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Research article

On a boundary control problem for a pseudo-parabolic equation

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Abstract: Previously, boundary control problems for parabolic type equations were considered. A portion of the thin rod boundary has a temperature-controlled heater. Its mode of operation should be found so that the average temperature in some region reaches a certain value. In this article, we consider the boundary control problem for the pseudo-parabolic equation. The value of the solution with the control parameter is given in the boundary of the interval. Control constraints are given such that the average value of the solution in considered domain takes a given value. The auxiliary problem is solved by the method of separation of variables, and the problem under consideration is reduced to the Volterra integral equation. The existence theorem of admissible control is proved by the Laplace transform method.

Keywords: Pseudo-parabolic equation; initial-boundary problem; admissible control; integral equation; Laplace transform
Mathematics Subject Classification: 35K70, 35K05

1. Introduction and statement of the problem

Consider the pseudo-parabolic equation in the domain $\Omega = \{(x,t) : 0 < x < l, \;\; t > 0\}$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial t \partial x}\Big(k(x)\frac{\partial u}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(k(x)\frac{\partial u}{\partial x}\Big), \quad (x,t) \in \Omega, \tag{1.1}$$

with boundary conditions

$$u(0,t) = \mu(t), \quad u(l,t) = 0, \quad t > 0, \tag{1.2}$$

and initial condition

$$u(x,0) = 0, \quad 0 \le x \le l. \tag{1.3}$$

Assume that the function $k(x) \in C^2([0,l])$ satisfies the conditions

$$k(x) > 0, \quad k'(x) \le 0, \quad 0 \le x \le l.$$

The condition (1.2) means that there is a magnitude of output given by a measurable real-valued function $\mu(t)$ (See [1–3] for more information).

Definition 1. If function $\mu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\mu(0) = 0$ and $|\mu(t)| \le 1$, we say that this function is an *admissible control*.

Problem B. For the given function $\theta(t)$ Problem B consists looking for the admissible control $\mu(t)$ such that the solution $u(x,t)$ of the initial-boundary problem (1.1)-(1.3) exists and for all $t \ge 0$ satisfies the equation

$$\int_0^l u(x,t)\,dx = \theta(t). \tag{1.4}$$

One of the models is the theory of incompressible simple fluids with decaying memory, which can be described by equation (1) (see [1]). In [2], stability, uniqueness, and availability of solutions of some classical problems for the considered equation were studied (see also [4, 5]). Point control problems for parabolic and pseudo-parabolic equations were considered. Some problems with distributed parameters impulse control problems for systems were studied in [3, 6]. More recent results concerned with this problem were established in [7–15]. Detailed information on the problems of optimal control for distributed parameter systems is given in [16] and in the monographs [17–20]. General numerical optimization and optimal boundary control have been studied in a great number of publications such as [21]. The practical approaches to optimal control of the heat conduction equation are described in publications like [22].

Control problems for parabolic type equations are considered in works [13, 14] and [15]. In this work, such control problems are considered for the pseudo-parabolic equation.

Consider the following eigenvalue problem

$$\frac{d}{dx}\Big(k(x)\frac{dv_k(x)}{dx}\Big) = -\lambda_k\, v_k(x), \quad 0 < x < l, \tag{1.5}$$

with boundary condition

$$v_k(0) = v_k(l) = 0, \quad 0 \le x \le l. \tag{1.6}$$

It is well-know that this problem is self-adjoint in $L_2(\Omega)$ and there exists a sequence of eigenvalues $\{\lambda_k\}$ so that $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \to \infty,\ k \to \infty$. The corresponding eigenfuction v_k form a complete orthonormal system $\{v_k\}_{k\epsilon N}$ in $L_2(\Omega)$ and these function belong to $C(\bar{\Omega})$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ (see, [23, 24]).

2. Main integral equation

Definition 2. By the solution of the problem (1.1)–(1.3) we understand the function $u(x,t)$ represented in the form

$$u(x,t) = \frac{l-x}{l}\mu(t) - v(x,t), \tag{2.1}$$

where the function $v(x,t) \in C_{x,t}^{2,1}(\Omega) \cap C(\bar{\Omega})$, $v_x \in C(\bar{\Omega})$ is the solution to the problem:

$$v_t = \frac{\partial^2}{\partial t \partial x}\Big(k(x)\frac{\partial v}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(k(x)\frac{\partial v}{\partial x}\Big) +$$

