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Research article

Continuous dependence on initial data and high energy blowup time estimate for porous elastic system

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Abstract: In this paper, we establish two conclusions about the continuous dependence on the initial data of the global solution to the initial boundary value problem of a porous elastic system for the linear damping case and the nonlinear damping case, respectively, which reflect the decay property of the dissipative system. Additionally, we estimate the lower bound of the blowup time at the arbitrary positive initial energy for nonlinear damping case.

Keywords: porous elastic system; continuous dependence; blowup time; arbitrary positive initial energy
Mathematics Subject Classification: 35L53, 35B30, 35B44

1. Introduction

We consider the initial boundary value problem of the following porous elastic system with nonlinear or linear weak damping terms and nonlinear source terms

$$\begin{cases} u_{tt} - \mu u_{xx} - b\phi_x + g_1(u_t) = f_1(u,\phi), & x \in (0,L),\ t \in [0,T), \\ \phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + g_2(\phi_t) = f_2(u,\phi), & x \in (0,L),\ t \in [0,T), \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in (0,L), \\ \phi(x,0) = \phi_0(x),\ \ \phi_t(x,0) = \phi_1(x), & x \in (0,L), \\ u(0,t) = u(L,t) = \phi(0,t) = \phi(L,t) = 0, & t \in [0,T), \end{cases} \tag{1.1}$$

where $u(x,t)$ and $\phi(x,t)$ are the displacement of the solid elastic material and the volume fraction, respectively, $\mu,\ b,\ \delta$ and ξ are coefficients with physical meaning satisfying

$$\mu > 0,\ b \neq 0,\ \delta > 0,\ \xi > 0 \text{ and } \mu\xi > b^2,$$

$u_0,\ u_1, \phi_0$ and ϕ_1 are given initial data, and the assumptions of weak damping terms $g_1,\ g_2$ and nonlinear source terms $f_1,\ f_2$ will be given in Section 2 by Assumption 2.1 and Assumption 2.2, respectively .

In the physical view, elastic solid with voids is an important extension of the classical elasticity theory. It allows the processing of porous solids in which the matrix material is elastic and the interstices are void of material (see [8, 20] and references therein). Porous media reflects the properties of many materials in the real world, including rocks, soil, wood, ceramics, pressed powder, bones, natural gas hydrates and so on. Due to the diversity of porous media and its special physical properties, such models were widely applied in the past few decades in the petroleum industry, engineering, etc (see [1, 12, 13, 16, 17, 19]).

As mathematical efforts, Goodman and Cowin [2, 8] established the continuum theory and the variational principle of granular materials. Then Nunziato and Cowin [3, 18] developed the linear and nonlinear theories of porous elastic materials. In recent years, the study of the porous elastic system also attracted a lot of attention [5–7, 21, 22]. We particularly mention that Freitas et.al. in [5] studied the problem (1.1) and proved the global existence and finite time blowup of solutions. Especially, they built up the continuous dependence on initial data of the local solution in the following version

$$\widehat{E}(t) \leq e^{C_0 t}\widehat{E}(0),\ C_0 > 0, \tag{1.2}$$

which can also be extended to the global solution with the same form. By denoting $z = (u, \phi)$ and $\tilde{z} = (\tilde{u}, \tilde{\phi})$ the global solutions to problem (1.1) corresponding to the initial data $z_0,\ z_1$ and $\tilde{z}_0,\ \tilde{z}_1$, respectively, $\widehat{E}(0)$ is the distance of two sets of different initial data

$$z_0, \tilde{z}_0 \in V := H_0^1(0, L) \times H_0^1(0, L),$$

and

$$z_1, \tilde{z}_1 \in L^2(0, L) \times L^2(0, L),$$

that is

$$\widehat{E}(0) := \frac{1}{2}\|z_1 - \tilde{z}_1\|_2^2 + \frac{1}{2}\|z_0 - \tilde{z}_0\|_V^2,$$

and $\widehat{E}(t)$ is the distance of solutions induced by these two sets of different initial data

$$\widehat{E}(t) := \frac{1}{2}\|z_t - \tilde{z}_t\|_2^2 + \frac{1}{2}\|z - \tilde{z}\|_V^2.$$

The growth estimate (1.2) indicates that the growth of the distance of solutions $\widehat{E}(t)$ is bounded by an exponential growth bound with time t. In other words, as the time t goes to infinity, the distance of solutions $\widehat{E}(t)$ of the system is bounded by a very large bound, by which it is hard to explain the solutions z and $\tilde{z}$ of such a dissipative system with the initial data z_0, z_1 and $\tilde{z}_0, \tilde{z}_1$, respectively, as both of them are expected to decay to zero as the time t goes to infinity. Hence, the estimate on the growth of the distance of solutions $\widehat{E}(t)$ is proposed to be improved to reflect the decay properties with time t to be consistent with the dissipative behavior of the system. To achieve this, the efforts in the present paper are illustrated by two new continuous dependence results on the initial data for the global-in-time solution. Especially, it is found that the system with the linear damping term behaves differently from that with the nonlinear damping term. Hence in the present paper, we adopt two different estimate strategies to deal with the problem and derive two different conclusions:

(i) For the linear damping case, i.e., $g_1(u_t)$ and $g_2(\phi_t)$ take the linear form and satisfy Assumption 2.1, we have

$$\widehat{E}(t) \leq C_1 \left(\widehat{E}(0) + C_2 \left(\widehat{E}(0)\right)^{\frac{a}{2}}\right)^{\rho} e^{-C_3 t}, \tag{1.3}$$

where the positive constants C_1, C_2, C_3, a, ρ are independent of initial data.

