \theta \leq 2\pi\} \subset \mathbb{C} \times R$, where R is a ring around the circle $\mathbb{R}/2\pi\mathbb{Z} \subset \mathbb{C}/2\pi\mathbb{Z}$.

In fact, we can extend it to a holomorphic first integral in Z, V. For this, one first defines it as $G(x) = \frac{1}{2}\left(x^2 + Q^2(x)\right)$ on the disc $\Sigma = \{(x,0) : x \in \mathbb{C}, |x| < \varepsilon\} \subset \mathbb{C}^2_{x,y}$, or $\Sigma = \{(z,\bar{z}) : z \in \mathbb{C}, |z| < \varepsilon\} \subset \mathbb{C}^2_{z,v}$, where Q is the extension of the square root of the return map. Outside of the disc Σ, this function is defined by the condition of being constant on the complex phase curves (Riemann surfaces). In particular, it is single-valued (see the second proof) and equals zero on the complex separatrices of the singular point. □

Second proof (due to Moussu [24]). Using the holomorphic Hadamard–Perron theorem, we can assume that the coordinate lines are invariant, i.e.,

$$\dot{z} = z(\mathrm{i}\omega + \ldots), \; \dot{v} = v(-\mathrm{i}\omega + \ldots).$$

Then, the phase curves outside of $W^- = \{z = 0\}$ are graphs of functions,

$$v = \varphi(z; v_0),$$

which satisfy the initial value problem

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{v\,(-\mathrm{i}\omega + \ldots)}{z\,(\mathrm{i}\omega + \ldots)},\ \varphi\,(z_0; v_0) = v_{0,}$$

where $z_0 \neq 0$ is fixed. The analytic continuation of solutions φ along the circle $\left\{z = \mathrm{e}^{\mathrm{i}\alpha} z_0 : 0 \leq \alpha \leq 2\pi\right\}$ defines the monodromy map $\mathcal{M} : D \longmapsto D$, $D = \{z_0\} \times \{|v_0| < \varepsilon\}$, as

$$\mathcal{M}\,(v_0) = \varphi\left(\mathrm{e}^{2\pi\mathrm{i}} z_0; v_0\right). \tag{4.9}$$

This map is holomorphic and we have

$$\mathcal{M}\,(v_0) = v_0 + e_k v_0^{k+1} + O\left(v_0^{k+2}\right),\ e_k = (4\pi z_0/\omega)\,a_k. \tag{4.10}$$

The center conditions from Theorem 1 mean that $\mathcal{M} = Id$.

Now, we can define a first integral $F\,(z, v)$ by putting

$$F\,(z_0, v_0) = z_0 v_0 \tag{4.11}$$

on D and continuing it analytically as constant on the phase curves. Because the monodromy is trivial, the function F is single-valued outside of the line $\{z = 0\}$. It is also bounded there. Hence, it is holomorphic.

The function F has a Morsean critical point at $z = v = 0$. By the Morse lemma [37], there exist analytic coordinates $Z = z + \ldots$, $V = V + \ldots$ such that $F = ZV$.

The vector field is parallel to the analytic Hamiltonian vector field $X_F = F'_V \frac{\partial}{\partial Z} - F'_Z \frac{\partial}{\partial V}$, i.e., it equals $\Phi \cdot X_F$, where $\Phi = \mathrm{i}\omega + \ldots$ is an analytic function (the orbital factor).

Finally, one can reduce Φ to an analytic function of one variable: $\Phi\,(Z, V) = \mathrm{i}M\,(ZV)$; we refer the reader to [30] for the proof. □

Final remark. The integrating factor M from Eq. (4.6) is also analytic. If it is constant, then the center is isochronous and the period does not depend on the periodic solution.

4.2. Second Lyapunov theorem

Recall that the spectrum of the linear part consists of $\lambda_{1,2} = \pm\mathrm{i}\omega$ and of $\mu_1, \ldots, \mu_m$ $(m \geq 1)$ in the left half-plane. Therefore, the resonant relations from Definition 1 are as follows:

$$\begin{aligned} \lambda_1 &= (p+1)\,\lambda_1 + p\lambda_2,\ \lambda_2 = q\lambda_1 + (q+1)\,\lambda_2, \\ \mu_j &= p\lambda_1 + p\lambda_2 + (l, \mu),\ \ |l| \geq 1. \end{aligned}$$

In the Poincaré–Dulac normal form, we have

$$\dot{Z} = Z\left\{\mathrm{i}\omega + \sum \left(a_p + \mathrm{i}b_p\right)(ZV)^p\right\}$$

and the corresponding equation for $\dot{V}$. Moreover,

$$\dot{Y}_j = \sum Y_r F_r\,(Y, ZV),\ j = 1, \ldots, m.$$

We see that the variables Z, V are formally separated from Y_j and the center condition from Theorem 2 reads as

$$a_p = 0,\ p = 1, 2, \ldots.$$

Moreover, we have the stable invariant manifold $W^s = \{Z = V = 0\}$ (which is analytic) and a formal center manifold $W^c = \{Y = 0\}$, about its analytic properties we cannot judge at this moment (see Example 1 above).

But, the eigenvalue $\lambda_1 = \mathrm{i}\omega$ is separated from the remaining eigenvalues $\mu_1, \ldots, \mu_m$, λ_2 by a line in $\mathbb{C}$ through 0, and, analogously, λ_2 is separated from the other eigenvalues. By the analytic Hadamard–Perron theorem, we have analytic invariant hypersurfaces: W^+ corresponding to $\lambda_2, \mu_1, \ldots, \mu_m$ and W^- corresponding to $\lambda_1, \mu_1, \ldots, \mu_m$; of course, $W^s = W^+ \cap W^-$. We also have two invariant lines L^+ and L^- corresponding to the eigenvalues λ_1 and λ_2, respectively; W^c is 'spanned' by L^+ and L^-.

We are ready to present the following:

Proof of Theorem 2. We would like to follow the second proof of Theorem 1.

Thus, we can assume the following analytic system:

$$\dot{z} = z\,(\mathrm{i}\omega + \ldots)\,,\ \dot{v} = v\,(-\mathrm{i}\omega + \ldots)\,,\ \dot{y} = Y\,(z, v, y)\,;$$

so, $W^+ = \{z = 0\}$ and $W^- = \{v = 0\}$. The phase curves outside of W^+ are graphs of functions of z.

Let us choose a hypersurface of initial conditions with the form

$$\Sigma = \{z_0\} \times \{|v| < \varepsilon\} \times \{|y| < \varepsilon\}$$

(point×disc×ball) of complex dimension $1 + m \geq 2$. The solutions to the equations for $\frac{\mathrm{d}v}{\mathrm{d}z}$ and for $\frac{\mathrm{d}y}{\mathrm{d}z}$ with the initial conditions $v\,(z_0) = v_0$ and $y\,(z_0) = y_0$ are of the form

$$v = \varphi\,(z; v_0, y_0) = z_0 v_0/z + \ldots,\ \ y = \chi\,(z; v_0, y_0)\,.$$

We also have the monodromy map $\mathcal{M} : \Sigma \longmapsto \Sigma$:

$$\mathcal{M}\,(v_0, y_0) = \left(\varphi\left(\mathrm{e}^{2\pi\mathrm{i}} z_0; v_0, y_0\right), \chi\left(\mathrm{e}^{2\pi\mathrm{i}} z_0; v_0, z_0\right)\right).$$

In contrast to the Poincaré–Lyapunov case, this map is far from the identity. It has the form

$$\mathcal{M} : (v, y) \longmapsto (v + \ldots, By + \ldots)\,,$$

where the spectrum of the matrix B lies in the open unit disc; B is a strong contraction. Therefore, we need an additional argument.

On the one hand, we can use the normal hyperbolicity theorem, which ensures the existence of a 2−dimensional invariant manifold $V^c = V^c_{z_0}$, which is the center manifold for $\mathcal{M}$. For this, we treat $\mathcal{M}$ as a perturbation of the monodromy map $\mathcal{M}_0$ associated with a polynomial truncation of the Poincaré–Dulac normal form (when $V^c = \{y = 0\}$. The manifold V^c is unique.

Let us construct the first integral F, beginning with its definition $F_0 = F|_\Sigma$ on Σ. This function should have the property

$$\mathcal{M}^* F_0 = F_0;$$

then, we will use the analytic continuation. Let

$$\partial\Sigma = \{z_0\} \times \{|v| < \varepsilon\} \times \{|y| = \varepsilon\}$$

(point×disc×sphere). We put

$$F_0|_{\partial\Sigma} = z_0 v.$$

We define the following family of maps $h^\alpha : \Sigma \longmapsto \Sigma, \alpha \geq 0$:

$$h^\alpha (v_0, y_0) = \left(\mathrm{e}^{\mathrm{i}\alpha}\varphi\left(\mathrm{e}^{\mathrm{i}\alpha}z_0; v_0, y_0\right), \chi\left(\mathrm{e}^{\mathrm{i}\alpha}; v_0, y_0\right)\right).$$

Since $\varphi\left(\mathrm{e}^{\mathrm{i}\alpha}z_0; v; y\right) \approx \mathrm{e}^{-\mathrm{i}\alpha}$, the maps h^α do not change much for v. But, they are contractions when acting on y. Moreover,

$$h^{2\pi} = \mathcal{M}.$$

If $(v, y) = h^\alpha (v_0, y_0), (v_0, y_0) \in \partial\Sigma$, then we put

$$F_0 (v, y) = z_0 v_0.$$

This defines a correct analytic function on $\Sigma \setminus V^c$, which is extended to an analytic function on Σ. Next, the latter function F_0 is extended to a neighborhood of the singular point, i.e., we get the first integral

$$F (z, v, y) = zv + \dots .$$

with the critical locus along the stable manifold $W^s = \{z = v = 0\}$. This follows from the center conditions.

By the parametric Morse lemma [37], there exist analytic coordinates $Z = z + \dots$, $V = v + \dots$, $Y = y + \dots$ such that

$$F = ZV.$$

From the center conditions, it also follows that

(i) $\mathcal{M}|_{V^c} = Id|_{V^c}$,

(ii) $W^c = \bigcup_{z_0} V^c_{z_0}$ is an analytic center manifold, $W^c = \{Y = 0\}$, which supports a family of periodic solutions

$$Z (t) = c\mathrm{e}^{\mathrm{i}t},\ V (t) = d\mathrm{e}^{-\mathrm{i}t},\ Y (t) \equiv 0.$$

The corresponding family

$$z (t) = c\mathrm{e}^{\mathrm{i}t} + \dots,\ v (t) = d\mathrm{e}^{-\mathrm{i}t} + \dots,\ y (t) = \dots$$

depends analytically on the parameters c, d (defined by the initial conditions). In the real domain, we have the family $z (t) = r\mathrm{e}^{\mathrm{i}t} + \dots$, $\bar{z} (t) = r\mathrm{e}^{-\mathrm{i}t} + \dots$, $y (t) = \dots$, with the real parameter r; this is the convergent family from [21]. □

Remark 5. *We see that the existence of an analytic family of periodic solutions, under the assumptions of Theorem 2 (and also of Theorem 3), implies the uniqueness and analyticity of the center manifold*

$W^c \simeq \left(\mathbb{C}^2, 0\right)$. *This surface is parametrized by complex parameters* c, d *and* $\zeta = (1+\delta)\,\mathrm{e}^{\mathrm{i}t}$ *(from a neighborhood of the unit circle):*

$$z = c\zeta + \ldots,\ v = d\zeta^{-1} + \ldots,\ y = \ldots.$$

(But, this situation is somewhat special In Example 5 below, we encounter a 1–parameter family of periodic solutions spreading non-smooth surfaces.)

Next, one can show that the Poincaré–Dulac normal form

$$\dot{Z} = \mathrm{i}Z \cdot M(ZV),\ \dot{V} = -\mathrm{i}V \cdot M(ZV),\ \dot{Y} = \Omega(Y, ZV),$$

where M *is a real series and* Ω *is polynomial in* Y *(with series in* VZ *as coefficients), is analytic.*

Finally, note that, in Example 1, the formal center manifold does not support a family of periodic solutions; formally, it has an unstable focus.

4.3. Third Lyapunov theorem

Recall that we have pure imaginary eigenvalues $\pm\mathrm{i}\omega_1, \ldots, \pm\mathrm{i}\omega_m$, $\omega_j > 0$. Let $\dot{z}_j = \mathrm{i}\omega_j z_j + \ldots$, $\dot{v}_j = -\mathrm{i}\omega_j v_j + \ldots$.

Let us describe the Poincaré–Dulac (P–D) normal form for such systems. Assume first that the frequencies ω_j are independent over $\mathbb{Q}$. Then, the only resonant relations are

$$\begin{aligned}
\mathrm{i}\omega_j &= k_1\mathrm{i}\omega_1 + k_1(-\mathrm{i}\omega_1) + \ldots + \left(1 + k_j\right)\left(\mathrm{i}\omega_j\right) + k_j\left(-\mathrm{i}\omega_j\right) \\
&\quad + \ldots + k_m\mathrm{i}\omega_m + k_m(-\mathrm{i}\omega_m), \\
-\mathrm{i}\omega_j &= k_1\mathrm{i}\omega_1 + k_1(-\mathrm{i}\omega_1) + \ldots + k_j\mathrm{i}\omega_j + \left(1 + k_j\right)\left(-\mathrm{i}\omega_j\right) \\
&\quad + \ldots + k_m\mathrm{i}\omega_m + k_m(-\mathrm{i}\omega_m).
\end{aligned}$$

It follows that the P–D normal form is

$$\dot{Z}_j = Z_j\left(\mathrm{i}\omega_j + f_j(Z_1V_1, \ldots, Z_mV_m)\right),\ \dot{V}_j = V_j\left(-\mathrm{i}\omega_j + g_j(Z_1V_1, \ldots, Z_mV_m)\right), \tag{4.12}$$

$j = 1, \ldots, m$.

