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Research article

Quantization of Hamiltonian and non-Hamiltonian systems

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Abstract: The quantization process was always tightly connected to the Hamiltonian formulation of classical mechanics. For non-Hamiltonian systems, traditional quantization algorithms turn out to be unsuitable. Numerous attempts to quantize non-Hamiltonian systems have shown that this problem is nontrivial and requires the development of new approaches. In this paper, we present the quantization methods that do not depend upon the Hamiltonian formulation of classical mechanics. Two approaches to the quantization of mechanical systems are considered: axiomatic and hydrodynamic. It is shown that the formal application of these approaches to the classical Hamilton-Jacobi theory allows obtaining the wave equation for the corresponding quantum system in natural way. Examples are considered that show the effectiveness of the proposed approaches, both for Hamiltonian and non-Hamiltonian systems. The spinor form of the relativistic Hamilton-Jacobi theory for classical particles is considered. It is shown that it naturally leads to the Dirac equation for the corresponding quantum particle and to its non-Hamiltonian generalization, the bispinor relativistic Kostin equation.

Keywords: Hamiltonian and non-Hamiltonian systems; quantization; Hamilton-Jacobi theory; Kostin equation; non-linear Dirac equation
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1. Introduction

The basic equation of nonrelativistic quantum mechanics, the Schrödinger equation, is a phenomenological equation: it is not derived from any first principles, but is, in fact, postulated. The traditional operator "derivation" of the Schrödinger equation from classical Hamiltonian mechanics [1,2] is nothing more than a spectacular mathematical technique and cannot be considered as a physical justification for this equation. This is due to the fact that the transition from the physical parameters of a classical system (energy E and generalized momenta P) to the corresponding mathematical operators according to the correspondence rules $E \to i\hbar\frac{\partial}{\partial t}$ and $P \to \frac{\hbar}{i}\frac{\partial}{\partial q}$ is formal and has no physical basis. The only justification for such an approach is that when a number of formalities [2] are fulfilled, it leads to correct results. However, there is no physical explanation for this.

Numerous attempts are known to derive the Schrödinger equation from some first principles [3–7], but none of them can be considered physically convincing.

The situation is even more complicated with the quantization of non-Hamiltonian systems, i.e., with the derivation of the wave equation for a quantum system whose classical counterpart is non-Hamiltonian. Currently, there are several attempts to solve this problem [8–11], however, all of them refer to special cases of non-Hamiltonian systems and do not give a general recipe for quantization of non-Hamiltonian systems.

In this paper, we propose a visual, but also formal "derivation" of the Schrödinger equation for Hamiltonian systems, close to the operator formalism and, in fact, justifying the operator formalism. As will be shown, this method is naturally generalized to a wide class of non-Hamiltonian systems, and in a number of cases allows obtaining the corresponding wave equations. Along with this approach, the hydrodynamic method of "deriving" the wave equations of nonrelativistic quantum mechanics and the method of transition from relativistic classical mechanics to relativistic quantum mechanics are also considered.

In the general case, the problem can be formulated as follows: there is a certain classical mechanical system described by known equations (for example, the Hamilton-Jacobi equation and/or Hamilton equations for Hamiltonian systems or, in the general case for a non-Hamiltonian system, Newton's equation). It is necessary to obtain the equations describing the quantum analogue of this system.

2. Hamiltonian systems

We will use the axiomatic approach. Let us introduce a number of postulates: (i) the state of a quantum system is described by a complex-valued wave function

$$\psi(t,q) = \sqrt{\rho}\exp(iS/\hbar) \tag{1}$$

which satisfies some partial differential equation (wave equation); here $\rho(t,q)$ and $S(t,q)$ are real-valued functions; (ii) the wave equation must be linear one with respect to the wave function $\psi(t,q)$; (iii) in the limit $\hbar \to 0$, the wave equation must lead to the Hamilton-Jacobi theory for the corresponding classical system, i.e. it should decompose into the Hamilton-Jacobi equation for the function $S(t,q)$, which plays the role of an action, and the continuity equation, in which the function $\rho(t,q)$ plays the role of the density of the Hamilton-Jacobi ensemble [12]. Note that the Hamilton-Jacobi equation can be written for any Hamiltonian system [12].

Some "reason" for such an approach can be the analysis of the connection between the Schrödinger equation and the classical theory of Hamilton-Jacobi [13–15].

Let the classical Hamilton-Jacobi ensemble [12] be described by the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+H\left(t,q,\frac{\partial S}{\partial q}\right)=0 \tag{2}$$

and the continuity equation in the configuration space

$$\frac{\partial \rho}{\partial t}+\sum_k \frac{\partial}{\partial q_k}\left(\rho \frac{\partial H(t,q,P)}{\partial P_k}\right)=0 \tag{3}$$

where the Hamiltonian $H(t,q,P)$, is in general an arbitrary function of the generalized momenta $P=\frac{\partial S}{\partial q}$.

Then it is easy to show that the equation

$$\left[i\hbar\frac{\partial}{\partial t}-H\left(t,q,\frac{\hbar}{i}\frac{\partial}{\partial q}\right)\right]\psi=0 \tag{4}$$

which is linear with respect to the wave function $\psi(t,q)$, in the limit $\hbar\to 0$ decomposes into the Hamilton-Jacobi equation (2) and the continuity equation (3).