(ii) For the nonlinear damping case, i.e., $g_1(u_t)$ and $g_2(\phi_t)$ take the nonlinear form and satisfy Assumption 2.1, we have

$$\widehat{E}(t) \leq C_5 \left(\widehat{E}(0) + C_6 \left(\widehat{E}(0)\right)^{\frac{b_0}{2}}\right)^{\kappa} e^{-C_7 t}, \tag{1.4}$$

where $0 < \kappa < 1$, and the positive constants C_5, C_6, C_7, b_0 are dependent of initial data.

By observing (1.3) and (1.4), we find that these two continuous dependence results can reasonably reflect the decay property of the dissipative system (1.1). The difference between (1.3) and (1.4) is that the parameters in (1.3) do not depend on the initial data, while the parameters in (1.4) depend on the initial data. Hence although (1.3) and (1.4) are in the similar form, we present and prove them separately.

Additionally, to develop the finite time blowup of the solution to problem (1.1) at the arbitrary positive initial energy level derived in [22], we estimate the lower bound of the blowup time in the present paper for the nonlinear weak damping case by noticing that the linear weak damping case was discussed in [22]. For more relative works on the blowup of solutions to the hyperbolic equations at high initial energy, please refer to [10,11,14,15,25]. We can also refer to [9,23,24] for the works about the blowup of solutions to parabolic equations.

The rest of the present paper is organized as follows. In Section 2, we give some notations, assumptions about damping terms and source terms, and functionals and manifolds for the potential well theory. In Section 3, we deal with the continuous dependence on initial data of the global solution for the linear weak damping case. In Section 4, we establish the continuous dependence on initial data of the global solution for the nonlinear weak damping case. In Section 5, we estimate the lower bound of blowup time at the arbitrarily positive initial energy level for the nonlinear weak damping case.

2. Preliminaries

2.1. Notations and assumptions

We denote the L^2-inner product by

$$(u, v) := \int_0^L uv \mathrm{d}x,$$

and the norm of $L^p(0, L)$ by

$$\|u\|_p := \left(\int_0^L |u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

As we are dealing with the system of two equations, for $z = (u, \phi)$ and $\hat{z} = (\hat{u}, \hat{\phi})$, we introduce

$$(z, \hat{z}) := (u, \hat{u}) + (\phi, \hat{\phi})$$

and

$$\|z\|_p := (\|u\|_p^p + \|\phi\|_p^p)^{\frac{1}{p}}. \tag{2.1}$$

Further, we consider the Hilbert space

$$V = H_0^1(0,L) \times H_0^1(0,L)$$

with inner products given by

$$(z,\hat{z})_V := \int_0^L \left(\mu u_x \hat{u}_x + \delta\phi_x\hat{\phi}_x + \xi\phi\hat{\phi} + b\left(u_x\hat{\phi} + \phi\hat{u}_x\right)\right) \mathrm{d}x \tag{2.2}$$

for $z = (u,\phi)$, $\hat{z} = (\hat{u},\hat{\phi})$, where μ, δ, ξ, b are the coefficients of the terms in the equations in problem (1.1). Therefore, we have

$$\|z\|_V^2 := \int_0^L \left(\mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right) \mathrm{d}x. \tag{2.3}$$

The norm $\|z\|_V$ is equivalent to the corresponding usual norm on V, i.e., $H_0^1(0,L) \times H_0^1(0,L)$, introduced in [20]. For $1 < q < +\infty$, we define

$$R_q := \sup_{z\in V\setminus\{0\}} \frac{\|z\|_q^q}{\|z\|_V^q}, \tag{2.4}$$

which means

$$\|z\|_q^q \le R_q\|z\|_V^q \tag{2.5}$$

for $z \in V$. Here, due to $H_0^1(0,L) \hookrightarrow L^q(0,L)$ for $1 < q < +\infty$, we see $0 < R_q < +\infty$. And we denote

$$\mathcal{F}(z) := (f_1(u,\phi), f_2(u,\phi))$$

and

$$\mathcal{G}(z_t) := (g_1(u_t), g_2(\phi_t)),$$

where $f_j(u,\phi)$, $j = 1,2$, are the source terms, and $g_1(u_t)$ and $g_2(\phi_t)$ are the damping terms in the equations in problem (1.1).

We give the following assumptions about damping terms, i.e., $g_1(u_t)$ and $g_2(\phi_t)$, and source terms, i.e., $f_j(u,\phi)$, $j = 1,2$, in the equations in problem (1.1).