If there are relations, like

$$p\omega_1 = q\omega_2, \quad \gcd(p, q) = 1, \tag{4.13}$$

then the P–D normal form will contain additional monomial terms, like

$$Z_j\left(Z_1^qV_2^p\right)^r\frac{\partial}{\partial Z_j},\ V_j\left(V_1^qZ_2^p\right)^r\frac{\partial}{\partial V_j},\ V_1^{rq-1}Z_2^{rp}\frac{\partial}{\partial Z_1}$$

(and some other in the case of (4.13)).

Anyway, the sublattice (in $\mathbb{Z}^m$) of relations $(k, \omega) = 0$ is finitely generated and the P–D normal form has finitely many series (formal functional moduli) depending on finitely many monomials.

Assume now that the frequency ω_1 is such that no other frequency ω_j is a multiple of it:

$$\omega_j/\omega_1 \notin \mathbb{Z},\ j \geq 2. \tag{4.14}$$

Lemma 1. *Under the which assumption, the P–D normal form is such that the equations*

$$Z_2 = V_2 = \ldots = Z_m = V_m = 0$$

define a formal invariant manifold (invariant surface) W_1.

In this case, the P–D normal form restricted to W_1 *is like in Eq. (4.2) with the center conditions* $a_k = 0,\, k = 1, 2, \ldots.$

Proof. The obstacles to the existence of such an invariant surface are implied by the following terms:

$$Z_1^k V_1^l \frac{\partial}{\partial Z_j},\ Z_1^k V_1^l \frac{\partial}{\partial V_j},\ (j \geq 2)\,.$$

They correspond to the resonant relations

$$\mathrm{i}\omega_j = (k-l)\,\mathrm{i}\omega_1,\ \ -\mathrm{i}\omega_j = (k-l)\,\mathrm{i}\omega_1.$$

By condition (4.14), there are no such relations. □

Below we give two proofs of Theorem 3. Let us being with Lyapunov's argument. For this, the following standard is needed.

Lemma 2. *Consider the linear equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \lambda x + a(\theta)\,,\ x \in \mathbb{C},$$

where $a(\theta)$ *is a periodic function with period* 2π. *If* λ *is not an integer multiple of* $\mathrm{i} = \sqrt{-1}$, *then this equation has a unique periodic solution with the period* 2π *defined by*

$$x = \phi(\theta) = \left(\mathrm{e}^{-2\pi\lambda} - 1\right)^{-1} \int_{\theta-2\pi}^{\theta} \mathrm{e}^{\lambda(\theta-\alpha)} a(\alpha)\,\mathrm{d}\alpha. \tag{4.15}$$

Proof. The solutions of the initial value problem with the initial conditions $x(0) = x_0$ are affine in x_0. After the period of 2π, these solutions define the monodromy map $x_0 \longmapsto \mathcal{M}(x_0)$, which is also affine:

$$\mathcal{M}(x_0) = \Lambda x_0 + A,$$

where $\Lambda = \mathrm{e}^{2\pi\lambda}$. By the assumption, $\Lambda \neq 1$. Hence, the map $\mathcal{M}$ has a unique fixed point corresponding to a unique 2π−periodic solution.

If $\mathrm{Re}(\lambda) < 0$, then this solution is as follows:[§]

$$x = \phi(\theta) = \int_{-\infty}^{\theta} \mathrm{e}^{\lambda(\theta-\alpha)} f(\alpha)\,\mathrm{d}\alpha.$$

After partition of the integration domain $(-\infty, \theta]$ into segments of length 2π, we get Eq. (4.15), which is valid for all $\lambda \in \mathbb{C}\diagdown 2\pi\mathbb{Z}$. □

[§]This formula was used by Lyapunov in his proof of Theorem 2.

Lyapunov's proof of Theorem 3. Let $z_1 = re^{\mathrm{i}\theta}$ and $v_1 = \bar{z}_1 = re^{-\mathrm{i}\theta}$, where r, θ are polar coordinates. The phase curves of our system are defined by the equations

$$\begin{aligned} r' &= f(r, \theta, \tilde{z}, \tilde{v}), \\ z_j' &= \mathrm{i}\left(\omega_j/\omega_1\right) z_j + g_j(r, \theta, \tilde{z}, \tilde{v}), \\ v_j' &= -\mathrm{i}\left(\omega_j/\omega_1\right) v_j + h_j(r, \theta, \tilde{z}, \tilde{v}), \end{aligned} \tag{4.16}$$

$j = 2, 3, \ldots$, where $\tilde{z} = (z_2, \ldots, z_m)$, $\tilde{v} = (v_2, \ldots, v_m)$ and $' = \mathrm{d}/\mathrm{d}\theta$. We can assume that f begins with quadratic terms in $r, \tilde{z}, \tilde{v}$. For simplicity, assume that the linearization matrix of the complex system is diagonal; the triangular case requires only slight modification of the proof. Thus, also g_j and h_j begin with quadratic terms.

We look for $2\pi-$periodic solutions to system (4.16) as power series with periodic coefficients:

$$\begin{aligned} r &= \phi^{(1)}(\theta) c + \phi^{(2)}(\theta) c^2 + \ldots \\ z_j &= \psi_j^{(2)} c^2 + \psi_j^{(3)} c^3 + \ldots, \\ v_j &= \chi_j^{(2)} c^2 + \chi_j^{(3)} c^3 + \ldots. \end{aligned} \tag{4.17}$$

We set the following initial conditions:

$$\phi^{(1)}(0) = 1,\ \phi^{(2)}(0) = 0, \phi^{(3)}(0) = 0, \ldots; \tag{4.18}$$

thus, we have that $r(0) = c$ and $\phi^{(1)}(\theta) \equiv 1$ (which is $2\pi-$periodic). Substituting the Ansatz (4.17) into system (4.18), we obtain a recurrent system of equations for $\phi^{(k)}$, $\psi^{(k)}$ and $\chi^{(k)}$.

We have

$$\mathrm{d}\phi^{(k)}/\mathrm{d}\theta = \Phi^{(k)}(\theta),\ k \geq 2,$$

where the functions $\Phi^{(k)}(\theta)$ are defined inductively and should be $2\pi-$periodic. Thus,

$$\phi^{(k)}(\theta) = \int_0^\theta \Phi^{(k)}(\alpha)\, \mathrm{d}\alpha, \tag{4.19}$$

where $\int_0^{2\pi} \Phi^{(k)}(\alpha)\, \mathrm{d}\alpha$ should equal zero. The latter vanishing conditions are guaranteed by the assumptions of the theorem, i.e., the center conditions.

The equations for $\psi_j^{(k)}$ take the following form:

$$\mathrm{d}\psi_j^{(k)}/\mathrm{d}\theta = \lambda_j \psi_j^{(k)} + \Psi_j^{(k)}(\theta),$$

where $\lambda_j = \mathrm{i}\omega_j/\omega_1$ and $\Psi_j^{(k)}(\theta)$ are defined inductively and are $2\pi-$periodic; analogous equations hold for $\chi_j^{(k)}$, but with λ_j replaced with $-\lambda_j$. The latter equations $d\psi_j^{(k)}/d\theta$ are like the equation from Lemma 2 with $\lambda = \lambda_j \notin \mathrm{i}\mathbb{Z}$. So, they have unique $2\pi-$periodic solutions defined in Eq. (4.15); no center conditions are needed here.

The series in Eq. (4.17) is convergent due to the analyticity of the right-hand sides of the equations of Eq. (4.16), the compactness of the domain $[0, 2\pi]$ of definition of this system and the simplicity of the integral formulas (4.15) and (4.19). In [21], suitable estimates are given. □

Second proof of Theorem 3. Recall that we have the formal invariant surface W_1, defined in Lemma 2. We will show that it is analytic.

For this, we use the Poincaré return map $\mathcal{P}$ defined by solutions after the period of 2π of system (4.16). We have

$$\mathcal{P}(r, z_2, \ldots, z_m) = (\rho, \zeta_1, \ldots, \zeta_m), \tag{4.20}$$

where $\rho = \rho(r, z_2, \ldots, z_m) = r + \ldots$, $\zeta_j = \zeta_j(r, z_2, \ldots, z_m) = \mu_j z_j + \ldots$, $\mu_j = e^{2\pi i\omega_j/\omega_1} \neq 1$. The periodic orbits of the period $\approx 2\pi/\omega_1$ correspond to the fixed points of the Poincaré map. The set of fixed points of $\mathcal{P}$ lies in the curve

$$\zeta_2 - z_2 = \ldots = \zeta_m - z_m = 0, \tag{4.21}$$

where $\zeta_j - z_j = \mu_j z_j + \ldots$ with $\mu_j \neq 0$. By the implicit function theorem, Eq. (4.21) defines a real analytic curve.

By the center conditions, this curve is $\mathcal{P}$–invariant and defines an analytic smooth surface supporting a family of periodic solutions. □

The following example demonstrates the importance of the condition (4.14).

Example 3. *The system*

$$\dot{z}_1 = iz_1,\ \dot{v}_1 = -iv_1,\ \dot{z}_2 = 2iz_2 + z_1^2,\ \dot{v}_2 = -2iv_2 + v_1^2$$

has a general solution with the form

$$z_1 = c_1 e^{it},\ v_1 = d_1 e^{-it},\ z_2 = \left(c_2 + c_1^2 t\right) e^{2it},\ v_2 = \left(d_2 + d_1^2 t\right) e^{-2it}.$$

Only the solutions with $c_1 = d_1 = 0$ are periodic, with a period of π (not 2π).

Remark 6. *(a) We see that the existence of a 1–parameter family of periodic solutions can be deduced from the P–D normal form. We need the following:*

(i) formal invariant surface, and

(ii) vanishing of the series of Poincaré–Lyapunov coefficients after restriction of the P–D normal form to this surface.

The following questions arise naturally in this context.

1. Are there other 1–parameter families of periodic solutions?

2. Are there many–parameter families of periodic solutions?

Concerning the first question, there should exist other formal invariant surfaces with vanishing corresponding Poincaré–Lyapunov quantities.

The answer to the second question can be positive in some resonant cases. For example, this holds for the following system:

$$\dot{z}_j = ip_j z_j \cdot M,\ \dot{v}_j = -ip_j v_j \cdot M, \tag{4.22}$$

with the integer p_j and the integrating factor $M = \omega + \ldots$. Here, the typical solution is periodic with the period $\frac{2\pi}{p\omega}(1 + \ldots)$, where $p = \gcd(p_1, \ldots, p_m)$.

But, when

$$\dot{z}_j = ip_j z_j \cdot M_j,\ \dot{v}_j = -ip_v v_j \cdot M_j, \tag{4.23}$$

where the factors $M_j = \omega + \ldots$ are independent, then the motions take place on the invariant 'tori' $\{z_1 v_1 = C_1, \ldots, z_m v_m = C_m\}$. In the real domain, where $v_j = \bar{z}_j$, the motion on the generic torus is quasi-periodic, or periodic with a very long period.

(b) Note that, in the case considered in this subsection, the system has two analytic invariant manifolds, W^+ and W^-, corresponding to the partition of the eigenvalues' set into two subsets $\{\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_m\}$ *and* $\{-\mathrm{i}\omega_1, \ldots, -\mathrm{i}\omega_m\}$ *. The system restricted to either of these submanifolds has analytic P–D normal form (the systems are in the Poincaré domain).*

5. Hamiltonian examples

Here, we discuss Theorem 3 (as well as the Weinstein- and Moser-type theorems) in the context of Hamiltonian systems. Again, we assume the imaginary eigenvalues $\pm\mathrm{i}\omega_1, \ldots, \pm\mathrm{i}\omega_m$, $\omega_j > 0$, where $m = \frac{n}{2}$ is the number of degrees of freedom. The Hamiltonian functions considered in this section are of the form

$$H = H_2 + H_3 + \ldots, \tag{5.1}$$

where H_2 is like in Eq. (2.5) and the functions H_j are homogeneous of degree j. Such an H_2 is obvious in the case that all frequencies ω_j are pairwise different. The opposite case is discussed below.

Recall that, in the third Lyapunov theorem, there are two assumptions: the existence of a formal invariant surface W_1 associated with the eigenvalues $\pm\mathrm{i}\omega_1$, and the vanishing of the series of the Poincaré–Lyapunov focus quantities after restriction of the system to W_1 (see Remark 6 above). But, in the Hamiltonian case, the situation is simpler.

Proposition 1. *If there exists a formal invariant surface W_1, then all of the focus quantities for the restriction of the system to W_1 vanish.*

Proof. Indeed, H restricted to W_1 is a formal first integral for the restricted system. So, in the real domain, all phase curves of that system are closed. □

People have tried to improve the third Lyapunov theorem in the Hamiltonian case. Recall that Weinstein ([35, Theorem 2.1] and Theorem 4 above) proved that, if H_2 is positive definite, i.e., all $\epsilon_j = 1$, then there exist at least $m = \frac{n}{2}$ families of periodic solutions.

Moreover, Moser, in [23, Theorem 4], claimed that, if H_2 restricted to the maximal linear invariant subspace E supporting periodic solutions of period $T = 2\pi/\omega$ of the linearized system is positive definite, then 'on each energy surface $H = \varepsilon^2$, the number of periodic orbits is at least $\frac{1}{2}\dim E$'. Recall that that result is wrong, but that some version of Moser's statement is valid provided that H_2 is positive definite (Theorem 5).