$$+\frac{k'(x)}{l}\mu(t)+\frac{k'(x)}{l}\mu'(t)+\frac{l-x}{l}\mu'(t),$$

with boundary conditions

$$v(0,t)=0, \quad v(l,t)=0,$$

and initial condition

$$v(x,0)=0.$$

Set

$$\beta_k=(\lambda_k a_k-b_k)\gamma_k, \tag{2.2}$$

where

$$a_k=\int_0^l \frac{l-x}{l}v_k(x)dx, \quad b_k=\int_0^l \frac{k'(x)}{l}v_k(x)dx, \tag{2.3}$$

and

$$\gamma_k=\int_0^l v_k(x)dx. \tag{2.4}$$

Consequently, we have

$$v(x,t)=\sum_{k=1}^{\infty}\frac{v_k(x)}{1+\lambda_k}\int_0^t e^{-\mu_k(t-s)}(\mu'(s)\,a_k+\mu'(s)\,b_k+\mu(s)\,b_k)\,ds, \tag{2.5}$$

where a_k, b_k defined by (2.3) and $\mu_k=\frac{\lambda_k}{1+\lambda_k}$.

From (2.1) and (2.5) we get the solution of the problem (1.1)–(1.3) (see, [23,25]):

$$u(x,t)=\frac{l-x}{l}\mu(t)-\sum_{k=1}^{\infty}\frac{v_k(x)}{1+\lambda_k}\int_0^t e^{-\mu_k(t-s)}(\mu'(s)\,a_k+\mu'(s)\,b_k+\mu(s)\,b_k)\,ds.$$

According to condition (1.4) and the solution of the problem (1.1)-(1.3), we may write

$$\theta=\int_0^l u(x,t)dx=\mu(t)\int_0^l\frac{l-x}{l}dx$$

$$-\sum_{k=1}^{\infty}\frac{1}{1+\lambda_k}\left(\int_0^t e^{-\mu_k(t-s)}(\mu'(s)\,a_k+\mu'(s)\,b_k+\mu(s)\,b_k)\,ds\right)\int_0^l v_k(x)dx$$

$$=\mu(t)\int_0^l\frac{l-x}{l}dx-\sum_{k=1}^{\infty}\frac{b_k\gamma_k}{1+\lambda_k}\int_0^t e^{-\mu_k(t-s)}\mu(s)ds$$

$$-\sum_{k=1}^{\infty}\frac{(a_k+b_k)\gamma_k}{1+\lambda_k}\int_0^t e^{-\mu_k(t-s)}\,\mu'(s)ds=\mu(t)\int_0^l\frac{l-x}{l}dx$$

$$-\sum_{k=1}^{\infty} \frac{b_k \gamma_k}{1+\lambda_k} \int\limits_0^t e^{-\mu_k(t-s)} \mu(s) ds - \mu(t) \sum_{k=1}^{\infty} \frac{(a_k+b_k)\gamma_k}{1+\lambda_k}$$

$$+\sum_{k=1}^{\infty} \frac{(a_k+b_k)\lambda_k \gamma_k}{(1+\lambda_k)^2} \int\limits_0^t e^{-\mu_k(t-s)} \mu(s) ds. \tag{2.6}$$

where γ_k defined by (2.4).

Note that

$$\int\limits_0^l \frac{l-x}{l} dx = \int\limits_0^l \Big(\sum_{k=1}^{\infty} a_k v_k(x) \Big) dx = \sum_{k=1}^{\infty} a_k \gamma_k. \tag{2.7}$$

Thus, from (2.6) and (2.7) we get

$$\theta(t) = \mu(t) \sum_{k=1}^{\infty} \frac{\beta_k}{1+\lambda_k} + \sum_{k=1}^{\infty} \frac{\beta_k}{(1+\lambda_k)^2} \int\limits_0^t e^{-\mu_k(t-s)} \mu(s) ds, \quad t>0, \tag{2.8}$$

where β_k defined by (2.2).

Set

$$B(t) = \sum_{k=1}^{\infty} \frac{\beta_k}{(1+\lambda_k)^2} e^{-\mu_k t}, \quad t>0, \tag{2.9}$$

and

$$\delta = \sum_{k=1}^{\infty} \frac{\beta_k}{1+\lambda_k}.$$

According to (2.8) and (2.9), we have the following integral equation

$$\delta \mu(t) + \int\limits_0^t B(t-s) \mu(s) ds = \theta(t), \quad t>0. \tag{2.10}$$

Proposition 1. *For the cofficients $\{\beta_k\}_{k=1}^{\infty}$ the estimate*

$$0 \le \beta_k \le C, \quad k=1,2,...$$

is valid.