Assumption 2.1. (Damping terms) Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be continuous, monotone increasing functions with $g_1(0) = g_2(0) = 0$. In addition, there exist positive constants $\alpha > 0$ and $\beta > 0$ such that
(i) for $|s| \ge 1$

$$\alpha|s|^{m+1} \le g_1(s)s \le \beta|s|^{m+1},\ m \ge 1; \tag{2.6}$$

and

$$\alpha|s|^{r+1} \le g_2(s)s \le \beta|s|^{r+1},\ r \ge 1; \tag{2.7}$$

(ii) for $|s| < 1$

$$\alpha|s|^{\hat{m}} \le |g_1(s)| \le \beta|s|^{\hat{m}},\ \hat{m} \ge 1; \tag{2.8}$$

and

$$\alpha|s|^{\hat{r}} \le |g_2(s)| \le \beta|s|^{\hat{r}},\ \hat{r} \ge 1. \tag{2.9}$$

Assumption 2.2. (Source terms) For the functions $f_j \in C^1(\mathbb{R}^2)$, $j = 1, 2$, there exists a positive constant C such that

$$|\nabla f_j(\eta)| \leq C\left(|\eta_1|^{p-1} + |\eta_2|^{p-1} + 1\right), \ p > 1. \tag{2.10}$$

where $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, $f_j(\eta) = f_j(\eta_1, \eta_2)$, $j = 1, 2$, and

$$\nabla f_j := \left(\frac{\partial f_j}{\partial \eta_1}, \frac{\partial f_j}{\partial \eta_2}\right).$$

There exists a nonnegative function $F \in C^2(\mathbb{R}^2)$ satisfying

$$\nabla F = \mathcal{F} \tag{2.11}$$

and

$$F(\lambda\eta) = \lambda^{p+1} F(\eta) \tag{2.12}$$

for all $\lambda > 0$, where $F(\eta) = F(\eta_1, \eta_2)$ and

$$\nabla F := \left(\frac{\partial F}{\partial \eta_1}, \frac{\partial F}{\partial \eta_2}\right). \tag{2.13}$$

According to [5], Assumption 2.2 implies that there exists a constant $M > 0$ such that

$$F(z) \leq M\left(|u|^{p+1} + |\phi|^{p+1}\right). \tag{2.14}$$

2.2. *Potential well*

Next, we recall some functionals and manifolds for the potential well theory. We recall the potential energy functional

$$J(z) := \frac{1}{2}\|z\|_V^2 - \int_0^L F(z)\mathrm{d}x \tag{2.15}$$

and the Nehari functional

$$I(z) := \|z\|_V^2 - (p+1)\int_0^L F(z)\mathrm{d}x.$$

The energy functional is defined as

$$\mathcal{E}(z(t), z_t(t)) := \frac{1}{2}\|z_t\|_2^2 + J(z). \tag{2.16}$$

And the Nehari manifold is defined as

$$\mathcal{N} := \{z \in V\backslash\{0\} |\ I(z) = 0\}.$$

Then we can define the depth of the potential well

$$d := \inf_{z\in\mathcal{N}} J(z).$$

By above, we introduce the stable manifold

$$W := \{z \in V |\ J(z) < d,\ I(z) > 0\} \cup \{0\}.$$

Next, since we need to apply the decay rate of the energy in investigating continuous dependence on the initial data of the solution, we recall the following notations used in the investigation of the decay rate of the energy in [5]

$$\hat{d} := \sup_{s\in[0,+\infty)} \mathcal{M}(s) = \mathcal{M}(s_0) = \frac{p-1}{2(p+1)} \left((p+1)\, MR_{p+1}\right)^{-\frac{2}{p-1}}, \tag{2.17}$$

where

$$\mathcal{M}(s) := \frac{1}{2}s^2 - MR_{p+1}s^{p+1}, \tag{2.18}$$

and $\mathcal{M}(s)$ attains the maximum value at

$$s_0 := \left((p+1)\, MR_{p+1}\right)^{-\frac{1}{p-1}}. \tag{2.19}$$

Here, Proposition 2.11 in [5] shows the fact $\hat{d} \le d$.

3. Continuous dependence on initial data of the global solution for linear weak damping case

In this section, we consider the model equations in (1.1) with the linear weak damping terms, i.e., $r = m = \hat{r} = \hat{m} = 1$. First, we need the following decay result of the energy.

Lemma 3.1. *(Decay of the energy) Let Assumption* 2.1 *and Assumption* 2.2 *hold with* $r = m = \hat{r} = \hat{m} = 1$. *For any* $0 < \sigma < 1$, *if* $\mathcal{E}(z_0, z_1) < \sigma\hat{d}$ *and* $z_0 \in W$, *then one has*

$$\mathcal{E}(z(t), z_t(t)) < K_0 e^{-\lambda_0 t} \tag{3.1}$$

for $t > 0$, *where* λ_0 *and* K_0 *will be defined in the proof.*

Proof. We define

$$H(t) := \mathcal{E}(z(t), z_t(t)) + \varepsilon(z, z_t),$$

where $\varepsilon > 0$. Here, according to Cauchy-Schwartz inequality, Young inequality, and (2.5), we have

$$\begin{aligned} H(t) \le& \mathcal{E}(z(t), z_t(t)) + \varepsilon\|z\|_2\|z_t\|_2 \\ \le& \mathcal{E}(z(t), z_t(t)) + \frac{\varepsilon}{2}\|z\|_2^2 + \frac{\varepsilon}{2}\|z_t\|_2^2 \\ \le& \mathcal{E}(z(t), z_t(t)) + \frac{\varepsilon}{2}R_2\|z\|_V^2 + \frac{\varepsilon}{2}\|z_t\|_2^2 \end{aligned}$$

$$\leq \mathcal{E}(z(t), z_t(t)) + \varepsilon \max\{R_2, 1\} \left(\frac{1}{2} \|z\|_V^2 + \frac{1}{2} \|z_t\|_2^2 \right) \tag{3.2}$$

and

$$\begin{aligned} H(t) \geq & \mathcal{E}(z(t), z_t(t)) - \varepsilon \|z\|_2 \|z_t\|_2 \\ \geq & \mathcal{E}(z(t), z_t(t)) - \varepsilon \max\{R_2, 1\} \left(\frac{1}{2} \|z\|_V^2 + \frac{1}{2} \|z_t\|_2^2 \right). \end{aligned} \tag{3.3}$$