Indeed, considering the operators $\frac{\partial}{\partial q_k}$ as ordinary variables, we expand the function $H\left(t,q,\frac{\hbar}{i}\frac{\partial}{\partial q}\right)$ formally in a power series:

$$H\left(t,q,\frac{\hbar}{i}\frac{\partial}{\partial q}\right)=H\left(t,q,P^0\right)+\sum_k H_k\left(t,q,P^0\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial q_k}-P_k^0\right)+$$
$$+\frac{1}{2}\sum_{k,n} H_{kn}\left(t,q,P^0\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial q_k}-P_k^0\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial q_n}-P_n^0\right)+\ldots$$

where $H_k\left(t,q,P^0\right)=\left.\frac{\partial H(t,q,P)}{\partial P_k}\right|_{P=P^0}$, $H_{kn}\left(t,q,P^0\right)=\left.\frac{\partial^2 H(t,q,P)}{\partial P_k \partial P_n}\right|_{P=P^0}$, $P^0(t,q)$ is some vector, the ellipsis means the terms of the expansion of the third and higher order with respect to $\left(\frac{\hbar}{i}\frac{\partial}{\partial q_k}-P_k^0\right)$.

Then equation (4) can be written in the form

$$i\hbar\frac{\partial \psi}{\partial t}-H\left(t,q,P^0\right)\psi-\sum_k H_k\left(t,q,P^0\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial q_k}-P_k^0\right)\psi+$$
$$-\frac{1}{2}\sum_{k,n} H_{kn}\left(t,q,P^0\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial q_k}-P_k^0\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial q_n}-P_n^0\right)\psi-\ldots=0$$

Substituting here the wave function in the form (1), we obtain

$$i\hbar\frac{1}{\sqrt{\rho}}\frac{\partial\sqrt{\rho}}{\partial t}\psi-\frac{\partial S}{\partial t}\psi-H\left(t,q,P^0\right)\psi-\psi\sum_k H_k\left(t,q,P^0\right)\left(\frac{\hbar}{i}\frac{1}{\sqrt{\rho}}\frac{\partial\sqrt{\rho}}{\partial q_k}+\frac{\partial S}{\partial q_k}-P_k^0\right)+$$
$$-\frac{1}{2}\psi\sum_{k,n}H_{kn}\left(t,q,P^0\right)\frac{\hbar}{i}\frac{\partial}{\partial q_k}\left(\frac{\hbar}{i}\frac{1}{\sqrt{\rho}}\frac{\partial\sqrt{\rho}}{\partial q_n}+\frac{\partial S}{\partial q_n}-P_n^0\right)+$$
$$-\frac{1}{2}\psi\sum_{k,n}H_{kn}\left(t,q,P^0\right)\left(\frac{\hbar}{i}\frac{1}{\sqrt{\rho}}\frac{\partial\sqrt{\rho}}{\partial q_k}+\frac{\partial S}{\partial q_k}-P_k^0\right)\left(\frac{\hbar}{i}\frac{1}{\sqrt{\rho}}\frac{\partial\sqrt{\rho}}{\partial q_n}+\frac{\partial S}{\partial q_n}-P_n^0\right)-...=0$$

Separating the real and imaginary parts, in the limit $\hbar\to 0$, one obtains

$$\begin{aligned}&\frac{i\hbar}{2\rho}[\frac{\partial\rho}{\partial t}+\sum_k H_k\left(t,q,P^0\right)\frac{\partial\rho}{\partial q_k}+\rho\sum_{k,n}H_{kn}\left(t,q,P^0\right)\frac{\partial}{\partial q_k}\left(\frac{\partial S}{\partial q_n}-P_n^0\right)+\\&+\sum_{k,n}H_{kn}\left(t,q,P^0\right)\frac{\partial\rho}{\partial q_k}\left(\frac{\partial S}{\partial q_n}-P_n^0\right)+...]-\\&-[\frac{\partial S}{\partial t}+H\left(t,q,P^0\right)+\sum_k H_k\left(t,q,P^0\right)\left(\frac{\partial S}{\partial q_k}-P_k^0\right)+\\&+\frac{1}{2}\sum_{k,n}H_{kn}\left(t,q,P^0\right)\left(\frac{\partial S}{\partial q_k}-P_k^0\right)\left(\frac{\partial S}{\partial q_n}-P_n^0\right)+...]+o(\hbar^2)=0\end{aligned}\tag{5}$$

where $o(\hbar^2)$ means small terms of orders of $\hbar^2$ and above. The terms in square brackets do not depend on $\hbar$.

It is easy to see that the square brackets contain the expansions in the series:

$$\sum_k H_k\left(t,q,P^0\right)\frac{\partial\rho}{\partial q_k}+\rho\sum_{k,n}H_{kn}\left(t,q,P^0\right)\frac{\partial}{\partial q_k}\left(\frac{\partial S}{\partial q_n}-P_n^0\right)+$$
$$+\sum_{k,n}H_{kn}\left(t,q,P^0\right)\frac{\partial\rho}{\partial q_k}\left(\frac{\partial S}{\partial q_n}-P_n^0\right)+...=\sum_k\frac{\partial}{\partial q_k}\left(\rho\frac{\partial H(t,q,P)}{\partial P_k}\right)$$

and

$$H\left(t,q,P^0\right)+\sum_k H_k\left(t,q,P^0\right)\left(\frac{\partial S}{\partial q_k}-P_k^0\right)+$$
$$+\frac{1}{2}\sum_{k,n}H_{kn}\left(t,q,P^0\right)\left(\frac{\partial S}{\partial q_k}-P_k^0\right)\left(\frac{\partial S}{\partial q_n}-P_n^0\right)+...=H(t,q,P)$$

where $P=\dfrac{\partial S}{\partial q}$.

Then equation (5) takes the form

$$\frac{i\hbar}{2\rho}\left[\frac{\partial\rho}{\partial t}+\sum_k\frac{\partial}{\partial q_k}\left(\rho\frac{\partial H(t,q,P)}{\partial P_k}\right)\right]-\left[\frac{\partial S}{\partial t}+H(t,q,P)\right]+o(\hbar^2)=0$$

and in the limit $\hbar\to 0$ splits into two real-valued equations (2) and (3).