Of course, the number m from the Weinstein theorem cannot be improved because the Hamiltonian system

$$\dot{z}_j = \mathrm{i}\omega_j z_j \cdot M_j\left(\left|z_j\right|^2\right), \ j = 1, \ldots, m,$$

with different factors M_j, has m families of periodic solutions with independently varying periods $T_j = T_j\left(\left|z_j\right|^2\right) \approx 2\pi/\omega_j$. Compare with Remark 6 above.

Recall also Example 3 above, where we observe 1 : 2 resonance between the frequencies and only one family of periodic solutions; but, there, the system is not Hamiltonian.

Example 4. *In the case of the Hamiltonian*[¶]

$$H = \frac{1}{2}\left\{|z_1|^2 - k|z_2|^2 + z_1^k z_2 + \bar{z}_1^k \bar{z}_2\right\},\ k \geq 2, \tag{5.2}$$

from [23, Example 2], we deal with the $k:1$ resonance, but with indefinite quadratic part. The corresponding equations for z_j are

$$\dot{z}_1 = \mathrm{i}z_1 + \mathrm{i}k\bar{z}_1^{k-1}\bar{z}_2,\ \dot{z}_2 = -k\mathrm{i}z_2 + \mathrm{i}\bar{z}_1^k.$$

Of course, the plane $W_2 = \{z_1 = v_1 = 0\}$ is invariant and supports a family of periodic solutions with the period $2\pi/k$.

But, the Lyapunov-type function $L = \frac{1}{2\mathrm{i}}\left(z_1^k z_2 - \bar{z}_1^k \bar{z}_2\right)$ satisfies

$$\dot{L} = |z_1|^{2(k-1)}\left(|z_1|^2 + k^2|z_2|^2\right),$$

which is positive in the domain $\{z_1 \neq 0\}$. It prevents the existence of other periodic solutions.

Surprisingly, this example contradicts Moser's original statement [23, Theorem 4]. Indeed, we have two invariant subspaces of the linear system supporting periodic solutions: $E_2 = W_2 = \{(0, z_2)\}$ and $E_1 = \{(z_1, 0)\}$. We have that $H_2|_{E_2} = -\frac{k}{2}|z_2|^2 \leq 0$ and $H_2|_{E_1} \geq 0$. But, only periodic solutions from E_2 are 'extended' to periodic solutions of the whole system.

We complete this example by analysis of some return map associated with the polar coordinates in the z_1-plane. Thus, we put

$$z_1 = r\mathrm{e}^{\mathrm{i}\theta},\ z_2 = w_2\mathrm{e}^{-\mathrm{i}k\theta}. \tag{5.3}$$

We have

$$\begin{aligned}\dot{r} &= \frac{1}{r}\mathrm{Re}\,(\dot{z}_1\bar{z}_1) = \frac{1}{r}\mathrm{Re}\left(\mathrm{i}r^2 + \mathrm{i}kr^k\bar{w}_2\right) = kr^{k-1}\mathrm{Im}w_2,\\ \dot{\theta} &= \mathrm{Im}\,(\dot{z}_1/z_1) = 1 + kr^{k-2}\mathrm{Re}w_2.\end{aligned}$$

Next, $\dot{w}_2 - \mathrm{i}kw_2\dot{\theta} = -\mathrm{i}kw_2 + \mathrm{i}r^k$ gives

$$\dot{w}_2 = \mathrm{i}r^{k-2}(r^2 + k^2w_2\mathrm{Re}w_2).$$

We get the system

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = k\frac{r^{k-1}\mathrm{Im}w_2}{1 + kr^{k-2}\mathrm{Re}w_2},\ \frac{\mathrm{d}w_2}{\mathrm{d}\theta} = \mathrm{i}r^{k-2}\frac{r^2 + k^2w_2\mathrm{Re}w_2}{1 + kr^{k-2}\mathrm{Re}w_2}, \tag{5.4}$$

which we call the return system. The twisted Poincaré map $(r, w_2) \longmapsto \mathcal{P}(r, w_2)$ is defined via the evaluation of solutions at the 'time' $\theta = 2\pi$ with the initial value (r, w_2) at the 'time' $\theta = 0$. The periodic solutions with the period $\approx 2\pi$ and outside of $W_2 = \{r = 0\}$ correspond to the singular points of the latter system (5.4) with $r \neq 0$. Thus, we have

$$\mathrm{Im}w_2 = 0,$$

i.e., $w_2 = \mathrm{Re}w_2$, and

$$r^2 + k^2w_2^2 = 0. \tag{5.5}$$

The latter equation does not have nontrivial solutions.

[¶]In [35], we find another Hamiltonian $H = \frac{1}{2}|z_1|^2 - |z_2| + \frac{1}{2}\mathrm{Im}\left(z_1^2 z_2\right)$ (due to Siegel [29]), also with only one family of periodic solutions.

The following example demonstrates that, in the case of a positively definite Hamiltonian, the situation is quite different.

Example 5. *(a) Consider the positive definite Hamiltonian*

$$H = \frac{1}{2}\left\{k\left|z_1\right|^2 + \left|z_2\right|^2 + z_1\bar{z}_2^k + \bar{z}_1 z_2^k\right\}, \tag{5.6}$$

generating the system

$$\dot{z}_1 = \mathrm{i}kz_1 + \mathrm{i}z_2^k,\ \dot{z}_2 = \mathrm{i}z_2 + \mathrm{i}kz_1\bar{z}_2^{k-1}. \tag{5.7}$$

Of course, the plane $W_1 = \{z_2 = 0\}$ is invariant and has a a family of periodic solutions with the period $2\pi/k$.

(b) To look for periodic solutions with the period $\approx 2\pi$, use the polar-type coordinates

$$z_2 = r\mathrm{e}^{\mathrm{i}\theta},\ z_1 = w_1\mathrm{e}^{\mathrm{i}k\theta}; \tag{5.8}$$

compare with Eq. (5.3). We get the return system

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -k\frac{r^{k-1}\mathrm{Im}w_1}{1 + kr^{k-2}\mathrm{Re}w_1},\ \frac{\mathrm{d}w_1}{\mathrm{d}\theta} = \mathrm{i}r^{k-2}\frac{r^2 - k^2w_1\mathrm{Re}w_1}{1 + kr^{k-2}\mathrm{Re}w_1}. \tag{5.9}$$

Again, the closed phase curves of the period $\approx 2\pi$ outside of $W_1 = \{r = 0\}$ correspond to the singular points of the vector field given by Eq. (5.9). We find that

$$w_1 = \mathrm{Re}w_1 = \pm r/k. \tag{5.10}$$

Therefore, we have three families of periodic solutions. The first family $\{z_1 = c\mathrm{e}^{\mathrm{i}kt}, z_2 = 0\}$ spans the invariant plane W_1, but none of the two other families of closed phase curves $\left\{z_1 = \pm(r/k)\mathrm{e}^{\mathrm{i}k\theta}, z_2 = r\mathrm{e}^{\mathrm{i}t\theta}\right\}$ span a smooth surface. These real surfaces (in $\mathbb{R}^4$) take the form

$$kz_1\left|z_2\right|^{k-1} = \pm z_2^k$$

and are singular; in $\mathbb{C}^4$, we get one complex singular surface $\{k^2z_1^2v_2^{k-1} = z_2^{k+1}, k^2v_1^2z_2^{k-1} = v_2^{k+1}\}$.

The number of families of periodic solutions is greater than in the Weinstein theorem.

(c) To explain the this difference we use the fact that the Hamiltonian system in the Birkhoff normal form is invariant with respect to the $\mathbb{S}^1$−action. In the present case, the action is as follows:

$$(z_1, z_2; \phi) \longmapsto \left(\mathrm{e}^{\mathrm{i}k\phi}z_1, \mathrm{e}^{\mathrm{i}\phi}z_2\right); \tag{5.11}$$

it is symplectic and generated by a corresponding Hamiltonian function (the momentum map), namely, $F = H_2 = \frac{1}{2}\left(k\left|z_1\right|^2 + \left|z_2\right|^2\right)$ (which is a first integral for system (5.7)). The symplectic reduction procedure (see [4]) says that, for each level surface $M_f = \{F = f\}$ (which is $\mathbb{S}^1$−invariant), the quotient space $N_f = M_f/\mathbb{S}^1$ with respect to the $\mathbb{S}^1$−action acquires a symplectic structure and we get a Hamiltonian vector field Y_f in this quotient space. The singular points of the latter vector field correspond to periodic solutions to system (5.7).

In this case, the hypersurface is a 3–sphere $M_f \simeq \mathbb{S}^3$, $f > 0$, and the quotient space N_f turns out to be the weighted projective space, $N_f \simeq \mathbb{P}_{k,1}$ (see Section 10.1). The weighted projective space $\mathbb{P}_{k,1}$

is homeomorphic with the standard complex projective line $\mathbb{P}_{k,1} \simeq \mathbb{P}^1$. *The natural charts in* N_f *are* $\zeta = z_2^k/z_1$ *and* $\eta = 1/\zeta = z_1/z_2^k$; *thus,* $z_1 = \eta z_2^k$ *and* $z_2 = (\zeta z_1)^{1/k}$.

In these charts, we get the systems

$$\dot{\zeta} = \mathrm{i}k^2 |\zeta|^{2(k-1)/k} |z_1|^{2(k-1)/k} - \mathrm{i}\zeta^2, \ \ \dot{\eta} = \mathrm{i}\left(1 - k^2 |z_2|^{2(k-1)} \eta^2\right). \tag{5.12}$$

Note that, in the ζ−*chart on* N_f, *we have that* $k|z_1|^2 + |\zeta|^{2/k} |z_1|^{2/k} = 2f$; *so, for* $\zeta \to 0$, *we find that* $|z_2|$ *is separated from zero and the first term on the right-hand side of the equation for* $\dot{\zeta}$ *is dominating. It follows that the latter system(s) (5.12) has three singular points:* $\zeta = 0$ *(with index* 0*) and* $\eta = \pm 1/k|z_2|^{k-1}$ *(both with index* 1*). As the Euler characteristic* $\chi\left(\mathbb{P}^1\right) = 2$, *this agrees with the Poincaré–Hopf formula.*

Note also that the equilibrium points of Y_f *correspond to the critical points of* H *as a function on* N_f. *This amounts to critical circles of the function* $G = H - F = \frac{1}{2}\left(z_1\bar{z}_2^k + \bar{z}_1 z_2^k\right)$, *restricted to* M_f. *With* λ *and the Lagrange multiplier, we get the equations*

$$\bar{z}_2^k = \lambda k \bar{z}_1, \ \ k\bar{z}_1 z_2^{k-1} = \lambda \bar{z}_2. \tag{5.13}$$

One solution is $z_2 = 0$, $\lambda = 0$ *and* $k|z_1|^2 = 2f$. *For the other solutions, we have* $\lambda^2 = |z_2|^{2(k-1)}$; *thus,* $\lambda = \pm |z_2|^{k-1}$, $z_1 = \pm\frac{1}{k}(z_2/|z_2|)^{k-1} z_1$ *and* $|z_2|^2 = 2f/\left(1 + 1/k^2\right)$.

(d) Finally, the Hamiltonian system (5.7) is completely integrable, with two independent first integrals (in involution). By the Liouville–Arnold theorem [3], the common level surfaces $\{F = f, \ G = g\}$, *which are compact, are tori (if smooth). There are also the action–angle variables, which are our objectives.*

Let $z_{1,2} = r_{1,2}\mathrm{e}^{\mathrm{i}\alpha_{1,2}}$ *and* $\beta = k\alpha_2 - \alpha_1$. *We have the differential equations*

$$\begin{aligned} \dot{r}_1 &= -r_2^k \sin\beta, \qquad \dot{r}_2 = kr_1 r_2^{k-1}\sin\beta, \\ \dot{\alpha}_1 &= k + r_1^{-1} r_2^k \cos\beta, \ \dot{\alpha}_2 = 1 + kr_1 r_2^{k-1}\cos\beta, \ . \end{aligned}$$

The levels of the first integrals take the form

$$kr_1^2 + r_2^2 = 2f, \ \ r_1 r_2^k \cos\beta = g.$$

This leads to $r_1 = g/\left(r_2^k\cos\beta\right)$, $\cos^2\beta = kg^2/r_2^{2k}\left(2f - r_2^2\right)$,

$$\sin^2\beta = \frac{2f\rho^k - \rho^{k+1} - kg^2}{\rho^k(2f - \rho)},$$

where $\rho = r_2^2$. *Finally, we arrive at the system*

$$\dot{\rho} = \pm 2k^{-1/2}\sqrt{2f\rho^k - \rho^{k+1} - kg^2}, \tag{5.14}$$

$$\dot{\alpha}_2 = 1 + kg/\rho. \tag{5.15}$$

We have an oval $\Gamma = \left\{k\sigma^2 = 4\left(2f\rho^k - \rho^{k+1} - kg^2\right)\right\}$ *in the* $(\rho,\sigma) = (\rho,\dot{\rho})$−*plane defined by Eq. (5.14). This is one of the circles generating the invariant torus. The corresponding Liouville–Arnold angle is*

$$\phi_1(\rho,\sigma) = \frac{2\pi}{T_1}\int_\gamma \frac{\mathrm{d}\rho}{\sigma}, \tag{5.16}$$

where $\gamma = \gamma(\rho, \sigma)$ *is a path in* Γ *from* $(\rho_0, \sigma_0) = (\rho_0, 0)$ *to* (ρ_0, σ) *(*ρ_0 *is the left root of the equation* $2f\rho^k - \rho^{k+1} = kg^2$*) and* $T_1 = \oint_\Gamma \frac{d\rho}{\sigma}$ *is the corresponding period. It is the period of the solution* $\rho = R(t)$ *to Eq. (5.14).*

Next, the second generating circle is parametrized by the angle α_1*. The corresponding solution to Eq. (5.15) takes the form* $\alpha_1 = A(t) = \alpha_0 + t + kg\int_0^t \frac{ds}{R(s)} = \alpha_1^{(0)} + t + kg\int_\gamma \frac{d\rho}{\rho\sigma}$*. We have the second period* $T_2 = A(T_1) - A(0) = T_1 + kg\oint_\Gamma \frac{d\rho}{\rho\sigma}$*. The second Liouville–Arnold angle equals*

$$\phi_2 = \frac{T_1}{T_2}\phi_1 + \frac{2\pi}{T_2}\left(\alpha_1 - \int_\gamma \left(1 - \frac{kg}{\rho}\right)\frac{d\rho}{\sigma}\right). \tag{5.17}$$

One can find corresponding action variables $I_j = I_j(F, G)$ *from the condition* $dI_1 \wedge d\phi_1 + dI_2 \wedge d\phi_2 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$*. For this, one has to solve some first order PDE; we omit the details.*

Consider now the situation with $1:1$ resonance and, more generally, with $1:1:\cdots:1$ resonances. Here, non-diagonal cells for the linear systems associated with homogeneous quadratic Hamiltonians H_2 are expected. Canonical forms of such Hamiltonians are given in [3, Appendix 6].