According to Theorem 2.12(iv) in [5], we know

$$\frac{1}{2} \|z\|_V^2 + \frac{1}{2} \|z_t\|_2^2 \leq \frac{p+1}{p-1} \mathcal{E}(z(t), z_t(t)), \tag{3.4}$$

which means that (3.2) and (3.3) turn to

$$H(t) \leq \mathcal{E}(z(t), z_t(t)) + \frac{\varepsilon \max\{R_2, 1\}(p+1)}{p-1} \mathcal{E}(z(t), z_t(t)) \tag{3.5}$$

and

$$H(t) \geq \mathcal{E}(z(t), z_t(t)) - \frac{\varepsilon \max\{R_2, 1\}(p+1)}{p-1} \mathcal{E}(z(t), z_t(t)). \tag{3.6}$$

According to (3.5) and (3.6), we know

$$\alpha_1 \mathcal{E}(z(t), z_t(t)) \leq H(t) \leq \alpha_2 \mathcal{E}(z(t), z_t(t)), \tag{3.7}$$

where

$$\alpha_1 := 1 - \frac{\varepsilon \max\{R_2, 1\}(p+1)}{p-1}$$

and

$$\alpha_2 := 1 + \frac{\varepsilon \max\{R_2, 1\}(p+1)}{p-1}.$$

We calculate the derivative of the auxiliary functional $H(t)$ with respect to time t as

$$H'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(z(t), z_t(t)) + \varepsilon \|z_t\|_2^2 + \varepsilon (z_{tt}, z). \tag{3.8}$$

In (3.8), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(z(t), z_t(t)) = & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|z_t\|_2^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|z\|_V^2 + \int_0^L \frac{\mathrm{d}}{\mathrm{d}t} F(z) \mathrm{d}x \\ = & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|u_t\|_2^2 + \|\phi_t\|_2^2 \right) + \frac{1}{2} \int_0^L \frac{\mathrm{d}}{\mathrm{d}t} \Big(\mu u_x^2 + \delta \phi_x^2 + \xi \phi^2 \\ & + 2 b u_x \phi \Big) \mathrm{d}x + \int_0^L \frac{\mathrm{d}}{\mathrm{d}t} F(z) \mathrm{d}x \\ = & \int_0^L (u_t u_{tt} + \phi_t \phi_{tt}) \, \mathrm{d}x + \int_0^L (\mu u_x u_{xt} + \delta \phi_x \phi_{xt} + \xi \phi \phi_t \end{aligned}$$

$$+bu_{xt}\phi + bu_x\phi_t)\,\mathrm{d}x + \int_0^L \nabla F(z)\cdot z_t \mathrm{d}x. \tag{3.9}$$

Here, the notation ∇F is defined by (2.13). Thus, according to (2.11), we know $\nabla F(z) = \mathcal{F}(z)$, which means (3.9) turns to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z(t), z_t(t)) \\
&= \int_0^L (u_t u_{tt} + \phi_t\phi_{tt})\,\mathrm{d}x + \int_0^L (\mu u_x u_{xt} + \delta\phi_x\phi_{xt} + \xi\phi\phi_t + bu_{xt}\phi + bu_x\phi_t)\,\mathrm{d}x \\
&\quad + \int_0^L \mathcal{F}(z)\cdot z_t \mathrm{d}x \\
&= \int_0^L (u_t u_{tt} + \phi_t\phi_{tt})\,\mathrm{d}x + \int_0^L (\mu u_x u_{xt} + \delta\phi_x\phi_{xt} + \xi\phi\phi_t + bu_{xt}\phi + bu_x\phi_t)\,\mathrm{d}x \\
&\quad + \int_0^L (f_1(u,\phi)u_t + f_2(u,\phi)\phi_t)\,\mathrm{d}x \\
&= \int_0^L (u_t u_{tt} + \phi_t\phi_{tt})\,\mathrm{d}x + \int_0^L (\mu u_x u_{xt} + \delta\phi_x\phi_{xt} + \xi\phi\phi_t + bu_x\phi_t)\,\mathrm{d}x \\
&\quad - \int_0^L bu_t\phi_x \mathrm{d}x + \int_0^L (f_1(u,\phi)u_t + f_2(u,\phi)\phi_t)\,\mathrm{d}x \\
&= \int_0^L (u_t u_{tt} + \mu u_x u_{xt} - bu_t\phi_x - f_1(u,\phi)u_t)\,\mathrm{d}x + \int_0^L (\phi_t\phi_{tt} + \delta\phi_x\phi_{xt} + bu_x\phi_t \\
&\quad + \xi\phi\phi_t - f_2(u,\phi)\phi_t)\,\mathrm{d}x.
\end{aligned} \tag{3.10}$$

Testing the both sides of the first equation in (1.1) by u_t and integrating both sides over $[0, L]$, we have

$$\int_0^L (u_t u_{tt} + \mu u_x u_{xt} - bu_t\phi_x - f_1(u,\phi)u_t)\,\mathrm{d}x = -\int_0^L g_1(u_t)u_t \mathrm{d}x. \tag{3.11}$$