Thus, equation (4) satisfies the postulates formulated above and can be considered as a wave equation describing the corresponding quantum system.

The above reasoning can be considered as some "substantiation" of the operator formalism in the derivation of the wave equations of quantum mechanics.

Note that this method of reasoning leads to a wave equation describing a quantum system if the Hamilton-Jacobi equation for the corresponding classical system is known.

3. Non-Hamiltonian systems

The approach described above can also be applied to a more general class of the system which are non-Hamiltonian systems if the requirement that the wave equation be linear with respect to the wave function is abandoned.

Let the classical Hamilton-Jacobi ensemble be described by a general Hamilton-Jacobi equation [12]

$$\frac{\partial S}{\partial t}+F\left(t,q,S,\frac{\partial S}{\partial q}\right)=0 \tag{6}$$

Note that the function $F\left(t,q,S,\frac{\partial S}{\partial q}\right)$ in Eq. (6) is not a Hamiltonian function of the system under consideration.

We assume that in this case the corresponding quantum system is also described by the wave function (1). Then, as is easy to show by analogy with the case of the Hamiltonian system considered above, the equation

$$\left[i\hbar\frac{\partial}{\partial t}-F\left(t,q,-i\hbar\ln(\psi/|\psi|),\frac{\hbar}{i}\frac{\partial}{\partial q}\right)\right]\psi=0 \tag{7}$$

in the limit $\hbar\to 0$ splits into the Hamilton-Jacobi equation (6) and the continuity equation

$$\frac{\partial\rho}{\partial t}+\sum_k\frac{\partial}{\partial q_k}\left(\rho\frac{\partial F(t,q,S,P)}{\partial P_k}\right)=0 \tag{8}$$

for some classical Hamilton-Jacobi ensemble. For this reason, Eq (7) should be considered as a "wave" equation describing the quantum analogue of the classical system (6). In this case, obviously, Eq (7) is no longer linear.

According to (8), the velocity of the Hamilton-Jacobi ensemble in the configuration space is determined by the relation

$$\dot{q}_k = \frac{\partial F(t,q,S,P)}{\partial P_k}$$

which generalizes the first equation of Hamilton [16].

As an example, we apply the described method to a nonrelativistic particle moving in a potential field in the presence of drag force. Let us consider the case when the drag force depends linearly on the momentum of the particle: $\mathbf{F} = -\frac{\partial U}{\partial \mathbf{q}} - (\beta/m)\mathbf{P}$, where U is the potential energy of the particle in an external field; β is the constant drag factor; m is the particle mass.

The Hamilton-Jacobi equation for this system has the form [12]

$$\frac{\partial S}{\partial t} + \frac{1}{2m}|\nabla S|^2 + \frac{\beta}{m}S + U(t,\mathbf{r}) = 0 \tag{9}$$

Equation (9) formally has the form (6), where

$$F(t,\mathbf{r},S,\nabla S) = \frac{1}{2m}|\nabla S|^2 + \frac{\beta}{m}S + U(t,\mathbf{r})$$

As a result, wave equation (7) takes the form

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - i\frac{\hbar\beta}{m}\psi\ln(\psi/|\psi|) + U(t,\mathbf{r})\psi = 0 \tag{10}$$

This equation can be rewritten as

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi - i\frac{\beta\hbar}{2m}\psi\ln\left(\psi/\psi^*\right) \tag{11}$$

and is the known Kostin equation [17,18], where ψ^* is the complex conjugate wave function.

4. Hydrodynamic quantization of nonrelativistic systems

The axiomatic approach described above, as well as the standard operator method of “deriving” the Schrödinger equation, are applicable only to those systems for which the Hamilton-Jacobi equation is known for the classical analogue. In particular, these methods are not applicable to arbitrary non-Hamiltonian systems that do not belong to the class (6).

Various approaches have been proposed for quantization of non-Hamiltonian systems [8–11,19–22], however, it seems to us that the most natural and understandable from a physical point of view is the hydrodynamic approach [20], which can be easily substantiated on the basis of the theory Hamilton-Jacobi ensemble [12].

Consider first a single quantum particle moving in an external potential field.

Substituting the wave function in the form (1) into the corresponding Schrödinger equation, and separating the real and imaginary parts, we obtain the equation

$$\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+U-U_q=0 \tag{12}$$

and the continuity equation

$$\frac{\partial \rho}{\partial t}+\mathrm{div}\left(\frac{1}{m}\rho\nabla S\right)=0 \tag{13}$$

where

$$U_q=-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \tag{14}$$

is the so-called quantum potential.

Equation (12) is formally the Hamilton-Jacobi equation for a particle, which is affected by an external classical potential U and an additional quantum potential (14), which has no classical analogue.

Equations (12), (13) describe an ensemble of identical noninteracting particles, which, by analogy with [12], we will call the quantum Hamilton-Jacobi ensemble.

Thus, we come to the conclusion that the quantum Hamilton-Jacobi ensemble moves under the action of an additional potential force

$$\mathbf{F}_q=-\nabla U_q \tag{15}$$

This allows formulating a general quantization rule for any mechanical nonrelativistic systems [20].

Let there be a classical mechanical system whose motion is described by Newton's equation

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}(t,\mathbf{r},\mathbf{v}) \tag{16}$$

where $\mathbf{F}(t,\mathbf{r},\mathbf{v})$ is the external force;

$$\frac{d\mathbf{r}}{dt}=\mathbf{v} \tag{17}$$

is the velocity of the particle.