Probably, the following easy statement was a motivation of Moser's work [23].

Proposition 2. *In the case of* $1:1:\ldots:1$ *resonance, i.e., with* $\omega_1 = \ldots = \omega_m$*, and of definite* H_2 *(positive definite or negative definite), the corresponding linearization matrix is diagonalizable.*

Proof. If there existed a pair of Jordan cells, then the general solution of the corresponding linear system would be unbounded. But, this would contradict the compactness of the level hypersurfaces $\{H_2 = h\}$. □

It is natural to look for families of periodic solutions in the cases with a positive definite H_2, i.e., with a diagonalizable linear part. How many such families should exist?

Consider first the positive definite H_2 and a homogeneous perturbation in the Birkhoff normal form. Thus, we have

$$H = \frac{1}{2}|z|^2 + H_{2d} = \frac{1}{2}z \cdot v + H_{2d}, \tag{5.18}$$

where $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and H_{2d} is a homogeneous polynomial depending only on the monomials $z_j\bar{z}_k$, $j, k = 1, \ldots, m$. Here, we have some freedom. Namely, we can apply linear changes of the variables z_j which leave H_2 invariant. The corresponding group is $U(m)$; this group preserves the Hermitian product $(z, \zeta) \longmapsto z \cdot \bar{\zeta}$, whose real part is the scalar product and whose imaginary part defines the symplectic form. Using these changes, one can reduce the our situation to the case with m invariant complex planes.

(Note that, in the case with the indefinite quadratic part, $H_2 = \frac{1}{2}|x|^2 - \frac{1}{2}|y|^2$, where $x = \left(x_1, \ldots, x_p\right) \in \mathbb{C}^p$ and $y = \left(y_1, \ldots, y_q\right) \in \mathbb{C}^q$, the corresponding group is $U(p, q)$ and the Birkhoff resonant terms depend on $x_j\bar{x}_k, y_j\bar{y}_k, x_jy_k$ and $\bar{x}_j\bar{y}_k$. Also, here, $m = p + q$ invariant planes can be found for $H_2 + H_{2d}$.)

Example 6. *A general quartic Hamiltonian with the form of Eq. (5.16) for* $m = 2$ *with two invariant complex planes is of the form*

$$H = \frac{1}{2}|z|^2 + \frac{a}{4}|z_1|^4 + \frac{b}{2}|z_1|^2|z_2|^2 + \frac{c}{4}|z_2|^4 + d \cdot \mathrm{Re} z_1^2\bar{z}_2^2, \tag{5.19}$$

with real constants $a, \dots, d$. The corresponding system becomes

$$\dot{z}_1 = \mathrm{i}z_1\left(1 + a|z_1|^2 + b|z_2|^2\right) + 2\mathrm{i}d\bar{z}_1 z_2^2,\ \dot{z}_2 = \mathrm{i}\left(1 + b|z_1|^2 + c|z_2|^2\right) + 2\mathrm{i}d z_1^2\bar{z}_2.$$

We study the twisted Poincaré map using the substitutions $z_1 = r\mathrm{e}^{\mathrm{i}\theta}$ and $z_2 = w\mathrm{e}^{\mathrm{i}\theta}$. It leads to the equations

$$\dot{r} = -2dr\mathrm{Im}w^2,\ \dot{w} = \mathrm{i}\left\{(b-a)r^2 w + (c-b)|w|^2 w + 2d\left(rw^2 - w\mathrm{Re}w^2\right)\right\}$$

(we do not write equations for $\mathrm{d}r/d\theta$ and $\mathrm{d}w/d\theta$).

The periodic solutions with period the $\approx 2\pi$ outside of $r = 0$ are given by $\mathrm{Im}w^2 = 0$ and $\dot{w} = 0$. Thus, either (1) $w = \mathrm{Re}w$, $|w|^2 = w^2$; or (2) $w = \mathrm{i}\cdot\mathrm{Im}w$, $|w|^2 = -w^2$. Let $\lambda = w/r$.

In case (1), we get the equation

$$r^2 w\left[b - a + 2d\lambda + (c - b - 2d)\lambda^2\right] = 0,$$

and, in case (2), we get

$$r^2 w\left[b - a + 2d\lambda + (b - c - 2d)\lambda^2\right] = 0.$$

It follows that, for generic values of the parameters, besides the solutions $r = 0$ and $w = 0$ (corresponding to periodic solutions of the Hamiltonian system in the invariant complex planes), we have four additional solutions. Thus, the total number of periodic solutions in the generic case is six.

On the other hand, by analogy with the previous example, the symplectic reduction, with $F = H_2$ and $G = H_{2d}$, leads to Hamiltonian vector field Y_f on $N_f \simeq \mathbb{P}^1$ with $\chi\left(N_f\right) = 2$. One can check that the additional two pairs of singular points of Y_f have opposite indices.

The above agrees with the following result of van Straten [34].

Theorem 14. *We have*

$$\sum_{m\geq 1} M(m-1, d)\,T^{m-1} = (1-T)^{-3/2}\left[1 - (2d-1)^2 T\right]^{-1/2}, \tag{5.20}$$

where, on the left-hand side stands the generating function for the numbers $M(m-1, d)$ of periodic modes for Hamiltonian systems with a generic Hamiltonian function in the Birkhoff normal form given by Eq. (5.18).

For $d = 2$, the right-hand side of Eq. (5.20) becomes $1 + 6T + \dots$, which agrees with Example 6; in fact, there are no such examples in [34].

Finally, we have the following Hamiltonian systems without periodic solutions for the expected period.

Example 7. *Assume the following Poisson brackets for the complex variables z_j:*

$$\left\{z_i, z_j\right\} = \left\{\bar{z}_i, \bar{z}_j\right\} = 0, \quad \{z_1, \bar{z}_2\} = \{\bar{z}_1, z_2\} = 2.$$

The Hamiltonian function equals

$$H = \frac{1}{2}\left\{|z_1|^2 + \mathrm{i}(z_1\bar{z}_2 - \bar{z}_1 z_2) + a|z_2|^4\right\}$$

and generates the system

$$\dot{z}_1 = \mathrm{i}z_1 + 4az_2 |z_2|^2, \quad \dot{z}_2 = \mathrm{i}z_2 - z_1.$$

Note that, for $a = 0$*, we get a nilpotent linear system with invariant plane* $\{z_1 = 0\}$ *that supports a family of periodic solutions; this Hamiltonian can be found also in [3, 7]. But, we assume that* $a < 0$*; hence, the planes* $\{z_1 = 0\}$ *and* $\{z_2 = 0\}$ *are not invariant.*

One should look for periodic solutions for the period $\approx 2\pi$*. Hence, we make the natural change (look at the previous examples):* $z_1 = r\mathrm{e}^{\mathrm{i}\theta}$, $z_2 = w\mathrm{e}^{\mathrm{i}\theta}$ *with* $r \geq 0$*. We get the equations*

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{4ar|w|^2 \operatorname{Re} w}{r + 4a|w|^2 \operatorname{Im} w}, \quad \frac{\mathrm{d}w}{\mathrm{d}\theta} = -\frac{r^2 + 4\mathrm{i}a|w|^2 \operatorname{Im} w}{r + 4a|w|^2 \operatorname{Im} w}.$$

The equations for the equilibrium points are given by $\operatorname{Re} w = 0$ *(i.e.,* $w = \mathrm{i}\zeta$*) and* $r^2 + 4ia|w|^2 \operatorname{Im} w = r^2 - 4a\zeta^4 = 0$.

We refer also to previous works [12, 33] devoted to bifurcations of Hamiltonian systems in the 1:1 resonant case.

In the last example from [22, Example 9.2], the linear part is diagonal.

Example 8. *Let*

$$H = \frac{1}{2}\left\{|z_1|^2 - |z_2|^2 + 2|z|^2 \operatorname{Re}(z_1 z_2)\right\}.$$

It generates the system

$$\dot{z}_1 = \mathrm{i}z_1 (1 + 2\operatorname{Re}(z_1 z_2)) + \mathrm{i}|z|^2 \bar{z}_2, \quad \dot{z}_2 = -\mathrm{i}z_2 (1 - 2\operatorname{Re}(z_1 z_2)) + \mathrm{i}|z|^2 \bar{z}_1.$$

One finds that

$$\mathrm{d}\operatorname{Im}(z_1 z_2)/\mathrm{d}t = 2[\operatorname{Re}(z_1 z_2)]^2 + |z|^4,$$

which excludes the existence of nontrivial periodic solutions.

6. Additional properties of the P–D normal form

Again, assume an analytic system (1.4) with the eigenvalues $\pm \mathrm{i}\omega_j$, $\omega_j > 0$. Recall the equivalence relation (1.9) on the set $\{\omega_1, \ldots, \omega_m\}$ of frequencies. For each equivalence class C_ν, we have a linear subspace $\mathcal{E}_\nu$ that is invariant for the linear part $\dot{x} = Ax$ of system (1.4). In fact, we can say more.

Proposition 3. *For each class* C_ν*, system (1.4) has an invariant analytic submanifold* $\mathcal{V}_\nu$ *tangent to* $\mathcal{E}_\nu$ *at the origin.*

Proof. First, it is rather obvious that there exists such an invariant manifold at the formal level. To prove its analyticity, we use the Poincaré return map (4.20), introduced in the second proof of Theorem 3.

Let $z_1, \bar{z}_1, \ldots, z_k, \bar{z}_k$ be the linear eigenfunctions associated with one equivalence class, say, C_1, and let $z_{k+1}, \bar{z}_{k+1}, \ldots$ be the remaining eigenfunctions.

We put $z_1 = re^{i\theta}$ and get Eq. (4.16) for the phase curves, as graphs of functions of θ. We have that $\omega_j = \left(p_j/q_j\right)\omega_1$ and $\gcd(p_j, q_j) = 1$. Let $\Theta = 2\pi\mathrm{gcm}\left(q_2, \ldots, q_k\right)$. The solutions to Eq. (4.16) after 'time' $\theta = \Theta$ define the Poincaré return map $\mathcal{P} : (r, z_2, \ldots, z_m) \longmapsto (\rho, \zeta_2, \ldots, \zeta_m)$ (like in Eq. 4.20)).

The linear part of this map has the form

$$(r, z_2, \ldots, z_m) \longmapsto (r, z_2, \ldots, z_k, \mu_{k+1} z_{k+1}, \ldots, \mu_m z_m),$$

$\mu_j = e^{i\Theta_j} \neq 1$ for $j > k$; in fact, μ_j are is not a root of unity.

Then, the equations $\zeta_{k+1} - z_{k+1} = \ldots = \zeta_m - z_m = 0$ define an analytic submanifold $\widetilde{V}_\nu$. It generates the submanifold $V_\nu = \bigcup_{\theta\in[0,1\pi]} g_0^\theta(\widetilde{V}_\nu)$, where $\{g_\vartheta^\theta\}$ is the 2–parameter family of diffeomorphisms defined by solutions to Eq. (4.6). □

Therefore, our problem is reduced to the case when there is only one equivalence class, i.e., that all of the frequencies are rationally related. Let us order these frequencies like in Eq. (1.10), i.e.,

$$\omega_1 = \ldots = \omega_{k_1} > \omega_{k_1+1} = \ldots = \omega_{k_1+k_2} > \ldots > \omega_{k_1+\ldots+k_{r-1}+1} = \ldots = \omega_{k_1+\ldots+k_r}.$$

Proposition 4. *In this situation, there exists a series $V_1 \subset V_2 \subset \ldots \subset V_{r-1} \subset \left(\mathbb{R}^{2m}, 0\right)$ of analytic invariant submanifolds (of dimensions $k_1, k_1 + k_2, \ldots$, tangent at the origin to linear subspaces) associated with corresponding groups of frequencies.*

Proof. It essentially repeats the previous proof. We find V_{r-1}; other submanifolds are obtained inductively.