And testing the both sides of the second equation in (1.1) by ϕ_t and integrating both sides over $[0, L]$, we have

$$\int_0^L (\phi_t\phi_{tt} + \delta\phi_x\phi_{xt} + bu_x\phi_t + \xi\phi\phi_t - f_2(u,\phi)\phi_t)\,\mathrm{d}x = -\int_0^L g_2(\phi_t)\phi_t \mathrm{d}x. \tag{3.12}$$

By substituting (3.11) and (3.12) into (3.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z(t), z_t(t)) = -\int_0^L g_1(u_t)u_t \mathrm{d}x - \int_0^L g_2(\phi_t)\phi_t \mathrm{d}x. \tag{3.13}$$

Next, we use Assumption 2.1 to deal with (3.13). In Assumption 2.1, for $|s| \geq 1$, according to (2.6) with $m = 1$ and (2.7) with $r = 1$, we know that

$$\alpha|s|^2 \leq g_j(s)s \leq \beta|s|^2,\ \ j = 1, 2. \tag{3.14}$$

Then taking the absolute value of (3.14) gives

$$\alpha|s| \leq \left|g_j(s)\right| \leq \beta|s|,\ \ j = 1, 2. \tag{3.15}$$

For $|s| < 1$, according to (2.8) with $\hat{m} = 1$ and (2.9) with $\hat{r} = 1$, we know that (3.15) also holds. Meanwhile, since $g_1(0) = g_2(0) = 0$ and $g_j(s)$, $j = 1, 2$, are assumed to be the increasing functions, for $j = 1, 2$, we know $g_j(s) > 0$ for $s > 0$ and $g_j(s) < 0$ for $s < 0$, which gives $g_j(s)s \geq 0$, $j = 1, 2$, for $s \in \mathbb{R}$. Thus, we have

$$\begin{aligned}
&\int_0^L g_1(u_t)u_t \mathrm{d}x + \int_0^L g_2(\phi_t)\phi_t \mathrm{d}x \\
=&\int_0^L |g_1(u_t)u_t|\, \mathrm{d}x + \int_0^L |g_2(\phi_t)\phi_t|\, \mathrm{d}x \\
\geq&\alpha\|u_t\|_2^2 + \alpha\|\phi_t\|_2^2 \\
=&\alpha\|z_t\|_2^2,
\end{aligned}$$

which makes (3.13) turn to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z(t), z_t(t)) \leq -\alpha\|z_t\|_2^2. \tag{3.16}$$

We deal with the term $\varepsilon\,(z_{tt}, z)$ in (3.8). Testing the both sides of the first equation in problem (1.1) by u and integrating both sides over $[0, L]$, we have

$$(u_{tt}, u) = -\mu\|u_x\|_2^2 - b\,(u_x, \phi) - (g_1(u_t), u) + (f_1(u, \phi), u)\,. \tag{3.17}$$

And testing the both sides of the second equation in problem (1.1) by ϕ and integrating both sides over $[0, L]$, we have

$$(\phi_{tt}, \phi) = -\delta\|\phi_x\|_2^2 - b\,(u_x, \phi) - \xi\|\phi\|_2^2 - (g_2(\phi_t), \phi) + (f_2(u, \phi), \phi)\,. \tag{3.18}$$

By (3.17) plus (3.18), we have

$$\begin{aligned}
(z_{tt}, z) =& -\int_0^L \left(\mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right)\mathrm{d}x - (g_1(u_t), u) - (g_2(\phi_t), \phi) \\
&+ (f_1(u, \phi), u) + (f_2(u, \phi), \phi) \\
=& -\|\phi\|_V^2 - (\mathcal{G}(z_t), z) + (\mathcal{F}(z), z) \\
\leq& -\|\phi\|_V^2 + |(\mathcal{G}(z_t), z)| + (\mathcal{F}(z), z)\,.
\end{aligned} \tag{3.19}$$

According to (3.16) and (3.19), we know that (3.8) turns to

$$H'(t) \leq -\alpha\|z_t\|_2^2 + \varepsilon\|z_t\|_2^2 - \varepsilon\|z\|_V^2 + \varepsilon\,|(\mathcal{G}(z_t), z)| + \varepsilon\,(\mathcal{F}(z), z)\,. \tag{3.20}$$

Next, we deal with the term $\varepsilon\,|(\mathcal{G}(z_t), z)|$ in (3.20). By using (3.15) and Hölder inequality, we know

$$\begin{aligned}
|(\mathcal{G}(z_t), z)| &= |(g_1(u_t), u) + (g_2(\phi_t), \phi)| \\
&\leq |(g_1(u_t), u)| + |(g_2(\phi_t), \phi)| \\
&\leq \int_0^L |g_1(u_t)||u|\mathrm{d}x + \int_0^L |g_2(\phi_t)||\phi|\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq\beta\int_0^L|u_t||u|\mathrm{d}x+\beta\int_0^L|\phi_t||\phi|\mathrm{d}x\\
&\leq\beta\|u_t\|_2\|u\|_2+\beta\|\phi_t\|_2\|\phi\|_2\\
&\leq 2\beta\|z_t\|_2\|z\|_2. \qquad (3.21)
\end{aligned}
$$