The classical Hamilton-Jacobi ensemble corresponding to this system is described by the Euler equation [12]

$$m\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}\right)=\mathbf{F}(t,\mathbf{r},\mathbf{v}) \tag{18}$$

and the continuity equation

$$\frac{\partial \rho}{\partial t}+\mathrm{div}\rho\mathbf{v}=0 \tag{19}$$

An additional “quantum force” (15) acts on the corresponding quantum particle, and formally the motion of a quantum particle can be described by Newton's equation

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}(t, \mathbf{r}, \mathbf{v}) - \nabla U_q$$

Then the corresponding quantum Hamilton-Jacobi ensemble is described by the Euler equation

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right) = \mathbf{F}(t, x, \mathbf{v}) - \nabla U_q \tag{20}$$

with quantum potential (14) and continuity equation (19).

We assert that the system of equations (14), (19) and (20) is an analogue of the Schrödinger equation for any (including non-Hamiltonian) nonrelativistic systems.

To demonstrate this, we consider a few examples.

4.1. Nonrelativistic quantum particle in a potential field

In this case $\mathbf{F} = -\nabla U$ and equation (20) takes the form

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right) = -\nabla U - \nabla U_q \tag{21}$$

By analogy with [12], we reduce Eq. (21) to the form

$$m\left(\frac{\partial \mathbf{v}}{\partial t} - [\mathbf{v}\,\mathrm{rot}\,\mathbf{v}]\right) = -\nabla\left(\frac{1}{2}m\mathbf{v}^2 + U + U_q\right) \tag{22}$$

This equation has a potential solution $\mathbf{v}(t, \mathbf{r}) = \frac{1}{m}\nabla S(t, \mathbf{r})$, where the function $S(t, \mathbf{r})$ satisfies equation (12), (14). Equations (12)-(14) are equivalent to the Schrödinger equation for the wave function (1), which proves the formal equivalence of equations (14), (19) and (20) to the Schrödinger equation.

4.2. Nonrelativistic quantum particle in an electromagnetic field

Consider a charged particle subjected to the Lorentz force $\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)$.

In this case, equation (20), by analogy with [12], can be written as

$$\frac{\partial}{\partial t}\left(m\mathbf{v} + \frac{q}{c}\mathbf{A}\right) + \nabla\left(\frac{mV^2}{2} + e\varphi + U_q\right) = [\mathbf{v}\,\mathrm{rot}\left(m\mathbf{v} + \frac{e}{c}\mathbf{A}\right)] \tag{23}$$

This equation has a potential solution $m\mathbf{v} + \frac{e}{c}\mathbf{A} = \nabla S$, where the function $S(t, \mathbf{r})$ satisfies the equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)^2 + e\varphi + U_q = 0 \tag{24}$$

It is easy to check that equation (24) with quantum potential (14), together with the continuity equation (19), is equivalent to the Schrödinger equation for a charged particle in an electromagnetic field

$$i\hbar\frac{\partial \psi}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)^2 + e\varphi\right]\psi$$

4.3. Nonrelativistic quantum particle in a potential field in the presence of drag force

If the particle is under the action of a classical force $\mathbf{F}(t, \mathbf{r}, \mathbf{v}) = -\nabla U - \beta\mathbf{v}$, then equation (20), by analogy with [12], can be written as

$$m\left(\frac{\partial \mathbf{v}}{\partial t} - [\mathbf{v}\,\mathrm{rot}\,\mathbf{v}]\right) = -\nabla\left(\frac{1}{2}m\mathbf{v}^2 + U + U_q\right) - \beta\mathbf{v}$$

This equation has a potential solution $\mathbf{v}(t, \mathbf{r}) = \frac{1}{m}\nabla S(t, \mathbf{r})$, where the function $S(t, \mathbf{r})$ satisfies the equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + U + \frac{\beta}{m}S + U_q = 0 \tag{25}$$

It is easy to see that equation (25) with quantum potential (14) together with the continuity equation (19) are equivalent to one complex-valued equation for the wave function (1)

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi + \frac{\beta}{m}S\psi$$

This equation can also be written in the form (10) or (11) and is the Kostin equation [17,18].

4.4. Nonrelativistic system of interacting quantum particles

Equations of motion of the classical particles have the form

$$m_i\frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i(t, \mathbf{r}_1, \ldots \mathbf{r}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N) \tag{26}$$

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i \tag{27}$$

where $\mathbf{F}_i(t, \mathbf{r}_1, ...\mathbf{r}_N, \mathbf{v}_1, ..., \mathbf{v}_N)$ is the force which acts on i th particle.

The Hamilton-Jacobi ensemble for a system of N interacting particles is characterized by a density $\rho(t, q)$ in the $3N$-dimensional configuration space $q = (\mathbf{r}_1, ..., \mathbf{r}_N)$. This density satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \nabla_i (\rho \mathbf{v}_i) = 0 \tag{28}$$

We assume that the particles are in an external classical potential field $U(t, q)$, which includes the interaction of particles with each other.

For the corresponding quantum system of many particles, the continuum equation of motion (20) has the form [12]

$$m_i \left(\frac{\partial \mathbf{v}_i}{\partial t} + \sum_{k=1}^{N} (\mathbf{v}_k \nabla_k) \mathbf{v}_i \right) = -\nabla_i U - \nabla_i U_q \tag{29}$$

where

$$U_q = -\frac{\hbar^2}{2} \sum_{i=1}^{N} \frac{1}{m_i} \frac{\nabla_i^2 \sqrt{\rho}}{\sqrt{\rho}} \tag{30}$$

is the quantum potential of a system of quantum particles, which is a generalization of quantum potential (14).