Let $z_1, \bar{z}_1, \ldots, z_l, \bar{z}_l$, $l = k_1+\ldots+k_{r-1}$ be the eigenfunctions associated with the eigenvalues $\pm i\omega_1, \ldots \pm i\omega_l$ and $z_{l+1}, \bar{z}_{l+1}, \ldots, z_m, \bar{z}_m$ be the remaining eigenvalues (associated with $\pm i\omega_m$). We put $z_l = re^{i\theta}$ and get the equations

$$\frac{dr}{d\theta} = \ldots, \quad \frac{dz_j}{d\theta} = i\frac{\omega_j}{\omega_l} z_j + \ldots,$$

$j \neq l$, for phase curves. Their solutions after the 'time' $\theta = 2\pi$ define the Poincaré map

$$\begin{aligned} \mathcal{P} \quad &: \quad (r, z_1, \ldots, z_{l-1}, z_{l+1}, \ldots z_m) \longmapsto (\rho, \zeta_1, \ldots, \zeta_{l-1}, \zeta_{l+1}, \ldots \zeta_m) \\ &= \quad (r + \ldots, z_1 + \ldots, z_{l-1} + \ldots, \mu z_{l+1} + \ldots, \ldots, \mu z_m + \ldots), \end{aligned}$$

where the coefficient $\mu \neq 1$ is a root of unity. The equations $\zeta_{l+1} - z_{l+1} = \ldots = \zeta_m - z_m = 0$ define a submanifold which is spread to the submanifold V_{r-1}. □

From now on, we focus our attention on the case with only one equivalence class for the frequencies. Thus, we can write

$$\omega_j = p_j\omega_0, \quad p_j \in \mathbb{N}, \quad \gcd\left(p_1, \ldots, p_m\right) = 1. \tag{6.1}$$

Proposition 5. *In the case of Eq. (6.1), the Poincaré–Dulac normal form is invariant with respect to the following action of the circle $\mathbb{S}^1$:*

$$z = (z_1, \ldots, z_m) \longmapsto \sigma^\phi(z) = \left(e^{ip_1\phi} z_1, \ldots, e^{ip_m\phi} z_m\right), \quad 0 \leq \phi \leq 2\pi. \tag{6.2}$$

Proof. Recall that a term $U = z_1^{k_1} \cdots z_m^{k_m} \overline{z}_1^{k_1} \cdots \overline{z}_m^{l_m} \partial/\partial z_j$ is resonant in the P–D normal form if and only if the resonant relation $(k_1 - l_1)\,\omega_1 + \ldots (k_m - l_m)\,\omega_m = \omega_j$ holds. But, the result of change, given by Eq. (6.2), on U is as follows:

$$U \longmapsto \left(\rho^{\phi}\right)^{*} U = \exp\left\{\mathrm{i}\left[(k_1 - l_1)\,p_1 + \ldots (k_m - l_m)\,p_m - p_j\right]\right\} U = U.$$

Only such terms are $\mathbb{S}^1$−invariant. □

7. Theorem 8 and its proof

Recall that the assumptions of the three Lyapunov's theorems are two-fold. One has an assumption about the eigenvalues of the linear part; it implies the existence of a formal invariant smooth surface (it is rather irrelevant in Theorem 1). The second assumption means that the focus quantities of the system restricted to the formal invariant surface vanish; it amounts to the existence of a family of periodic solutions at a formal level. We underline that this vanishing condition admits a precise definition in terms of the P–D normal form; but, there are other approaches.

To provide a precise assumption of Theorem 8, we definitely need the P–D theorem. It roughly states that the P–D normal form predicts the existence of a 1–parameter family of periodic solutions for the period $\approx 2\pi/\omega_l$ at a formal level. Consider the equivalence class of frequencies which contains ω_l. By Proposition 3, we can focus our attention on this class, i.e., that we have the situation of Eq. (1.10). By Proposition 4, there is an invariant submanifold associated with the frequencies ω_j such that $\omega_j \geq \omega_l$. Therefore, we can assume that

$$\omega_1 \geq \omega_2 \geq \ldots \geq \omega_l, \tag{7.1}$$

i.e., $m = l$, besides property (1.10).

Assume the Poincaré–Dulac normal form (with the coordinates Z_j). Following Examples 4 and 5, introduce the change

$$Z_l = R\mathrm{e}^{\mathrm{i}\Theta}, \;\; Z_j = W_j \mathrm{e}^{\mathrm{i}p_j\Theta/p_l}, \;\; (j < l). \tag{7.2}$$

Lemma 3. *We have the following return system for phase curves:*

$$\frac{\mathrm{d}R}{\mathrm{d}\Theta} = \Pi\left(R, W, \overline{W}\right) = \ldots, \;\; \frac{\mathrm{d}W_j}{\mathrm{d}\Theta} = \Lambda_j\left(R, W, \overline{W}\right) = \ldots, \;\; (j < l), \tag{7.3}$$

where the right-hand sides do not depend on Θ *and the dots mean nonlinear terms.*

Proof. We have that $\dot{\Theta} = \omega_l + \ldots$, $\dot{R} = \ldots$ and $\dot{W}_j = \mathrm{i}\omega_j W_j + \ldots$, $(j < l)$, where the right-hand sides do not depend on Θ, due to the P–D normal form. □

Definition 4. *We say that the vector field given by (1.4) satisfies a center condition if system (7.3) has a formal curve* Γ, *staring at* $(R, W) = (0, 0)$, *of (non-isolated) equilibrium points.*

Theorem 8 states that, in this case, there exists a genuine continuous 1–parameter family of periodic solutions to system (1.4).

Proof of Theorem 8. Analogously to the change given by Eq. (7.2), introduce the change

$$z_m = r e^{i\theta}, \;\; z_j = w_j e^{i p_j \theta / p_m}, \;\; (j < m).$$

We get a system of equations $\frac{dr}{d\theta} = \ldots, \frac{dw_j}{d\theta} = \ldots$ (the return system) with analytic right-hand sides. The periodic solutions of the period $\approx 2\pi/\omega_m$ correspond to the fixed points of the twisted Poincaré map $\mathcal{P}$, which is defined via solutions to the latter system $dr/d\theta$ and $dw_j/d\theta$ after the 'time' $\theta = 2\pi$.

The equation $\mathcal{P}(r, w) - (r, w) = 0$ has a zero locus along an analytic curve γ which is formally defined as the curve Γ from Definition 4 (after the change $(R, W) \longmapsto (r, w)$). □

Remark 7. *The statement of Theorem 8 can be generalized to the case when system (7.3) has a formal singular locus* Γ *of higher dimension, say, d. Then, the equation* $\mathcal{P} - Id = 0$ *has an analytic zero locus* γ *of dimension d; it corresponds to a d–parameter family of periodic solutions for the period* $\approx 2\pi/\omega_m$.

8. Systems with a first integral

The assumptions of Theorems 1, 2, 3 and 8 involve an infinite number of conditions, related to either the vanishing of all focus quantities after restriction to a formal invariant plane or the existence of a curve of non-isolated singular points for system (7.3). These phenomena have infinite codimension.

In this section, we avoid the infinite number of conditions by imposing an additional restriction to the class of considered vector fields and the existence of an analytic first integral with a regular leading part. This restriction is also of infinite codimension, but it is natural in some applications.

8.1. Generalization of Theorem 7

Theorem 15. *Assume that analytic system (1.4) has pure imaginary eigenvalues* $\pm i\omega_j$, $\omega_j > 0$, *and that it has a first integral F such that* D^2F *is definite on a linear subspace* E_{j_0} *(of dimension* ≥ 2*) corresponding to one pair of eigenvalues* $\pm i\tilde{\omega}_{j_0}$ *such that, for any other frequency* $\tilde{\omega}_j \neq \tilde{\omega}_{j_0}$, *one has* $\tilde{\omega}_j/\tilde{\omega}_{j_0} \notin \mathbb{Z}$. *Then, there exists at least one 1–parameter family of periodic solutions for the period* $\approx 2\pi/\tilde{\omega}_{j_0}$.

Proof. By Proposition 4, we reduce the situation to one equivalence class of frequencies containing ω_{j_0}; thus, we can assume that the class C_1 and $\tilde{\omega}_{j_0} = \tilde{\omega}_1$ in Eq. (1.10). By Proposition 5, we can assume $E_1 = V_1 = \mathbb{R}^{2m}$, $q = m$ and there is a definite first integral $F(z, \bar{z})$, say, with positive definite $D^2F(0, 0)$.

Assume first that the P–D normal form is analytic. So, let the system be in this form; denote the corresponding vector field X (with variables $z_j, \bar{z}_j$). Recall that such a system is invariant with respect to the corresponding $\mathbb{S}^1$–action $\{\rho^\phi\}$ (Proposition 6). By averaging,

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\sigma^\phi\right)^* F d\phi; \tag{8.1}$$

we get that the first integral can be chosen also as $\mathbb{S}^1-$invariant: $F \circ \sigma^{\phi} = F$.

The level hypersurfaces $M_f = \{F(z, \bar{z}) = 0\}$, $f > 0$, are diffeomorphic with a $(2m-1)-$dimensional sphere. They are invariant for the vector field X. Like in [23], we obtain vector fields Y_f on the quotient varieties $N_f = M_f/\mathbb{S}^1$. The vector fields Y_f' are diffeomorphic with the complex projective space:

$$N_f \simeq \mathbb{P}^{m-1}.$$

So, the Euler characteristic

$$\chi\left(N_f\right) = \chi\left(\mathbb{P}^{m-1}\right) = m \neq 0.$$

By the Poincaré–Hopf theorem [14], $\sum i_{y_j} Y_f = \chi\left(N_f\right)$ (where $i_{y_j} Y_f$ denotes the indices of Y_f at the isolated singular points y_j), and the vector field Y_f has at least one singular point (if its singular points are isolated). This amounts to the existence of a periodic solution of X on M_f. This implies the existence of a 1−parameter family of periodic solutions like in the thesis of Theorem 15.

Let us present our approach without the assumption of analyticity of the P–D normal form.

Suppose that there are no families of periodic solutions as above. We shall construct a real vector field on the projective space $\mathbb{P}^{m-1}$, which is a P–D approximation $\widetilde{V}^{\text{ap}}$ of the vector field V.

We take an $\mathbb{S}^1-$invariant approximation F^{ap} of the first integral F; we take the average given by Eq. (8.1).

We project $\widetilde{V}^{\text{ap}}$ onto the tangent spaces of the level hypersurfaces $M_f^{\text{ap}} = \{F^{\text{ap}} = f\}$, $f > 0$, along the radii, such that the projection is $\mathbb{S}^1-$equivariant; we get a regular vector field V^{ap} in the P–D normal form.

Next, we quotient M_f^{ap} and V^{ap} by $\mathbb{S}^1$ and obtain a vector field Y^{ap} on $N_f^{\text{ap}} = M_f^{\text{ap}}/\mathbb{S}^1 \simeq \mathbb{P}^{m-1}$ without singular points. This contradicts the Poincaré–Hopf theorem. □

8.2. The Weinstein theorem

In the proof of Theorem 15 (in the previous section), we have used, essentially, the Poincaré–Hopf formula. But, that formula does not give us much information about the singular points of the quotient vector field Y_f on the quotient variety $N_f = M_f/\mathbb{S}^1$; it is because there is no restriction for the values of the indices. In particular, we cannot get any bound for the number of singular points of Y_f.

Probably, the only topological tool that could be used in this problem is the Lusternik–Schnirelmann category. The corresponding theorem (Theorem 17 from Section 10.2) provides the estimate from below for the number of critical points of a sufficiently smooth function on a manifold based on the category of this manifold.

Weinstein [35] used this idea in the case of germs of Hamiltonian vector fields X_H generated by Hamiltonian functions $H(z, \bar{z})$ such that $D^2H(0, 0)$ are definite, and in fact, positive definite. He skillfully constructed a function on the level hypersurface $L_h = \{H(z, \bar{z}) = h\}$, $h > 0$, which has a critical locus at the set of periodic phase curves of X_H in L_h. His construction is not direct and involves many technical details.

Below, we propose a more direct proof.

Proof of Theorem 4. Recall that we have a Hamiltonian vector field X_H in $\left(\mathbb{R}^{2m}, 0\right)$ with nonzero pure imaginary eigenvalues of the linear part and with a positive definite D^2H. Thus, $H = H_2 + H_3 + \ldots$, where

$$H_2 = \frac{1}{2} \sum \omega_j \left| z_j \right|^2$$

with $\omega_j > 0$ (compare with Eq. (2.5)). Here, we can reduce H to the Birkhoff normal form, which is an analogue of the P–D normal form. Like in the proof of Theorem 15 (in the previous section), we can reduce the situation to one equivalence class of frequencies, i.e., we assume Eq. (7.1).

Assume first that the Birkhoff normal form is analytic. So, let H be in this form.

Proposition 6. *The corresponding action* $\{\sigma^\phi\}$ *of* $\mathbb{S}^1$ *on* $\left(\mathbb{R}^{2m}, 0\right)$ *defines a symmetry of the vector field* X_H. *This action is symplectic; in fact, it is Hamiltonian with*

$$F = H_2$$

as the momentum map.

Proof. We have to prove the second statement. Note that the phase flow generated by the X_F is $g^t(z) = \left(e^{i\omega_1 t} z_1, \ldots, e^{i\omega_m t} z_m\right)$ and coincides with the action $\{\sigma^\phi\}$. □

Now, we apply the symplectic reduction. Recall that, in the case of the general (non-abelian) symmetry group, the symplectic reduction concerns only the zero level of the momentum map (see [4]). Here, the group $\mathbb{S}^1$ is abelian and the symplectic reduction works for any level of the momentum function; in fact, it is the simplest case of the symplectic reduction.