Then, We deal with $\varepsilon(\mathcal{F}(z),z)$ in (3.20). Here, we first need to give

$$
\mathcal{F}(z)\cdot z=(p+1)F(z). \qquad (3.22)
$$

For all $\lambda>0$, taking the derivative of both sides of (2.12) with respect to λ, we know

$$
\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda z)=\nabla F(\lambda z)\cdot z=\frac{\mathrm{d}}{\mathrm{d}\lambda}\lambda^{p+1}F(z)=(p+1)\lambda^pF(z), \qquad (3.23)
$$

where ∇F is defined by (2.13). By taking $\lambda=1$ in (3.23) and using (2.11), we obtain (3.22). According to (3.22) and (2.14), we have

$$
\begin{aligned}
(\mathcal{F}(z),z)=&\int_0^L\mathcal{F}(z)\cdot z\mathrm{d}x\\
=&(p+1)\int_0^L F(z)\mathrm{d}x\\
\leq&(p+1)M\|z\|_{p+1}^{p+1}. \qquad (3.24)
\end{aligned}
$$

By using (2.5), (3.24) turns to

$$
(\mathcal{F}(z),z)\leq(p+1)MR_{p+1}\|z\|_V^{p+1}=(p+1)MR_{p+1}\|z\|_V^{p-1}\|z\|_V^2. \qquad (3.25)
$$

Then, we estimate the term $\|z\|_V^{p-1}$ in (3.25). According to Theorem 2.12 (ii) in [5], we know $z(t)\in W$ for $t>0$. By using $I(z(t))>0$, i.e., $z(t)\in W$, we have

$$
(p+1)\int_0^L F(z(t))\mathrm{d}x<\|z(t)\|_V^2,
$$

which means

$$
\begin{aligned}
J(z(t))=&\frac{1}{2}\|z(t)\|_V^2-\int_0^L F(z(t))\mathrm{d}x\\
>&\frac{1}{2}\|z(t)\|_V^2-\frac{1}{p+1}\|z(t)\|_V^2\\
=&\frac{p-1}{2(p+1)}\|z(t)\|_V^2. \qquad (3.26)
\end{aligned}
$$

Meanwhile, according to (3.16), i.e.,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(z(t),z_t(t))\leq 0,
$$

we have $\mathcal{E}(z(t), z_t(t)) \leq \mathcal{E}(z_0, z_1)$. Thus, we know

$$\frac{p-1}{2(p+1)}\|z(t)\|_V^2 \leq J(z(t)) \leq \mathcal{E}(z(t), z_t(t)) \leq \mathcal{E}(z_0, z_1), \tag{3.27}$$

i.e.,

$$\|z(t)\|_V^{p-1} \leq \left(\frac{2(p+1)}{p-1}\mathcal{E}(z_0, z_1)\right)^{\frac{p-1}{2}},$$

for $t > 0$, which implies that (3.25) turns to

$$(\mathcal{F}(z), z) \leq (p+1)MR_{p+1}\left(\frac{2(p+1)}{p-1}\mathcal{E}(z_0, z_1)\right)^{\frac{p-1}{2}}\|z\|_V^2. \tag{3.28}$$

Due to $\mathcal{E}(z_0, z_1) < \sigma\hat{d}$ being assumed, we know that (3.28) turns to

$$(\mathcal{F}(z), z) \leq \sigma^{\frac{p-1}{2}}\|z\|_V^2, \tag{3.29}$$

where $\hat{d}$ is defined by (2.17). Substituting (3.21) and (3.29) into (3.20), we have

$$H'(t) \leq -\alpha\|z_t\|_2^2 + \varepsilon\|z_t\|_2^2 + \varepsilon\sigma^{\frac{p-1}{2}}\|z\|_V^2 - \varepsilon\|z\|_V^2 + 2\varepsilon\beta\|z_t\|_2\|z\|_2. \tag{3.30}$$

By using Young inequality for $\delta_0 > 0$ and inequality (2.5) for $q = 2$, we know that (3.30) turns to

$$\begin{aligned} H'(t) \leq& -\alpha\|z_t\|_2^2 + \varepsilon\|z_t\|_2^2 + \varepsilon\sigma^{\frac{p-1}{2}}\|z\|_V^2 - \varepsilon\|z\|_V^2 + \frac{\varepsilon\beta}{\delta_0}\|z_t\|_2^2 + \varepsilon\beta\delta_0R_2\|z\|_V^2 \\ =& -\left(\alpha - \varepsilon - \frac{\varepsilon\beta}{\delta_0}\right)\|z_t\|_2^2 - \varepsilon\left(1 - \sigma^{\frac{p-1}{2}} - \beta\delta_0R_2\right)\|z\|_V^2. \end{aligned} \tag{3.31}$$

In (3.31), we choose $\delta_0 > 0$ to make $1 - \sigma^{\frac{p-1}{2}} - \beta\delta_0R_2 > 0$ hold, where $1 - \sigma^{\frac{p-1}{2}} > 0$ due to $\sigma \in (0, 1)$. Then, we select $\varepsilon > 0$ such that $\alpha - \varepsilon - \frac{\varepsilon\beta}{\delta_0} > 0$ and

$$\alpha_1 = 1 - \frac{\varepsilon\max\{R_2, 1\}(p+1)}{p-1} > 0.$$

To deal with (3.31), we first have

$$\begin{aligned} &\left(\alpha - \varepsilon - \frac{\varepsilon\beta}{\delta_0}\right)\|z_t\|_2^2 + \varepsilon\left(1 - \sigma^{\frac{p-1}{2}} - \beta\delta_0R_2\right)\|z\|_V^2 \\ =&2\left(\alpha - \varepsilon - \frac{\varepsilon\beta}{\delta_0}\right)\frac{1}{2}\|z_t\|_2^2 + 2\varepsilon\left(1 - \sigma^{\frac{p-1}{2}} - \beta\delta_0R_2\right)\frac{1}{2}\|z\|_V^2 \\ \geq& \min\left\{2\left(\alpha - \varepsilon - \frac{\varepsilon\beta}{\delta_0}\right), 2\varepsilon\left(1 - \sigma^{\frac{p-1}{2}} - \beta\delta_0R_2\right)\right\}\left(\frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\|z\|_V^2\right). \end{aligned} \tag{3.32}$$