Proof. **Step 1.** Now we use (1.5) and (2.3). Then consider the following equality

$$\lambda_k a_k = \int\limits_0^l \frac{l-x}{l} \lambda_k v_k(x) dx = -\int\limits_0^l \frac{l-x}{l} \frac{d}{dx}\Big(k(x) \frac{dv_k(x)}{dx}\Big) dx$$

$$= -\Big(\frac{l-x}{l} k(x) v_k'(x) \Big|_{x=0}^{x=l} + \frac{1}{l} \int\limits_0^l k(x) v_k'(x) dx \Big) = k(0) v_k'(0) - \frac{1}{l} \int\limits_0^l k(x) v_k'(x) dx$$

$$= k(0)v_k'(0) - \frac{1}{l}\Big(k(l)v_k(l) - k(0)v_k(0)\Big) + \int_0^l \frac{k'(x)}{l} v_k(x)dx$$

$$= k(0)v_k'(0) + b_k.$$

Then we have

$$\lambda_k a_k - b_k = k(0)v_k'(0). \tag{2.11}$$

Step 2. Now we integrate the Eq. (1.5) from 0 to x

$$k(x)v_k'(x) - k(0)v_k'(0) = -\lambda_k \int_0^x v_k(\tau)d\tau,$$

and according to $k(x) > 0,\ x \in [0, l]$, we can write

$$v_k'(x) - \frac{1}{k(x)}k(0)v_k'(0) = -\frac{\lambda_k}{k(x)} \int_0^x v_k(\tau)d\tau. \tag{2.12}$$

Thus, we integrate the Eq. (2.12) from 0 to l. Then we have

$$v_k(l) - v_k(0) - k(0)v_k'(0) \int_0^l \frac{dx}{k(x)} = -\lambda_k \int_0^l \frac{1}{k(x)}\Big(\int_0^x v_k(\tau)d\tau\Big)dx. \tag{2.13}$$

From (1.6) and (2.13) we get

$$k(0)v_k'(0) \int_0^l \frac{dx}{k(x)} = \lambda_k \int_0^l \frac{1}{k(x)}\Big(\int_0^x v_k(\tau)d\tau\Big)dx.$$

Then

$$k(0)v_k'(0) = \lambda_k \int_0^l G(\tau)v_k(\tau)d\tau, \tag{2.14}$$

where

$$G(\tau) = \int_\tau^l \frac{dx}{k(x)}\Big(\int_0^l \frac{dx}{k(x)}\Big)^{-1}.$$

According to $G(\tau) > 0$ and from (2.14) we have (see, [24])

$$v_k'(0) \int_0^l v_k(\tau)d\tau \geq 0. \tag{2.15}$$

Consequently, from (2.11) and (2.15) we get the following estimate

$$\beta_k = (\lambda_k b_k - a_k)\gamma_k = k(0)v_k'(0)\cdot\int_0^l v_k(x)dx \geq 0.$$

Step 3. It is clear that if $k(x) \in C^1([0,l])$, we may write the estimate (see, [24, 26])

$$\max_{0\leq x\leq l}|v_k'(x)| \leq C\lambda_k^{1/2}.$$

Therefore,

$$|v_k'(0)| \leq C\lambda_k^{1/2}, \quad |v_k'(l)| \leq C\lambda_k^{1/2}, \tag{2.16}$$

Then from Eq. (1.5), we can write

$$k(l)v_k'(l) - k(0)v_k'(0) = -\lambda_k \int_0^l v_k(x)dx = -\lambda_k\gamma_k. \tag{2.17}$$

According to (2.16) and (2.17) we have the estimate

$$|\gamma_k| \leq \left|\frac{1}{\lambda_k}\Big(k(l)v_k'(l) - k(0)v_k'(0)\Big)\right| \leq C\lambda_k^{-1/2}.$$