So, we take the manifolds

$$M_f = \{F(z, \bar{z}) = f\}, \quad f > 0,$$

which are diffeomorphic with $\mathbb{S}^{2m-1}$, and their quotients

$$N_f = M_f / \mathbb{S}^1$$

of dimension $2m - 2$. The varieties N_f are equipped with a natural symplectic structure and support $\pi_* H = \pi_* F + \pi_* G$ fields Y_f obtained from X_H. Each vector field Y_f is Hamiltonian with the Hamiltonian function $\pi^* H = \pi^* F + \pi^* G$, where $\pi : M_f \longmapsto N_f$ is the projection and

$$G = H - H_2$$

contains higher-order terms.

Proposition 7. *The periodic orbits of the* X_H *in* M_f *of period* $\approx 2\pi/\omega_k$, *for some* $1 \leq k \leq m$, *correspond to the critical points of the function* $\pi^* G$ *on* N_f.

Proof. The periodic orbits from this proposition are somewhat distinguished. They are defined via the return vector fields U_k of type (7.3), i.e., they are associated with a fixed choice of the frequency ω_k; $z_k = r e^{i p_k \theta}$ and $z_j = w_j e^{i p_j \theta}$ for $j \neq k$. (We do not consider possible orbits of a very long period.) The singular points of U_k correspond to such periodic orbits.

But, the singular points of U_k, when considered on M_f, are the equilibrium points of Y_f. They are the critical points of $\pi_* F$, i.e., of $\pi_* G$, on N_f. Here, there is some subtlety associated with the fact that

N_f is singular, but one can use the local charts $\left[w_1^{(k)}:\ldots:r^{(k)}:\ldots:w_m^{(k)}\right]$ in N_f defined in Eq. (8.2) below. □

Since M_f is a sphere, its quotient N_f is the weighted projective space $\mathbb{P}_{p_1,\ldots,p_m}$. By the Lusternik–Schnirelmann theorem (Theorem 16 in Section 10.2), the number of critical points of the function π^*G is bounded from below by the category of N_f. But, $\mathbb{P}_{p_1,\ldots,p_m}$ is homologically 'similar' to $\mathbb{P}^{m-1}$ (see Section 10.1) and $\mathrm{cat}\mathbb{P}_{p_1,\ldots,p_m}\geq m$ (see Theorem 17 from Section 10.2).[‖]

This gives the estimate $\geq m$ for the number of isolated periodic orbits.

Now, we present our argumentation without the assumption of the analyticity of the Birkhoff normal form. We have essentially two tools at our disposal (we have used them in the proof of Theorem 8). Again, we assume only one equivalence class of the frequencies.

The first tool is the Birkhoff normal form, which we assume only to be formal.

The other is the Poincaré return map (as defined in the second proof of Theorem 3, see Eq. (4.20)), or, rather, the twisted Poincaré map (as defined in the proof of Theorem 8 and in Examples 4–6).

In fact, we will use a collection of twisted Poincaré maps $\mathcal{P}^{(k)}$, $k=1,\ldots,m$, via the substitutions

$$z_k=r^{(k)}\mathrm{e}^{\mathrm{i}\theta},\ \ z_j=w_j^{(k)}\mathrm{e}^{\mathrm{i}p_j\theta/p_k}\ \ (j\neq k),$$

as well as the solutions after the 'time' $\theta=2\pi$ of corresponding differential systems. The variables $r^{(k)}\in(\mathbb{R},0)$ and $w_j^{(k)}\in(\mathbb{C},0)$ parametrize corresponding twisted Poincaré sections.

As before, we assume that $\omega_j=p_j\omega_0$, with relatively prime positive integers such that $p_1\geq p_2\geq\ldots\geq p_m$. The twisted Poincaré map differs from the standard Poincaré map (where $z_j=w_j$) in the coordinates w_j such that $\omega_j>\omega_k$. In the standard case we have that $z_j\longmapsto\mathrm{e}^{\mathrm{i}p_j/p_k}z_j+\ldots$ and, in the twisted case, we have that $w_j\longmapsto w_j+\ldots$.

If the system (1.7) were in an analytic Poincaré–Dulac–Birkfoff normal form with an $\mathbb{S}^1$–invariant first integral F, then the variables $r^{(k)}$ and $w_j^{(k)}$, when restricted to $M_f=\{F=f\}$, would form a local chart

$$\left[w_1^{(k)}:\ldots:r^{(k)}:\ldots:w_m^{(k)}\right]\tag{8.2}$$

in the quotient variety $N_f=M_f/\mathbb{S}^1$, or an affine chart $w_1^{(k)}/r^{(k)},\ldots,w_m^{(k)}/r^{(k)}$, $r^{(k)}>0$.

Recall that the 1–parameter families of periodic solutions of system (1.7), whose corresponding vector field we denote by X_H, of period $\approx 2\pi/\omega_k$, correspond to 1–dimensional curves of fixed points of the twisted Poincaré map $\mathcal{P}^{(k)}$ (see the above proof of Theorem 7). They are analytic curves γ defined by the analytic equation $\mathcal{P}_k-Id=0$. We assume that this equation has a zero locus of dimension 1; otherwise, we have infinitely many 1–parameter families of such periodic solutions.

From the analytic geometry, we know that (germs of) analytic sets are analytically equivalent (diffeomorphic) with algebraic sets. This holds in the case of complex analytic sets and in the case of real analytic sets. For example, a germ of an analytic curve is defined by a corresponding Puiseux series. The proof uses the Weierstrass preparation theorem (see [37]). Moreover, if such a germ is defined by a system of germs of analytic functions, then an approximation of these functions by polynomials (sufficiently long jets) define an algebraic set locally diffeomorphic with the original germ.

We apply this to the germs γ of curves defined by the equation $\mathcal{P}^{(k)}-Id=0$. These equations admit the approximation $\mathcal{P}_\varepsilon^{(k)}-Id=0$, where the twisted Poincaré map $\mathcal{P}_k^{(\varepsilon)}$ is defined via a Hamiltonian

‖The category of quotient spaces was studied also by Weinstein in [36].

vector field X_{H^ε} generated by a polynomial approximation H^ε of H in the Birkhoff normal form. Then, the curve γ will correspond to a curve $\gamma_\varepsilon \subset \left\{\mathcal{P}_\varepsilon^{(k)} - Id = 0\right\}$. The number of the latter curves is estimated like above. □

8.3. The Moser-type theorem about Hamiltonian systems

Recall also that, in the original version of the work [23, Theorem 4], Moser assumed only that $H_2|_{E\setminus 0} > 0$, where E is the linear subspace associated with one frequency for the linear system, but his example from [23, Example 2] (or our Example 4 above) provides a contradiction to this restricted version.

It seems that the crucial point in Moser's approach is his lemma (Lemma 1) from [23]. There, he somehow associates with any small vector $\xi \in E\setminus 0$ a solution to the Hamiltonian system and claims that some of them are periodic; probably, therein lies the flaw of his argument. Finally, he uses the Lusternik–Schnirelmann category for a suitable function on $\mathbb{P}^{r-1}$.

Our approach is more direct. In part, it repeats the above proof of the Weinstein theorem.

Proof of Theorem 5. Much of his proof repeats the corresponding part of the proof of Theorem 4 in the previous section. In particular, we can assume that the Hamiltonian function is in an analytic Birkhoff normal form. Recall that we assume only one equivalence class among the frequencies.

Thus, we have the additional first integral $F = H_2$, which defines the momentum map for the action of the group $\mathbb{S}^1$.

Recall that, by Proposition 5, we have the filtration $V_1 \subset \ldots \subset V_{r-1} \subset V_r = \left(\mathbb{R}^{2m}, 0\right)$ of submanifolds that are invariant for the vector field X_H such that $V_j \setminus V_{j-1}$ is associated with one frequency. This filtration induces corresponding filtrations in $M_f = \{F = f\}$ and in $N_f = M_f/\mathbb{S}^1$:

$$V_1 \cap M_f/\mathbb{S}^1 = \mathbb{P}^{(1)} \subset \ldots \subset V_r \cap M_f/\mathbb{S}^1 = \mathbb{P}^{(r)},$$

where $\mathbb{P}^{(j)}$ denotes weighted projective spaces with complex dimensions $\dim V_j - 1$.

We have to show that $G = H - F$, as a function on N_f, has at least $\frac{1}{2}\dim V_j = \frac{1}{2}(k_1 + \ldots k_j)$ critical points in $\mathbb{P}^{(j)}$. But, this follows from Theorems 16 and 17 in Section 10.2. □

8.4. Nonlinear functional analysis approach

Topological theorems of nonlinear functional analysis are well suited to deal with nonlinear PDEs. But, some specialists began to use them to prove (or reprove) some qualitative results in the theory of ODEs; see [22] for an example.

In particular, Szulkin [32] developed a functional analytic approach to the question of the accumulation of small-amplitude periodic solutions to Hamiltonian systems. His method was followed in papers by Rybicki with various collaborators [10, 15, 25, 26, 31].

The first trick in the analysis of periodic solutions for the period $T = 2\pi/\omega$, which usually varies with variation of the solution, is to normalize this period to 2π. This is done by replacing the Hamiltonian system $\dot{x} = X_H(x)$ with the following family of Hamiltonian systems:

$$\dot{y} = \lambda X_H(y), \tag{8.3}$$

where $\lambda > 0$ is a parameter chosen (for a given periodic solution $x = \varphi(t)$) such that the corresponding solution $y = \varphi(\lambda t)$ to Eq. (8.3) has the period 2π. One says that a family $y = \psi_\lambda(t)$ of $2\pi-$periodic solutions to Eq. (8.3) emanates from $(\lambda, y) = (\lambda_0, 0)$, $\lambda_0 = 1/\omega_j$, when $x = \psi_\lambda(t/\lambda)$, with λ close to λ_0, denotes solutions to $\dot{x} = X_H$ with the period $\approx 2\pi/\omega_j$.

Next, it is not difficult to show that the $2\pi-$periodic solutions to system (8.3), where $y = (q, p)$, correspond to the stationary paths of the following functional (see also [23]):

$$\Phi(\gamma) = \Phi_\lambda(\gamma) = \int_0^{2\pi} p\mathrm{d}q + \lambda H\mathrm{d}t. \tag{8.4}$$

This functional is defined on the $2\pi-$periodic path $\gamma : \mathbb{S}^1 \longmapsto \mathbb{R}^{2m}$, $\gamma(t) = (q(t), p(t))$. The author considers Φ as a functional on the Sobolev space $\mathcal{H} = H^{1/2}\left(\mathbb{S}^1, \mathbb{R}^{2m}\right)$; using complex vector-variables $z = q + \mathrm{i}p$ and $\bar{z} = q - \mathrm{i}p$, we have that $z = \gamma(t) = \sum_{k\in\mathbb{Z}} z^{(k)}\mathrm{e}^{\mathrm{i}kt}$, $z^{(k)} = \left(z_1^{(k)}, \ldots, z_m^{(k)}\right)$ and $\|\gamma\|^2 = \pi\left\{\left|z^{(0)}\right|^2 + \sum k\left|z^{(k)}\right|^2\right\}$.

The first part of the functional (8.4) takes the form

$$\langle \boldsymbol{L}\gamma, \gamma\rangle = \int_0^{2\pi} p\mathrm{d}q = \frac{1}{2}\mathrm{Im}\int_0^{2\pi} z\mathrm{d}\bar{z} = \pi\sum k\left|z^{(k)}\right|^2.$$

With the splitting $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+ = \{(\gamma_-, \gamma_0, \gamma_+)\}$ of the above Hilbert space into subspaces generated by negative, zero and positive modes, we have that $\langle \boldsymbol{L}\gamma, \gamma\rangle = \|\gamma_+\|^2 - \|\gamma_-\|^2$. In particular, the subspace $\mathcal{H}_{0,\pm}$ is invariant for $\boldsymbol{L}$.

Assume that $H = H_2 + \ldots$, with $H_2 = \frac{1}{2}(Az, \bar{z})$. We have the quadratic functional

$$\langle \boldsymbol{B}\gamma, \gamma\rangle = \int H_2\mathrm{d}t;$$

if $H_2 = \frac{1}{2}\sum_j \epsilon_j\omega_j\left|z_j\right|^2$, $\omega_j > 0, \epsilon_j = \pm 1$, like in Eq. (1.8), then $\langle \boldsymbol{B}\gamma, \gamma\rangle = \pi\left\{\sum_{j,k}\epsilon_j\omega_j\left|z_j^{(k)}\right|^2\right\}$. It turns out (see [32]) that the variational derivative $\frac{\delta}{\delta\gamma}\int H\mathrm{d}t$ is a compact operator in $\mathcal{H}$, and that the functional Φ satisfies the so-called Palais–Smale condition (which we do not define here).

Consider the operator $\boldsymbol{L} + \lambda\boldsymbol{B}$ and its restrictions $T^{(k)} = (\boldsymbol{L} + \lambda\boldsymbol{B})|_{\mathcal{H}^{(k)}}$ to the subspace $\mathcal{H}^{(k)} = \mathrm{span}\left\{\mathrm{e}^{\mathrm{i}kt}, \mathrm{e}^{-\mathrm{i}kt}\right\} \simeq \mathbb{R}^{2m}$, $k > 0$, which are as follows:

$$T^{(k)}(z^{(k)}, z^{(-k)}) = \left(\mathrm{i}\left(kI_m + \lambda A\right)z^{(k)}, \mathrm{i}\left(-kI_m + \lambda A\right)z^{(-k)}\right).$$

We see that the operator $T^{(k)} = T^{(k)}(\lambda A)$ is non-invertible if and only if $\lambda = k/\omega_j$ for some j; this is the case when the linear system $\dot{x} = X_{H_2}(x)$ has a family of periodic solutions for the period $2\pi/\omega_j$.