According to Theorem 2.12 (iv) in [5], (3.32) turns to

$$\left(\alpha - \varepsilon - \frac{\varepsilon\beta}{\delta_0}\right)\|z_t\|_2^2 + \varepsilon\left(1 - \sigma^{\frac{p-1}{2}} - \beta\delta_0R_2\right)\|z\|_V^2$$

$$\geq \min\left\{2\left(\alpha-\varepsilon-\frac{\varepsilon\beta}{\delta_0}\right), 2\varepsilon\left(1-\sigma^{\frac{p-1}{2}}-\beta\delta_0R_2\right)\right\}\mathcal{E}(z(t),z_t(t)). \tag{3.33}$$

Due to (3.7), i.e., $H(t) \leq \alpha_2\mathcal{E}(z(t), z_t(t))$, (3.33) turns to

$$\begin{aligned}&\left(\alpha-\varepsilon-\frac{\varepsilon\beta}{\delta_0}\right)\|z_t\|_2^2+\varepsilon\left(1-\sigma^{\frac{p-1}{2}}-\beta\delta_0R_2\right)\|z\|_V^2\\ \geq&\frac{\min\left\{2\left(\alpha-\varepsilon-\frac{\varepsilon\beta}{\delta_0}\right), 2\varepsilon\left(1-\sigma^{\frac{p-1}{2}}-\beta\delta_0R_2\right)\right\}}{\alpha_2}H(t).\end{aligned} \tag{3.34}$$

Thus, we know that (3.31) implies

$$H'(t) \leq -\lambda_0 H(t), \tag{3.35}$$

where

$$\lambda_0 := \frac{\min\left\{2\left(\alpha-\varepsilon-\frac{\varepsilon\beta}{\delta_0}\right), 2\varepsilon\left(1-\sigma^{\frac{p-1}{2}}-\beta\delta_0R_2\right)\right\}}{\alpha_2}. \tag{3.36}$$

By using Gronwall's inequality, (3.35) gives

$$H(t) \leq e^{-\lambda_0 t}H(0). \tag{3.37}$$

According to (3.7), (3.37), and the assumptions $\mathcal{E}(z_0, z_1) < \sigma\hat{d}$ and $0 < \sigma < 1$, we have

$$\mathcal{E}(z(t), z_t(t)) \leq \frac{\alpha_2\mathcal{E}(z_0,z_1)}{\alpha_1}e^{-\lambda_0 t} < \frac{\alpha_2\sigma\hat{d}}{\alpha_1}e^{-\lambda_0 t} < K_0e^{-\lambda_0 t}, \tag{3.38}$$

where

$$K_0 := \frac{\alpha_2\hat{d}}{\alpha_1}. \tag{3.39}$$

Theorem 3.2. *(Continuous dependence on initial data for linear weak damping case) Let Assumption* 2.1 *and Assumption* 2.2 *hold with $r = m = \hat{r} = \hat{m} = 1$. For any $0 < \sigma < 1$, suppose $\mathcal{E}(z_0, z_1) < \sigma\hat{d}$, $z_0 \in W$, $\mathcal{E}(\tilde{z}_0, \tilde{z}_1) < \sigma\hat{d}$ and $\tilde{z}_0 \in W$. Let $z = (u, \phi)$ and $\tilde{z} = (\tilde{u}, \tilde{\phi})$ be the global solutions to problem (1.1) with the initial data z_0, z_1, and $\tilde{z}_0$, $\tilde{z}_1$, respectively. Then one has*

$$\widehat{E}(t) \leq C_1\left(\widehat{E}(0)+C_2\left(\widehat{E}(0)\right)^{\frac{a}{2}}\right)^{\rho}e^{-C_3 t}, \tag{3.40}$$

where

$$C_1 := \left(1+\frac{C_4e^{\frac{C_4}{\lambda_0(p-1)}}}{\lambda_0(p-1)}\right)^{\rho}\left(\frac{4(p+1)K_0}{p-1}\right)^{1-\rho},$$

$$C_2 := 2^{\frac{a}{2}}\frac{N}{\lambda_1},$$

$$C_3 := \lambda_0(1-\rho),$$

$$C_4 := 4^3 C R_4^{\frac{1}{2}} R_{4(p-1)}^{\frac{1}{2}} \left(\frac{2(p+1)K_0}{p-1}\right)^{p-1}, \tag{3.41}$$

$$0 < a < \min\left\{\frac{2\lambda_0}{\bar{M}C+\lambda_0}, 1\right\},$$

$$0 < \rho < 1,$$

λ_0 and K_0 are defined by (3.36) and (3.39), respectively, $R_{4(p-1)}$ is the best embedding constant defined in (2.4) taking $q = 4(p-1)$,