Equation (29) has a potential solution $\mathbf{v}_i(t, q) = \frac{1}{m_i} \nabla_i S(t, q),\ i = 1, ..., N$, where the function $S(t, q)$ satisfies the equation

$$\frac{\partial S}{\partial t} + \sum_{k=1}^{N} \frac{1}{2m_k} \left| \nabla_k S \right|^2 + U + U_q = 0 \tag{31}$$

It is easy to check directly that equation (31) with quantum potential (30), together with the continuity equation (28), is equivalent to the many-particle Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \sum_{i=1}^{N} \frac{1}{m_i} \nabla_i^2 \psi + U(t, q) \psi$$

for the wave function (1) defined in $3N$-dimensional configuration space $q = (\mathbf{r}_1, ..., \mathbf{r}_N)$.

5. Quantization of relativistic systems

The description of physical objects in relativistic quantum mechanics is fundamentally different from the description of their classical counterparts.

In classical physics, all parameters describing physical objects are real-valued, and, depending on the nature of the physical object, they can be scalar, vector, or, in general, tensor functions of spatial coordinates and time.

In quantum mechanics, on the contrary, all physical objects are described by complex-valued wave functions, which, depending on the nature of the object, are either scalars or spinors of various ranks.

In classical physics, there is not a single physical object that would be described by spinors in natural way. For this reason, it is believed that the spinor form of writing the equations of motion is a feature of the exclusively quantum mechanical description of microworld objects, and has no analogues in classical physics. This is primarily due to the fact that the rank of a spinor that describes a quantum object is related to the value of the spin of this object, a purely quantum property that classical particles do not have.

At the same time, the spinor form of writing the laws of nature is in fact not an exclusive property of quantum objects, but is only one of the alternative forms of writing equations describing physical objects.

For example, Maxwell equations describing the classical electromagnetic field can be written in spinor form, similar to the Dirac equation [23,24].

In this connection, it is of interest to consider the spinor formulation of classical mechanics. This allows establishing a closer connection between relativistic classical mechanics and relativistic quantum mechanics, in particular with the Dirac equation.

5.1. Derivation of Hamilton-Jacobi theory in classical relativistic mechanics

In classical relativistic mechanics, the 4-velocity of a particle $u^\mu = \dfrac{dx^\mu}{ds}$, by definition, satisfies the condition [25]

$$u_\mu u^\mu = 1 \tag{32}$$

where $u^\mu = (u^0, \mathbf{v}/c)$, $u_\mu = g_{\mu\nu} u^\nu$, $g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$, $\mu,\nu = 0,1,2,3$.

The equations of motion of a charged particle in an electromagnetic field in a four-dimensional form are [25]

$$\frac{du^\mu}{ds} = \frac{e}{mc^2} F^{\mu\nu} u_\nu \tag{33}$$

where e is the electric charge of the particle, m is its rest mass,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \tag{34}$$

is the electromagnetic field tensor; A_μ is the 4-potential of the electromagnetic field.

Let us pass to the relativistic Hamilton-Jacobi ensemble [12]. This means that we consider a 4-velocity field $u^\mu(x)$ and hence $\frac{du^\mu}{ds} = \frac{dx^\nu}{ds}\frac{\partial u^\mu}{\partial x^\nu} = u^\nu \frac{\partial u^\mu}{\partial x^\nu}$. Equation (33) corresponds to the Lagrangian description of the ensemble; it can be rewritten in the form of the 4-dimensional Euler equation

$$u^\nu \frac{\partial u_\mu}{\partial x^\nu} = \frac{e}{mc^2} F_{\mu\nu} u^\nu \tag{35}$$

To this equation, it is necessary to add the continuity equation for the Hamilton-Jacobi ensemble

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 \tag{36}$$

where $J^\mu = \rho_0 u^\mu$ is the particle flux density in the Hamilton-Jacobi ensemble; ρ_0 is a 4-scalar, which is related to the density ρ of the Hamilton-Jacobi ensemble in the considered inertial reference frame by the relation [25]

$$\rho = \rho_0 \frac{cdt}{ds}$$

Equations (35) and (36) are a general relativistic formulation of the Hamilton-Jacobi theory for a charged particle in an electromagnetic field.

From them, it is easy to obtain the well-known relativistic Hamilton-Jacobi equation.

Taking into account (34), we rewrite (35) in the form

$$u^\nu \left[\frac{\partial}{\partial x^\nu}\left(u_\mu + \frac{e}{mc^2} A_\mu \right) - \frac{e}{mc^2}\frac{\partial A_\nu}{\partial x^\mu} \right] = 0 \tag{37}$$

It follows from relation (32) that

$$u^\nu \frac{\partial u_\nu}{\partial x^\mu} = 0 \tag{38}$$

Subtract from the left side of (37) the expression $u^\nu \frac{\partial u_\nu}{\partial x^\mu}$; taking into account (38), one obtains

$$u^{\nu}\left[\frac{\partial}{\partial x^{\nu}}\left(u_{\mu}+\frac{e}{mc^{2}}A_{\mu}\right)-\frac{\partial}{\partial x^{\mu}}\left(u_{\nu}+\frac{e}{mc^{2}}A_{\nu}\right)\right]=0 \tag{39}$$

Obviously, one of the solutions to equation (39) is

$$mcu_{\mu}+\frac{e}{c}A_{\mu}=-\frac{\partial S}{\partial x^{\mu}}$$

where S is the 4-scalar.

Thus

$$mcu_{\mu}=-\frac{\partial S}{\partial x^{\mu}}-\frac{e}{c}A_{\mu} \tag{40}$$

Substituting (40) into (32) and (36), we obtain the relativistic Hamilton-Jacobi equation [25]

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^{\mu}}+\frac{e}{c}A_{\mu}\right)\left(\frac{\partial S}{\partial x^{\nu}}+\frac{e}{c}A_{\nu}\right)=m^{2}c^{2} \tag{41}$$

and the continuity equation for the Hamilton-Jacobi ensemble

$$g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\left[\rho_{0}\left(\frac{\partial S}{\partial x^{\mu}}+\frac{e}{c}A_{\mu}\right)\right]=0 \tag{42}$$

Equations (41) and (42) are the traditional formulation of the relativistic Hamilton-Jacobi theory for a classical charged particle, which naturally follows from the general formulation of the theory of motion of the Hamilton-Jacobi ensemble (35) and (36).