Consider the splitting $\mathcal{H} = \mathcal{F} \oplus \widetilde{\mathcal{F}}$, where $\mathcal{F} = \mathcal{H}_-$ and $\widetilde{\mathcal{F}} = \mathcal{H}_0 \oplus \mathcal{H}_+$. The subspace $\mathcal{F}$ is invariant for $\boldsymbol{L}$ and $\boldsymbol{L}|_{\mathcal{F}} < 0$ has a quadratic form. The operator $\boldsymbol{B}$ is a bounded self-adjoint operator in $\mathcal{H}$. Let $\mathcal{E}_n = \mathrm{span}\left\{\mathrm{e}^{\mathrm{i}kt} : k = 1, \ldots, n\right\} \oplus \widetilde{\mathcal{F}}$ be subspaces of $\mathcal{H}$ (invariant for $\boldsymbol{L}$) and $\boldsymbol{P}_n : \mathcal{H} \longmapsto \mathcal{E}_n$ be corresponding orthogonal projections. We have the Morse indices $M^-\left((\boldsymbol{L} + \lambda\boldsymbol{P}_n\boldsymbol{B})|_{\mathcal{E}_n}\right)$, i.e., the numbers of negative eigenvalues. One defines another index

$$M^-(\lambda\boldsymbol{B}) = \lim_{n\to\infty}\left\{M^-\left((\boldsymbol{L} + \lambda\boldsymbol{P}_n\boldsymbol{B})|_{\mathcal{E}_n}\right) - n\right\}.$$

(It has a cohomological interpretation; some cohomology groups $H^q_{\mathcal{F}}(W, W^-)$, which we do not define, vanish for $q \neq M^-(\lambda \boldsymbol{B})$ and are equal to $\mathbb{R}$ otherwise; see [32, Proposition 2.3].)

An abstract statement, i.e., [32, Theorem 2.3], says the following:

Proposition 8. *If (i) $\boldsymbol{L} + \lambda \boldsymbol{B}$ is invertible for $\lambda \in I \setminus \lambda_0$, where I is a neighborhood of λ_0, and (ii) the function $\lambda \longmapsto M^-(\lambda \boldsymbol{B})$ has a jump at λ_0, then either (a) the path $\gamma(t) \equiv 0$ is a non-isolated critical path of Φ_{λ_0}, or, (b) for all λ's in $I \setminus \lambda_0$ from one side of λ_0, the functional Φ_λ has a critical path $\gamma_\lambda(t) \not\equiv 0$ near $\gamma(t) \equiv 0$.*

The quantity $M^-(\lambda \boldsymbol{B})$ is explicitly calculated in terms of the matrix λA; we have

$$M^-(\lambda \boldsymbol{B}) = i^-(\lambda A) := M^-(-\lambda A) + \sum_{k>0} \left\{ M^-\left(T^{(k)}\right) - 2m \right\}$$

(see [32, Proposition 3.3]).

Assume the situation like in Eq. (1.10), i.e., that we have r different frequencies, $\tilde{\omega}_1 = \omega_1$ (of multiplicity k_1), $\tilde{\omega}_2 = \omega_{k_1+1}$ (of multiplicity k_2), etc., possibly belonging to different equivalence classes. Let us fix $\tilde{\omega}_l$. It turns out that, if $\tilde{\omega}_j / \tilde{\omega}_l \notin \mathbb{N}$ for all $j \neq l$, then $M^-(\lambda \boldsymbol{B}) = i^-(\lambda A)$ jumps at $\lambda = 1/\tilde{\omega}_l$ if and only if $M^-\left(T^{(1)}(\lambda A)\right)$ jumps.

Next, $M^-\left(T^{(1)}(\lambda A)\right)$ jumps at $\lambda = 1/\tilde{\omega}_l$ if and only if

$$M^-\left(A|_{E_l}\right) \neq M^+\left(A|_{E_l}\right) := M^-\left(-A|_{E_l}\right), \tag{8.5}$$

where E_l is the invariant subspace for A corresponding to the eigenvalues $\pm i\tilde{\omega}_l$ (see [32, Proposition 3.6]). Condition (8.5) means that, if H_2 is like in Eq. (1.8) and $\tilde{\omega}_l = \omega_{k_1+\ldots+k_s+1}$, then $\epsilon_{k_1+\ldots+k_{l-1}+1} + \ldots + \epsilon_{k_1+\ldots+k_l} \neq 0$ for ϵ_i from Eq. (1.8).

This yields the Szulkin's Theorem 6 from Introduction.

We see that the first assumption of Theorem 6, i.e., $\tilde{\omega}_j / \tilde{\omega}_l \notin \mathbb{N}$, is like in the third Lyapunov theorem. But, the assumption, i.e., inequality (8.5), is new; it replaces the definiteness assumption of H_2 from Theorems 4 and 5. Note also that, in this topological approach, one does not assume the analyticity of the Hamiltonian function. But, when one additionally assumes the analyticity of the Hamiltonian function, then the statement of Theorem 6 can be strenghtened: there exists at least one continuous family of periodic solutions.

At this moment , the method to reprove this result using the proposed approach is unknown by the author. But, in all examples considered by the author, my tools seem to be more effective. In fact, examples of where only the Szulkin theorem works are not known by the author.

9. Applications

9.1. Quasi-periodic movement near geostationary orbit

Geostationary orbits of celestial objects, like GPS satellites, are determined by the property by which they are fixed with respect to an Earth's observer. The following definitions were taken from Strzelecki's thesis [31].

Regarding the position and momentum framework of $q = (x, y, z)$, $p = \left(p_x, p_y, p_z\right)$ associated with the moving object, we get a Hamiltonian system with the following Hamiltonian:

$$H = \frac{1}{2}|p|^2 + \nu\left(xp_y - yp_x\right) + V, \tag{9.1}$$

where ν is the Earth's angular velocity and the gravity potential (generated by an ellipsoid of rotation) equals

$$V = V(\rho, z) = -r^{-1}\left(1 - 2cr^{-2}P_2(z/r)\right) = -r^{-1} - cr^{-3} + 3cz^2r^{-5},$$

where $P_2(\lambda) = \frac{1}{2}\left(3\lambda^2 - 1\right)$ is the quadratic Legendre polynomial, $r = |q| = \sqrt{\rho^2 + z^2}$ and $c > 0$ is a constant evaluated experimentally. The corresponding Hamiltonian equations

$$\dot{x} = p_x - \nu y,\ \dot{y} = p_y + \nu x,\ \dot{z} = p_z,\ \dot{p}_x = -V'_x - \nu p_y,\ \dot{p}_y = -V'_y + \nu p_x,\ \dot{p}_z = -V'_z \tag{9.2}$$

imply the following second-order Newtonian equations:

$$\ddot{x} = -V'_x - 2\nu\dot{y} + \nu^2 x,\ \ddot{y} = -V'_y + 2\nu\dot{x} + \nu^2 y,\ \ddot{z} = -V'_z;$$

here, $(-2\nu\dot{y}, 2\nu\dot{x}, 0)$ and $\left(\nu^2 x, \nu^2 y, 0\right)$ are the Coriolis and the centrifugal forces caused by the rotation of the coordinate frame.**

The function (9.1) is invariant with respect to the simultaneous rotations in the $\vec{\rho} = (x, y)$ and $\vec{\sigma} = \left(p_x, p_y\right)$ planes. This $\mathbb{S}^1$−action is symplectic, and the corresponding momentum map is the vertical component of the classical angular momentum, $F = xp_y - yp_x = \vec{\rho} \times \vec{\sigma}$. The symplectic reduction (see Section 10.1) means the introduction of the coordinates $\rho = |\vec{\rho}|, \phi, z, \zeta = p_3$, where ϕ is the angle between the vectors $\vec{\rho}$ and $\vec{\sigma}$. Thus, $F = \rho\sigma\sin\phi$, $\sigma = |\vec{\sigma}|$. On the hypersurface $M_f = \{F = f\}$, where $f \neq 0$, we have

$$\sigma = f/(\rho\sin\phi).$$

Next, $\frac{d}{dt}\vec{\rho} = \vec{\sigma} + \nu(-q_2, q_1)$ implies $\dot{\rho} = \frac{1}{\rho}\vec{\rho}\cdot\vec{\sigma} = \sigma\cos\phi$ and $\dot{\sigma} = -V'_\rho\cos\phi$. So, we get the reduced differential system on $N_f = M_f/\mathbb{S}^1$:

$$\dot{\rho} = f\rho^{-1}\cot\phi,\ \dot{\phi} = -f\rho^{-2} + \rho V'_\rho\sin^2\phi/f,\ \dot{z} = \zeta,\ \dot{\zeta} = -V'_z. \tag{9.3}$$

The equilibrium points are found as follows. First, $\zeta_0 = z_0 = 0$ since V depends on z^2. We have $\cos\phi = 0$, i.e., $\phi = \pm\frac{\pi}{2}$, and we choose $\phi_0 = \frac{\pi}{2}$. Next, from Eq. (9.2), we get $\sigma_0 = \nu\rho_0$ and $f = \nu\rho_0^2$. Thus, $\dot{\phi} = 0$ implies $-\nu + \frac{1}{\nu\rho_0}\left(\rho_0^{-2} + 3c\rho_0^{-4}\right) = 0$, and we get the following equation for ρ_0:

$$\nu^2\rho_0^5 - \rho_0^2 - 3c = 0;$$

for $c > 0$, it has a unique positive solution.

Let us linearize system (9.3) at this equilibrium point. First, $\cot'(\pi/2) = -1$, Next, $\beta = \partial\dot{\phi}/\partial\rho = 2f\rho_0^{-3} + \frac{2c}{f}\rho_0^{-3} > 0$. Finally, for a small z, we have that $V \approx -\rho_0^{-1}\left(1 - z^2/2\rho_0^2\right) - c\rho_0^{-3}\left(1 - 3z^2/2\rho_0^2\right) + 3cz^2\rho_0^{-5} = \text{const} + \frac{1}{2}\left(\rho_0^{-3} + 15\rho_0^{-5}\right)z^2 = \text{const} + \gamma z^2$, $\gamma > 0$. So, the linearization, with $\rho_1 = \rho - \rho_0$, $\rho_0 = \rho - \rho_0, \ldots$, is as follows:

$$\dot{\rho}_1 \approx -\alpha\phi_1,\ \dot{\phi}_1 \approx \beta\rho_1,\ \dot{z}_1 = \zeta_1,\ \dot{\zeta}_1 \approx \gamma z_1,$$

**Such systems were already considered by Lyapunov [21, Section 45] with positive constants $\alpha = f/\rho_0, \beta, \gamma$. It follows that there is a pair of pure imaginary eigenvalues $\pm i\omega = \pm i\sqrt{\alpha\beta}$ and a pair of real eigenvalues $\pm\sqrt{\gamma}$. Since the system is Hamiltonian, we have a 1–parameter family of periodic solutions for the period $\approx 2\pi/\omega$ on the 2–dimensional center manifold $W^c \approx \{z_1 = \zeta_1 = 0\}$.

But, for the Hamiltonian system with three degrees of freedom (before the symplectic reduction), we have, in fact, invariant tori, with the additional angle $\psi = \arg(x + iy)$. The movement on these tori is quasi-periodic; we have

$$\dot{\psi} \approx \delta,$$

$\delta = f\rho_0^{-2}$.

In Strzelecki's approach [31], the same result was obtained using the so-called G−equivariant Conley index.

9.2. *Restricted four-body problem*

In the restricted four-body problem, we deal with three heavy bodies, S (from the sun), J (from Jupiter) and P (from a planet), and one light body A (an asteroid). The heavy bodies lie in the vertices of an equilateral triangle which rotates with constant angular velocity (equal $\nu = -1$) about their center of mass and in a fixed plane. The asteroid moves in the gravity field generated by the principal bodies and does not influence their movement. ††

As in the previous section, one passes to the coordinate–momentum frame wherein the heavy bodies rest. Thus, we have the Hamiltonian function like in Eq. (9.1), but with $\nu = 1$ and

$$V(q) = -\frac{m_S}{|q - q_S|} - \frac{m_J}{|q - q_J|} - \frac{m_P}{|q - q_P|}; \tag{9.4}$$

here, $q = 0$ is the center of mass of the system. We assume two degrees of freedom (the case with three degrees of freedom is only slightly more complex; see [15]), i.e., that A lies in the plane SJP; so, $q = (x, y)$ and $p = \left(p_x, p_y\right)$.

It is useful to rewrite the Hamiltonian as

$$H = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}\left(p_y + x\right)^2 + W(x, y),$$

where

$$W = -\left(x^2 + y^2\right)/2 + V$$

is the effective potential. The critical points of H are defined by $p_x = y_j$ and $p_y = -x_j$, where $\left(x_j, y_j\right)$ are critical points of W. By putting $q_1 = x - x_j$, $q_2 = y - y_j$, $p_1 = p_x + y_j$, $p_2 = p_y - x_j$ (and $\tilde{q} = (q_1, q_2)$, $\tilde{p} = (p_1 p_2)$), the quadratic part of the expansion of the Hamiltonian at such a critical point becomes

$$H_2(\tilde{q}, \tilde{p}) = \frac{1}{2}\left\{(p_1 - q_2)^2 + (p_2 + q_1)^2 - aq_1^2 - 2bq_1q_2 - cq_2^2\right\}.$$

One finds (see [20]) that the characteristic polynomial of the corresponding linear system becomes

$$P(\lambda) = \lambda^4 + a_2\lambda^2 + a_4,$$
$$a_2 = 4 - a - c,\ a_4 = ac - b^2.$$

††Recall that periodic solutions near the triangular libration point L_4 for the restricted three-body problem, i.e., with only S, J and A, were studied by Schmidt [28]. I would like to thank one of the reviewers for this reference

Note that the characteristic polynomial of the Hessian matrix $D^2W\left(x_j, y_j\right) = \begin{pmatrix} -a & -b \\ -b & -c \end{pmatrix}$ equals

$$R(\beta) = \beta^2 + (a+c)\beta + ac - b^2.$$

If $Q(\mu) = \mu^2 + a_2\mu + a_4$, then we have the following relations between zeros of the polynomials Q and R :

$$\mu_1\mu_2 = \beta_1\beta_2,\ \mu_1 + \mu_2 = -(\beta_1 + \beta_2 + 4). \tag{9.5}$$

We have the following generic possibilities for the zeros of P (in [15], these cases are expressed in terms of β_1, β_2) :

(i) $\mu_1 = \bar{\mu}_2 \in \mathbb{C}\setminus\mathbb{R}$, i.e., $a_4 > 0$ and $\Delta = a_2^2 - 4a_4 < 0$; then, $\lambda_j \in \mathbb{C}\setminus\mathbb{R}\setminus \mathrm{i}\mathbb{R}$.