Then

$$\beta_k \leq k(0)\left|v_k'(0)\gamma_k\right| \leq C.$$

Proposition 2. *A function $B(t)$ is continuous on the half-line $t \geq 0$.*

Proof. Indeed, according to Proposition 1 and (2.9), we can write

$$0 < B(t) \leq \text{const}\sum_{k=1}^{\infty}\frac{1}{(1+\lambda_k)^2}.$$

3. Main result

Denote by $W(M)$ the set of function $\theta \in W_2^2(-\infty,+\infty)$, $\theta(t) = 0$ for $t \leq 0$ which satisfies the condition

$$\|\theta\|_{W_2^2(R_+)} \leq M.$$

Theorem 1. *There exists $M > 0$ such that for any function $\theta \in W(M)$ the solution $\mu(t)$ of the equation (2.10) exists, and satisfies condition*

$$|\mu(t)| \leq 1.$$

We write integral equation (2.10)

$$\delta\,\mu(t) + \int\limits_0^t B(t-s)\mu(s)ds = \theta(t), \quad t > 0.$$

By definition of the Laplace transform we have

$$\widetilde{\mu}(p) = \int\limits_0^\infty e^{-pt}\,\mu(t)\,dt.$$

Applying the Laplace transform to the second kind Volterra integral equation (2.10) and taking into account the properties of the transform convolution we get

$$\widetilde{\theta}(p) = \delta\,\widetilde{\mu}(p) + \widetilde{B}(p)\,\widetilde{\mu}(p).$$

Consequently, we obtain

$$\widetilde{\mu}(p) = \frac{\widetilde{\theta}(p)}{\delta + \widetilde{B}(p)}, \quad \text{where } p = a + i\xi, \quad a > 0,$$

and

$$\mu(t) = \frac{1}{2\pi i}\int\limits_{a-i\xi}^{a+i\xi} \frac{\widetilde{\theta}(p)}{\delta + \widetilde{B}(p)}\,e^{pt}dp = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \frac{\widetilde{\theta}(a+i\xi)}{\delta + \widetilde{B}(a+i\xi)}\,e^{(a+i\xi)t}d\xi. \tag{3.1}$$

Then we can write

$$\widetilde{B}(p) = \int\limits_0^\infty B(t)e^{-pt}\,dt = \sum_{k=1}^\infty \frac{\beta_k}{(1+\lambda_k)^2}\int\limits_0^\infty e^{-(p+\mu_k)t}\,dt = \sum_{k=1}^\infty \frac{\rho_k}{p+\mu_k},$$

where $\rho_k = \frac{\beta_k}{(1+\lambda_k)^2} \geq 0$ and

$$\widetilde{B}(a+i\xi) = \sum_{k=1}^\infty \frac{\rho_k}{a+\mu_k+i\xi} = \sum_{k=1}^\infty \frac{\rho_k\,(a+\mu_k)}{(a+\mu_k)^2+\xi^2} - i\xi\sum_{k=1}^\infty \frac{\rho_k}{(a+\mu_k)^2+\xi^2}.$$

It is clear that

$$(a+\mu_k)^2 + \xi^2 \leq [(a+\mu_k)^2 + 1](1+\xi^2),$$

and we have the inequality

$$\frac{1}{(a+\mu_k)^2+\xi^2} \geq \frac{1}{1+\xi^2}\,\frac{1}{(a+\mu_k)^2+1}. \tag{3.2}$$

Consequently, according to (3.2) we can obtain the estimates

$$|\mathrm{Re}(\delta + \widetilde{B}(a+i\xi))| = \delta + \sum_{k=1}^\infty \frac{\rho_k\,(a+\mu_k)}{(a+\mu_k)^2+\xi^2}$$

$$\geq \frac{1}{1+\xi^2}\sum_{k=1}^{\infty}\frac{\rho_k\,(a+\mu_k)}{(a+\mu_k)^2+1}=\frac{C_{1a}}{1+\xi^2}, \tag{3.3}$$

and

$$|\mathrm{Im}(\delta+\widetilde{B}(a+i\xi))| = |\xi|\sum_{k=1}^{\infty}\frac{\rho_k}{(a+\mu_k)^2+\xi^2}$$

$$\geq \frac{|\xi|}{1+\xi^2}\sum_{k=1}^{\infty}\frac{\rho_k}{(a+\mu_k)^2+1}=\frac{C_{2a}\,|\xi|}{1+\xi^2}, \tag{3.4}$$

where C_{1a}, C_{2a} as follows

$$C_{1a}=\sum_{k=1}^{\infty}\frac{\rho_k\,(a+\mu_k)}{(a+\mu_k)^2+1},\ \ C_{2a}=\sum_{k=1}^{\infty}\frac{\rho_k}{(a+\mu_k)^2+1}.$$