It is easy to check that the Klein-Gordon equation

$$g^{\mu\nu}\left(i\hbar\partial_{\mu}-\frac{e}{c}A_{\mu}\right)\left(i\hbar\partial_{\nu}-\frac{e}{c}A_{\nu}\right)\psi=m^{2}c^{2} \tag{43}$$

with respect to the wave function

$$\psi=\sqrt{\rho_{0}}\,\exp(iS/\hbar) \tag{44}$$

obtained by the formal replacement $\frac{\partial S}{\partial x^{\mu}}\to -i\hbar\partial_{\mu}$ in the Hamilton-Jacobi equation (41), satisfies the postulates of Section 2: at $\hbar\to 0$, equation (43) splits into the Hamilton-Jacobi equation (41) and the continuity equation (42), which describe the classical Hamilton-Jacobi ensemble.

5.2. Spinor representation of Hamilton-Jacobi theory for classical charged particle in electromagnetic field

Let us show that the relativistic theory of the Hamilton-Jacobi ensemble for a classical charged particle, considered in the previous section, can be written in the spinor form, which is completely equivalent to its traditional formulation (41), (42).

Consider the equation

$$(\gamma^{\mu}u_{\mu}-1)\,\Psi=0 \tag{45}$$

where 1 is the identity matrix; γ^{μ} are the Dirac matrices; $\Psi=\mathrm{col}(\psi_1,\psi_2,\psi_3,\psi_4)$ is the some column matrix.

Relations (45) can be considered as a system of linear homogeneous algebraic equations with respect to unknown parameters $\psi_1,\psi_2,\psi_3,\psi_4$.

If relation (45) is assumed to be Lorentz invariant, then it follows from the transformation properties of u_{μ} and γ^{μ} that the matrix Ψ is a bispinor.

Using α-matrices

$$\alpha_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}$$

equation (45) can be rewritten as [26]

$$(u^0-\boldsymbol{\alpha}\mathbf{u}-\alpha_0)\Psi=0 \tag{46}$$

In expanded form, the system of equations (46) looks as follows:

$$\begin{aligned}
&(u^0-1)\psi_1-u_3\psi_3-(u_1-iu_2)\psi_4=0\\
&(u^0-1)\psi_2-(u_1+iu_2)\psi_3+u_3\psi_4=0\\
&u_3\psi_1+(u_1-iu_2)\psi_2-(u^0+1)\psi_3=0\\
&(u_1+iu_2)\psi_1-u_3\psi_2-(u^0+1)\psi_4=0
\end{aligned} \tag{47}$$

The determinant of this system of equations is

$$\Delta=(u_{\mu}u^{\mu}-1)^2$$

In order for the system of equations (47) to have nontrivial solutions $\Psi\neq 0$, the determinant of the system must be equal to zero: $(u_{\mu}u^{\mu}-1)^2=0$.

As a result, we come to the conclusion that the system of equations (45) has nontrivial solutions $\Psi\neq 0$ only if condition (32) is satisfied. In other words, for nonzero matrices Ψ, relation (32) is a trivial consequence of equation (45).

The same result can be obtained more formally using the standard representation [26].

We introduce spinors $\varphi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ and $\chi=\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$. Then $\Psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$. Taking this into account, equation (46) decomposes into two spinor equations

$$(u^0 - 1)\varphi - \mathbf{u\sigma}\chi = 0 \tag{48}$$

$$(u^0 + 1)\chi - \mathbf{u\sigma}\varphi = 0 \tag{49}$$

From equation (49), one obtains

$$\chi = \frac{\mathbf{u\sigma}\varphi}{u^0 + 1} \tag{50}$$

Substituting χ from (50) into equation (48), one obtains

$$(u^0 - 1)\varphi - (\mathbf{u\sigma})\frac{(\mathbf{u\sigma})\varphi}{u^0 + 1} = 0$$

Taking into account that $(\mathbf{u\sigma})(\mathbf{u\sigma}) = \mathbf{u}^2$ [26], one obtains

$$(u_\mu u^\mu - 1)\varphi = 0$$

For $\varphi \neq 0$ we again arrive at equation (32).

Substituting relation (40) into equation (45), for the classical Hamilton-Jacobi ensemble, one obtains

$$\left(\gamma^\mu \left(\frac{\partial S}{\partial x^\mu} + \frac{e}{c} A_\mu\right) + mc\right)\Psi = 0 \tag{51}$$

The equivalence of equations (32) and (45) also means the equivalence of the Hamilton-Jacobi equation (41) and the spinor equation (51). Thus, equation (51) for nonzero matrices Ψ is an equivalent (spinor) form of the relativistic Hamilton-Jacobi equation for a classical charged particle in an external electromagnetic field.

As is known [2,26], from the bispinor Ψ and the Dirac matrix γ^μ, one can construct a real-valued 4-vector

$$J^\mu \equiv (J^0, \mathbf{J}) = \overline{\Psi}\gamma^\mu\Psi \tag{52}$$

where $\overline{\Psi} = \Psi^+\gamma^0$, Ψ^+ is the matrix Hermitian conjugate to the matrix Ψ.