(ii) $\mu_1, \mu_2 > 0$, i.e., $a_4 > 0$, $a_2 < 0$ and $\Delta > 0$; then, $\lambda_{1,2} = \pm\sqrt{\mu_1}$ and $\lambda_{3,4} = \pm\sqrt{\mu_2}$.

(iii) $\mu_1 < 0 < \mu_2$, i.e., $a_4 < 0$ and $\Delta > 0$; then, $\lambda_{1,2} = \pm\mathrm{i}\sqrt{|\mu_1|}$ and $\lambda_{3,4} = \pm\sqrt{\mu_2}$.

(iv) $\mu_1 < \mu_2 < 0$, i.e., $a_4 > 0$, $a_2 > 0$ and $\Delta > 0$; then, $\lambda_{1,2} = \pm\mathrm{i}\sqrt{|\mu_1|}$ and $\lambda_{3,4} = \pm\mathrm{i}\sqrt{|\mu_2|}$.

We note also the following non-generic case of $0:1$ resonance (considered in [15]):

(v) $\mu_1 < 0 = \mu_2$, i.e., $a_4 = 0$ and $a_2 > 0$; then, $\lambda_{1,2} = \pm\mathrm{i}\sqrt{|\mu_1|}$ and $\lambda_{3,4} = 0$.

Using only the above information about the eigenvalues λ_j, we can state the following.

(1) In cases (i) and (ii), there are no families of periodic solutions.

(2) In case (iii), there is one 1–parameter family of periodic solutions for the period $\approx 2\pi/\sqrt{|\mu_1|}$.

(3) In case (iv), there exists at least one 1–parameter family of periodic solutions for the period $\approx 2\pi/\sqrt{|\mu_1|}$*; moreover, if* $\sqrt{|\mu_1|/|\mu_2|} > 1$ *is not an integer, then there is another 1–parameter family for the period* $\approx 2\pi/\sqrt{|\mu_2|}$.

Analogues of these statements can be found in [15, Theorem 3.1]. But, in the same theorem, we find the following statement:

If in case (v), additionally, the index $i_{(x_j,y_j)}\nabla W$ *(the authors call it the Brouwer index) is nonzero, then there exists a branch of closed orbits.*

We shall justify this using a refined version of the Birkhoff normal form in the $0:1$ resonant case:

$$H = \frac{1}{2}\left\{|Z|^2\,\Omega\left(Y, |Z|^2\right) - X^2 + \Phi(Y)\right\},$$

where $\Omega = \omega + \ldots$ and $\Phi = a_kY^k + \ldots$, $a_k \neq 0$. In the case of one degree of freedom, this result is due to Baider and Sanders [5] (some coefficients in the series $\Phi(Y)$ vanish); the above case is a generalization of [7, Eq. (3.4)]. The corresponding differential system becomes

$$\dot{Z} = \mathrm{i}(\omega + \ldots)Z,\ \dot{X} = ka_kY^{k-1}(1 + O(Y)) + |Z|^2\,\Psi\left(Y, |Z|^2\right),\ \dot{Y} = X.$$

As usual, when looking for periodic phase curves, we make the substitution $Z = re^{\mathrm{i}\theta}$. We get the system

$$\mathrm{d}r/\mathrm{d}\theta = 0,\ \mathrm{d}X/\mathrm{d}\theta = cY^{k-1}(1 + \ldots) + r^2\Xi\left(Y, r^2\right),\ \mathrm{d}Y/\mathrm{d}\theta = X/(\omega + \ldots),$$

$c = ka_k/\omega \neq 0$, with equilibrium points defined by $X = 0$ and $cY^{k-1}(1 + \ldots) + r^2\Xi\left(Y, r^2\right) = 0$. The sufficient condition for the existence of a nontrivial solution to the this system is that the exponent

$k - 1$ is odd. But, this amounts to the property that the index of X_H at this singular point is nonzero; equivalently, the index of ∇W at $\left(x_j, y_j\right)$ is nonzero.

In the case of the potential given by Eq. (9.4) for the planar restricted four-body problem, the authors of [15] referred to works of other authors when looking for libration points (see [2, 6, 19] for example). One assumes $m_S + m_J + m_P = 3\sqrt{3}$ (normalization); then, the sides of the equilateral triangle $\mathcal{T}$ equal $\sqrt{3}$.

One draws three circles of radius $\sqrt{3}$ with centers at S, J and P and three straight lines which extend the sides of $\mathcal{T}$. Then, one obtains 16 bounded regions, but only seven of them contain libration points (critical points of W). These (open) domains are as follows: the triangle $\mathcal{T}$; three circular sectors $\mathcal{D}_S$, $\mathcal{D}_J$ and $\mathcal{D}_P$ with vertices at S, J and P, respectively; three circular triangles O_{SJ}, O_{JP} and O_{PS} opposite to the corresponding sides of $\mathcal{T}$ (see [15, Theorem 4.1(i)]).

Moreover, in the case of equal masses, there are 10 critical points of W, all nondegenerate: the minimum $x = y = 0$ and three saddle points in $\mathcal{T}$, one local maximum in each $O_{\#\#}$ and one saddle in each $\mathcal{D}_\#$ (see [15, Lemma 4.2]). In the case of non-equal masses, one calculates the indices of ∇W along the boundaries of the above domains (called the Brouwer degree) and finds that, in each of the above seven domains, there exists a libration point with a nonzero index of ∇W.

By the above analysis of the eigenvalues $\lambda_j = \pm\sqrt{\mu_l}$, using the relations of Eq. (9.5) between μ_k and the eigenvalues β_m of $D^2 W$, one concludes the existence of a 1–parameter family of periodic solutions for each of the seven above libration points.

It would be interesting to study periodic solutions in the 1 : 1 resonance case, i.e., when $a_2 > 0$ and $\Delta = 0$ ($\mu_1 = \mu_2 < 0$).

Finally, we note that the restricted four-body problem was studied from the points of view of the Lyapunov stability of the libration points, i.e., for case (iv); the existence of a family of periodic solutions does not imply it. Here, one uses the KAM theorem and fourth-order Birkhoff normal form. In [20], the generic Lyapunov stability for case (iv) was established for the following situations:

when m_S and m_J dominate over m_P and the libration points lie near P;

when $m_S = m_J$ and the libration points lie in a symmetry line of $\mathcal{T}$;

when m_S dominates over m_J and m_P and the libration points lie near the circle with its center at S and radius $|SJ|$.

10. Appendices

10.1. Weighted projective spaces

Let $p_1, \ldots, p_m$ be relatively prime positive integers, called the weights. Consider the following action of $\mathbb{C}^*$ on $\mathbb{C}^m$:

$$(\lambda, z) \longmapsto \lambda \cdot z = (\lambda^{p_1} z_1, \ldots, \lambda^{p_m} z_m).$$

By definition, the quotient of $\mathbb{C}^m \setminus 0$ by $\mathbb{C}^*$ is the weighted projective space,

$$\mathbb{P}_{p_1,\ldots,p_m} = (\mathbb{C}^* \setminus 0) / \mathbb{C}^*. \tag{10.1}$$

Its complex dimension is $m - 1$ and real dimension is $2m - 2$.

From the algebraic geometry point of view (see [11]), it is the projective spectrum $\mathrm{Proj}\mathbf{S}(P)$, $P = (p_1, \ldots, p_m)$, of the graded polynomial algebra $\mathbf{S}(P) = \mathbb{C}[T_1, \ldots, T_m]$ such that $\deg T_j = p_j$. These spaces are important from the complex geometry point of view. In particular, the morphism $z_j \longmapsto T_j = z_j^{p_j}$ defines the formal isomorphism

$$\mathbb{P}_{p_1,\ldots,p_m} \simeq \mathbb{P}^{m-1}/\mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_m}.$$

But, we are more interested in their topological properties. Consider the standard sphere $\mathbb{S}^{2m-1} = \left\{\sum |z_j|^2 = 1\right\}$. Each orbit $\mathbb{C}^* \cdot z$, $z \neq 0$, intersects this sphere along a circle. Therefore, we have

$$\mathbb{P}_{p_1,\ldots,p_m} = \mathbb{S}^{2m-1}/\mathbb{S}^1,$$

where the action of the circle $\mathbb{S}^1 = \left\{e^{i\phi}\right\}$ is like in Eq. (6.2), i.e., $\sigma^\phi \cdot z = \left(e^{ip_1\phi}z_1, \ldots, e^{ip_m\phi}z_m\right)$.

Let us find the representation of our weighted projective space as a kind of CW complex (see [14]). This complex consists of 'cells' of (real) dimensions $2m-2, 2m-4, \ldots, 2, 0$.

The 'cell' of maximal dimension σ^{2m-2} is defined by $\{z_m \neq 0\}$ in $\mathbb{S}^{2m-1}$. We rotate the variable z_m to get $\arg z_m = 0$, i.e., $z_m = r_m > 0$. We have the following description of this 'cell':

$$\sigma^{2m-2} = D^{2m-2}/\Delta,$$

where $D^{2m-2} = \left\{r_m = \sqrt{1 - |z_1|^2 - \ldots - |z_{m-1}|^2}\right\}$ and $\Delta \subset \mathbb{S}^1$ is the stabilizer of the set $\{\arg z_m = 0\}$. The above upper semi-sphere is diffeomorphic with the standard $(2m-2)$ −dimensional unit ball, and its quotient is singular but with the singular locus of (real) codimension ≥ 4 (complex codimension is ≥ 2). This cell defines the fundamental cycle of $\mathbb{P}_{p_1,\ldots,p_m}$.

The boundary of σ^{2m-2} is the weighted projective space $\mathbb{P}_{p_1,\ldots,p_{m-1}}$. Note that now the weights can be not relatively prime; but, if $p_j = r\tilde{p}_j$, then the spaces $\mathbb{P}_{p_1,\ldots,p_{m-1}}$ and $\mathbb{P}_{\tilde{p}_1,\ldots,\tilde{p}_{m-1}}$ are isomorphic (see [11]).

In this way, we successively construct the lower-dimensional cells, as well as cells in the singular locus of the action of the group Δ on D^{2m-2}. By the dimensional argument, the boundaries of these 'cells' are zero in the homological sense; so, the only nonzero homology groups have even dimensions. We distinguish the $(2m-4)$ −dimensional cycles $\left[\sigma_1^{2m-4}\right], \ldots, \left[\sigma_m^{2m-4}\right]$ corresponding to the hyperplane sections $\{z_1 = 0\}, \ldots, \{z_m = 0\}$ respectively. The intersection of any collection of $m-1$ of the above cycles is nonempty and ,hence, $\mathrm{cat}\mathbb{P}_{p_1,\ldots,p_m} \geq m$ (see Theorem 17 below).

Recall [14] that the homological length of a manifold M is the maximal number k such that there exist homology classes $\rho_1, \ldots, \rho_k \in H_*(M, \mathbb{Z})$, $\dim\rho_j < \dim M$, such that their intersection is a nonzero cycle.[‡‡]

Finally, we note that, in our analysis, we can replace the standard sphere with any manifold of the form

$$\left\{c_1 |z_1|^2 + \ldots + c_m |z_m|^2 = 1\right\}$$

for positive constants c_j.

[‡‡]In fact, in [14], one finds the definition of the cohomological length (or the cup length) as the maximal number k such that there exist cohomology classes $a_1, \ldots, a_k$, $\dim a_j > 0$, such that $a_1 \cup \ldots \cup a_k \neq 0$. These cohomology classes are Poincaré-dual to the cycles $\rho_1, \ldots, \rho_k$.

But, in the proof of Theorem 17 below given in [14], only intersections of the cycles are used.

10.2. Lusternik–Schnirelmann category

By definition (see [14]), the Lusternik–Schnirelmann category $\mathrm{cat}_X A$ of A with respect to X is the smallest number k for which there exist closed subsets $A_1, \ldots, A_k$ of X such that

(i) $A = \bigcup A_j$ and

(ii) each A_j is contractible in X.

If X is connected, then we define $\mathrm{cat}_X X = \mathrm{cat} X$.

In this work, we have used the following properties of this notion.

Theorem 16. *(Lusternik–Schnirelmann) Let M be a compact closed (without boundary) connected manifold, and let f be a smooth function on M with isolated critical points (or bifurcational points). Then, $k \geq \mathrm{cat} M$.*

Theorem 17. *If M is a manifold, then $\mathrm{cat} M$ is bounded from below by 1 plus the homological length of M.*

For the proofs, we refer the reader to a previous monograph [14].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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