From (3.3) and (3.4), we have the estimate

$$|\delta+\widetilde{B}(a+i\xi)|^2 = |\mathrm{Re}(\delta+\widetilde{B}(a+i\xi))|^2 + |\mathrm{Im}(\delta+\widetilde{B}(a+i\xi))|^2 \geq \frac{\min(C_{1a}^2, C_{2a}^2)}{1+\xi^2},$$

and

$$|\delta+\widetilde{B}(a+i\xi)| \geq \frac{C_a}{\sqrt{1+\xi^2}}, \quad \text{where} \quad C_a=\min(C_{1a}, C_{2a}). \tag{3.5}$$

Then, by passing to the limit at $a\to 0$ from (3.1), we can obtain the equality

$$\mu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\theta}(i\xi)}{\delta+\widetilde{B}(i\xi)}\,e^{i\xi t}d\xi. \tag{3.6}$$

Lemma 1. *Let $\theta(t)\in W(M)$. Then for the image of the function $\theta(t)$ the following inequality*

$$\int_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|\,\sqrt{1+\xi^2}d\xi \leq C\,\|\theta\|_{W_2^2(R_+)},$$

is valid.

Proof. We use integration by parts in the integral representing the image of the given function $\theta(t)$

$$\widetilde{\theta}(a+i\xi)=\int_0^{\infty}e^{-(a+i\xi)t}\theta(t)\,dt = -\theta(t)\,\frac{e^{-(a+i\xi)t}}{a+i\xi}\Big|_{t=0}^{t=\infty}+\frac{1}{a+i\xi}\int_0^{\infty}e^{-(a+i\xi)t}\,\theta'(t)\,dt.$$

Then using the obtained inequality and multiplying by the corresponding coefficient we get

$$(a+i\xi)\,\widetilde{\theta}(a+i\xi)=\int_0^{\infty}e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

and for $a \to 0$ we have

$$i\xi\,\widetilde{\theta}(i\xi) = \int_0^\infty e^{-i\xi t}\,\theta'(t)\,dt.$$

Also, we can write the following equality

$$(i\xi)^2\,\widetilde{\theta}(i\xi) = \int_0^\infty e^{-i\xi t}\,\theta''(t)\,dt.$$

Then we have

$$\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2 d\xi \le C_1\,||\theta||^2_{W_2^2(R_+)}. \tag{3.7}$$

Consequently, according to (3.7) we get the following estimate

$$\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\,\sqrt{1+\xi^2}d\xi = \int_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|(1+\xi^2)}{\sqrt{1+\xi^2}}$$

$$\le \Big(\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2 d\xi\Big)^{1/2}\Big(\int_{-\infty}^{+\infty} \frac{1}{1+\xi^2}d\xi\Big)^{1/2} \le C\,||\theta||_{W_2^2(R_+)}.$$

Proof of the Theorem 1. We prove that $\mu \in W_2^1(\mathbb{R}_+)$. Indeed, according to (3.5) and (3.6), we obtain

$$\int_{-\infty}^{+\infty} |\widetilde{\mu}(\xi)|^2(1+|\xi|^2)\,d\xi \;=\; \int_{-\infty}^{+\infty} \left|\frac{\widetilde{\theta}(i\xi)}{\delta + \widetilde{B}(i\xi)}\right|^2 (1+|\xi|^2)\,d\xi$$

$$\le C\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+|\xi|^2)^2\,d\xi \;=\; C||\theta||^2_{W_2^2(\mathbb{R})}.$$

Further,

$$|\mu(t)-\mu(s)| \;=\; \left|\int_s^t \mu'(\tau)\,d\tau\right| \;\le\; ||\mu'||_{L_2}\,\sqrt{t-s}.$$

Hence, $\mu \in \operatorname{Lip}\alpha$, where $\alpha = 1/2$. Then from (3.5), (3.6) and (3.7), we have

$$|\mu(t)| \le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|}{|\delta + \widetilde{B}(i\xi)|}d\xi \le \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\,\sqrt{1+\xi^2}d\xi$$

$$\le \frac{C}{2\pi C_0}||\theta||_{W_2^2(R_+)} \le \frac{C\,M}{2\pi C_0} = 1,$$

as M we took

$$M = \frac{2\pi C_0}{C}.$$

4. Conclusion

An auxiliary boundary value problem for the pseudo-parabolic equation was considered. The restriction for the admissible control is given in the integral form. By the separation variables method, the desired problem was reduced to Volterra's integral equation. The last equation was solved by the Laplace transform method. Theorem on the existence of an admissible control is proved. Later, it is also interesting to consider this problem in the $n-$ dimensional domain. We assume that the methods used in the present problem can also be used in the $n-$ dimensional domain.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare there is no conflict of interest.

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