Using the spinors φ and χ, as well as the connection between them (50), after simple transformations one obtains

$$J^0 = \frac{2|\varphi|^2}{u^0+1} u^0, \ \mathbf{J} = \frac{2|\varphi|^2}{u^0+1} \mathbf{u}$$

or

$$J^\mu = \frac{2|\phi|^2}{u^0+1} u^\mu \tag{53}$$

Similarly, a 4-scalar can be formed from the bispinor Ψ and the Dirac matrices γ^μ [2,26]

$$\rho_0 = \overline{\Psi}\Psi$$

which can be converted to the form

$$\rho_0 = \frac{2|\varphi|^2}{u^0+1} \tag{54}$$

Using (53) and (54), one can write

$$J^\mu = \rho_0 u^\mu \tag{55}$$

Taking into account (40), one obtains

$$J_\mu = -\frac{\rho_0}{mc}\left(\frac{\partial S}{\partial x^\mu} + \frac{e}{c} A_\mu\right) \tag{56}$$

Equation (45) implies that the bispinor Ψ is defined up to a factor, which can be an arbitrary 4-scalar function. This allows us to choose the 4-scalar ρ_0 in such a way that the 4-vector (55) determines the particle flux density in the Hamilton-Jacobi ensemble, which satisfies the continuity equation (36) or, in expanded form, equation (42).

Taking into account (54) and (40), relation (56) can be rewritten using the spinor φ in the form

$$J_\mu = 2|\varphi|^2 \frac{\left(\frac{\partial S}{\partial x^\mu} + \frac{e}{c} A_\mu\right)}{\frac{\partial S}{\partial x^0} + \frac{e}{c} A_0 - mc} \tag{57}$$

If the action S is known (for example, found by solving the Hamilton-Jacobi Eq (41)), then the continuity equation (36) with the 4-vector (57) allows finding the modulus of the spinor φ, and, using relation (50), the modulus of the spinor χ.

Thus, we have shown that for a classical charged particle moving in an external electromagnetic field, the Hamilton-Jacobi theory can be formulated in spinor form, while the corresponding Hamilton-Jacobi ensemble is described by the Hamilton-Jacobi equation in the spinor representation (51) and the continuity equation (36) for the particle flux density (57), which is expressed in terms of the bispinor Ψ and the Dirac matrix by the same expression (52) as in the Dirac theory.

Note that in the non-relativistic limit, when $|\mathbf{u}| << 1$ and $1-u^0 << 1$, it will be $\chi << \varphi$, taking into account that the density of the Hamilton-Jacobi ensemble $\rho = \rho_0 u^0$, one obtains

$$\rho \approx \rho_0 \approx |\varphi|^2$$

The connection of the spinor form of writing (36), (51) and (52) of the theory of the classical Hamilton-Jacobi ensemble with the Dirac equation is obvious.

Indeed, we represent the components of the bispinor Ψ as

$$\psi_n = |\psi_n| \exp(i\alpha_n + iS_n/\hbar) \tag{58}$$

where S_n and α_n are real-valued functions having finite limits at $\hbar \to 0$; $n = 1,2,3,4$.

We set the conditions:

$$\lim_{\hbar\to 0} \hbar\partial_\mu |\psi_n| = 0,\ \lim_{\hbar\to 0} \hbar\partial_\mu S_n = 0,\ \lim_{\hbar\to 0} \hbar\partial_\mu \alpha_n = 0,\ \lim_{\hbar\to 0} \hbar\alpha_n = 0 \tag{59}$$

Then, it is easy to show that the Dirac equation

$$\left(\gamma^\mu \left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right) - mc\right)\Psi = 0$$

obtained by a formal replacement $\frac{\partial S}{\partial x^\mu} \to -i\hbar\partial_\mu$ in equation (51), at $\hbar \to 0$ decomposes into equation (51), which is the spinor form of the Hamilton-Jacobi equation for a classical particle, and the continuity equation (36), (52).

Thus, we see that the postulates introduced in Section 2 are valid both in the nonrelativistic and relativistic cases, which is the justification for the operator formalism in relativistic quantum mechanics.

5.3. Non-Hamiltonian relativistic systems

In the general case, the relativistic equation of particle motion has the form [25]

$$mc\frac{du^\mu}{ds} = F^\mu \tag{60}$$

where the 4-force F^μ satisfies the condition

$$u_\mu F^\mu = 0 \tag{61}$$

In the case of particle motion in an external electromagnetic field, the 4-force (33), (34) automatically satisfies condition (61).

We consider a relativistic motion of a particle in an external electromagnetic field in the presence of drag force. The corresponding nonrelativistic case was considered in [12].

In the relativistic case, equation (60), taking into account (33), has the form

$$mc\frac{du^{\mu}}{ds}=\frac{e}{c}F^{\mu\nu}u_{\nu}+f^{\mu} \tag{62}$$

where the drag 4-force $f^{\mu}=(f^{0},\mathbf{f})$, according to (61), satisfies the condition

$$u_{\mu}f^{\mu}=0 \tag{63}$$

The simple 4-vector f^{μ} generalizing the nonrelativistic drag force [12] and satisfying condition (63) has the form

$$f^{\mu}=\beta u_{\nu}\left[\left(u^{\nu}+\frac{e}{mc^{2}}A^{\nu}\right)U^{\mu}-\left(u^{\mu}+\frac{e}{mc^{2}}A^{\mu}\right)U^{\nu}\right] \tag{64}$$

where $U^{\mu}=(U^{0},\mathbf{V}/c)$ is the constant 4-vector; β is the constant 4-scalar (drag factor).

It is easy to check that the 4-force (64) in the non-relativistic approximation ($\mathbf{v}^{2}/c^{2}<<1$ and $\mathbf{U}^{2}/c^{2}<<1$) transforms into a linear resistance law

$$\mathbf{f}=-\frac{\beta}{mc}\left[m(\mathbf{v}-\mathbf{V})+\frac{e}{c}\mathbf{A}\right] \tag{65}$$

considered in [12] for a particle moving in an external electromagnetic field.

Obviously, the system described by equations (62), (64) is non-Hamiltonian one.

For the Hamilton-Jacobi ensemble, using the continuum description, we reduce equation (62) to the Euler equation

$$u^{\nu}\frac{\partial u_{\mu}}{\partial x^{\nu}}=\frac{e}{mc^{2}}F_{\mu\nu}u^{\nu}+\frac{1}{mc}f_{\mu} \tag{66}$$

Equations (66), (34) together with the continuity Eq (36) form the Hamilton-Jacobi theory [12] for the relativistic motion of a particle in the presence of a “linear” drag force.

Substituting (34) and (64) into equation (66), we transform it by analogy with Section 5.1 to the form

$$u^{\nu}\left(\frac{\partial P_{\mu}}{\partial x^{\nu}}-\frac{\partial P_{\nu}}{\partial x^{\mu}}-\frac{\beta}{mc}(P_{\nu}U_{\mu}-P_{\mu}U_{\nu})\right)=0 \tag{67}$$

where

$$P_\mu = mcu_\mu + \frac{e}{c} A_\mu \tag{68}$$

Equation (67) has a simple solution

$$P_\mu = -\frac{\partial S}{\partial x^\mu} - \frac{\beta}{mc} SU_\mu \tag{69}$$

where S is the 4-scalar function.

Taking into account (68), one obtains

$$mcu_\mu = -\frac{\partial S}{\partial x^\mu} - \frac{e}{c} A_\mu - \frac{\beta}{mc} SU_\mu \tag{70}$$

This solution generalizes relation (40) and passes into it for $\beta = 0$.

Using (32), we obtain the relativistic Hamilton-Jacobi equation for a particle moving in an external electromagnetic field in the presence of a drag force (64)

$$g^{\mu\nu} \left(\frac{\partial S}{\partial x^\mu} + \frac{\beta}{mc} SU_\mu + \frac{e}{c} A_\mu \right) \left(\frac{\partial S}{\partial x^\nu} + \frac{\beta}{mc} SU_\nu + \frac{e}{c} A_\nu \right) = m^2 c^2 \tag{71}$$

This equation generalizes the Hamilton-Jacobi equation (41) to the case of drag force (64) and together with the continuity equation (36) describes the classical Hamilton-Jacobi ensemble.

Using the postulates formulated in Section 2, it is easy to obtain a relativistic wave equation for the scalar wave function (44) for the corresponding spinless quantum particle; to do this, it suffices to make the replacements $\frac{\partial S}{\partial x^\mu} \to -i\hbar \partial_\mu$ and $S = -i\hbar \ln(\psi/|\psi|)$. As a result, we obtain the equation

$$g^{\mu\nu} \left(i\hbar\partial_\mu - \frac{e}{c} A_\mu + \frac{i\hbar\beta}{mc} U_\mu \ln(\psi/|\psi|) \right) \left(i\hbar\partial_\nu - \frac{e}{c} A_\nu + \frac{i\hbar\beta}{mc} U_\nu \ln(\psi/|\psi|) \right) \psi = m^2 c^2 \tag{72}$$

which generalizes the Klein-Gordon equation (43) to the case of drag force (64) and is a relativistic version of the Kostin equation (10).

In the case of the spinor representation of the Hamilton-Jacobi equation (45), we substitute (70) into (45). As a result, one obtains

$$\left[\gamma^\mu \left(\frac{\partial S}{\partial x^\mu} + \frac{e}{c} A_\mu + \frac{\beta}{mc} SU_\mu \right) + mc \right] \Psi = 0 \tag{73}$$

The spinor equation (73) together with the continuity equation (36), (70) describes the classical Hamilton-Jacobi ensemble in the presence of drag force (64), and, as shown above, at $\Psi \neq 0$ is equivalent to equation (71).

To obtain the wave equation for the corresponding quantum particle, we introduce a unit spinor Φ satisfying the condition

$$\bar{\Phi}\Phi = 1 \tag{74}$$

where $\overline{\Phi} = \Phi^{+}\gamma^{0}$, Φ^{+} is the matrix Hermitian conjugate to the matrix Φ.

By definition, $\overline{\Phi}\Psi$ is a 4-scalar [26].

Obviously, the function $-i\hbar\ln\left(\overline{\Phi}\Psi/|\Psi|\right)$, where $|\Psi|^{2} = \overline{\Psi}\Psi$, in the limit $\hbar \to 0$ is equivalent to the action S.

Then the corresponding wave equation for a quantum particle with spin ½ with respect to the wave function (58), (59) according to the postulates of Section 2 will be obtained by replacing $\frac{\partial S}{\partial x^{\mu}} \to -i\hbar\partial_{\mu}$ and $S \to -i\hbar\ln\left(\overline{\Phi}\Psi/|\Psi|\right)$ in Eq (73). As a result, one obtains a nonlinear equation

$$\left[\gamma^{\mu}\left(i\hbar\partial_{\mu} - \frac{e}{c}A_{\mu} + \frac{i\hbar\beta}{mc}U_{\mu}\ln(\overline{\Phi}\Psi/|\Psi|)\right) - mc\right]\Psi = 0 \quad (75)$$

Equation (75) generalizes the Dirac equation to the case of a non-Hamiltonian quantum system under the action of the drag force (64) and can be called the bispinor Kostin equation or the Dirac-Kostin equation.

6. Conclusions

In this paper, we present the quantization methods that allow obtaining wave equations for both Hamiltonian and non-Hamiltonian systems. Using these methods, we have obtained nonlinear wave equations for both non-relativistic and relativistic non-Hamiltonian systems. We hope that these approaches can be useful for studying non-Hamiltonian (in particular, dissipative) quantum mechanical systems, such as mesoscopic systems.

In the next articles of this series, we plan to apply the considered quantization methods to complex non-Hamiltonian systems.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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