$$\lambda_1 := \frac{\lambda_0(2-a) - a\bar{M}C}{2},$$

$$N := 2^{1-a} C \left(2K_0\right)^{\frac{2-a}{2}} + 2^{3-a} C R_2^{\frac{1}{2}} \left(\frac{2(p+1)K_0}{p-1}\right)^{\frac{1}{2}} \left(2K_0\right)^{\frac{1-a}{2}},$$

and

$$\bar{M} := \max\left\{2^{\frac{3}{2}} 5^2 R_4^{\frac{1}{2}} \left(2R_{4(p-1)} \left(\frac{2(p+1)\sigma\hat{d}}{p-1}\right)^{2(p-1)} + L\right)^{\frac{1}{2}}, 1\right\}. \tag{3.42}$$

Proof. We denote $w := z - \tilde{z}$. According to the proof of Theorem 2.5 in [5], we notice that

$$\widehat{E}(t) \le \widehat{E}(0) + \int_0^t \int_0^L (\mathcal{F}(z(\tau)) - \mathcal{F}(\tilde{z}(\tau))) w_t(\tau) \mathrm{d}x \mathrm{d}\tau \tag{3.43}$$

holds by Assumption 2.1 and Assumption 2.2. In the following, we shall finish this proof by considering the following two steps. In Step I, we shall derive a similar estimate of the growth of $\widehat{E}(t)$ to (135) in [5]. As we build this estimate for the global solution instead of the local solution treated in [5], we have to rebuild all the necessary estimates based on the conditions for global existence theory.

Step I: Global estimate of $\widehat{E}(t)$ for global solution.

We estimate the term $\int_0^t \int_0^L (\mathcal{F}(z(\tau)) - \mathcal{F}(\tilde{z}(\tau)))\, w_t(\tau) \mathrm{d}x \mathrm{d}\tau$ in (3.43) as follows

$$\begin{aligned}
&\int_0^L (\mathcal{F}(z(t)) - \mathcal{F}(\tilde{z}(t))) \cdot w_t \mathrm{d}x \\
=& \int_0^L (f_1(z) - f_1(\tilde{z}))\,(u_t - \tilde{u}_t) \mathrm{d}x \\
&+ \int_0^L (f_2(z) - f_2(\tilde{z}))\,(\phi_t - \tilde{\phi}_t) \mathrm{d}x \\
\le& \int_0^L |f_1(z) - f_1(\tilde{z})||u_t - \tilde{u}_t| \mathrm{d}x
\end{aligned}$$

$$+\int_0^L |f_2(z)-f_2(\tilde{z})||\phi_t-\tilde{\phi}_t|\mathrm{d}x. \tag{3.44}$$

Here, according to the proof of Lemma 3.2 in [5], we notice that (2.10) in Assumption 2.2 gives

$$|f_j(z)-f_j(\tilde{z})| \le C|z-\tilde{z}|(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1),\ \ j=1,2, \tag{3.45}$$

which means (3.44) turns to

$$\begin{aligned}
&\int_0^L (\mathcal{F}(z(t))-\mathcal{F}(\tilde{z}(t)))\cdot w_t\mathrm{d}x\\
\le&\underbrace{\int_0^L C|z-\tilde{z}|(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)|u_t-\tilde{u}_t|\mathrm{d}x}_{:=A_1}\\
&+\underbrace{\int_0^L C|z-\tilde{z}|(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)|\phi_t-\tilde{\phi}_t|\mathrm{d}x}_{:=A_2}.
\end{aligned} \tag{3.46}$$

Next, we deal with A_1 and A_2 separately. For A_1, by Hölder inequality and Young inequality, we have

$$\begin{aligned}
A_1 \le& C\left(\int_0^L |z-\tilde{z}|^2(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\left(\int_0^L |u_t-\tilde{u}_t|^2\mathrm{d}x\right)^{\frac{1}{2}}\\
\le&\frac{C}{2}\int_0^L |z-\tilde{z}|^2(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)^2\mathrm{d}x\\
&+\frac{C}{2}\int_0^L |u_t-\tilde{u}_t|^2\mathrm{d}x.
\end{aligned} \tag{3.47}$$

By the similar process, we can deal with A_2 as

$$\begin{aligned}
A_2 \le&\frac{C}{2}\int_0^L |z-\tilde{z}|^2(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)^2\mathrm{d}x\\
&+\frac{C}{2}\int_0^L |\phi_t-\tilde{\phi}_t|^2\mathrm{d}x.
\end{aligned} \tag{3.48}$$

According to (3.47), (3.48) and Hölder inequality, we know that (3.46) turns to

$$\begin{aligned}
&\int_0^L (\mathcal{F}(z(t))-\mathcal{F}(\tilde{z}(t)))\cdot w_t\mathrm{d}x\\
\le& C\int_0^L |z-\tilde{z}|^2(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)^2\mathrm{d}x+\frac{C}{2}\|w_t\|_2^2\\
\le& C\left(\int_0^L |z-\tilde{z}|^4\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^L (|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)^4\mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\frac{C}{2}\|w_t\|_2^2. \tag{3.49}$$

In (3.49), by noticing $z=(u,\phi)$, $\tilde{z}=(\tilde{u},\tilde{\phi})$, we see that

$$\begin{aligned}
&\left(\int_0^L |z-\tilde{z}|^4 \mathrm{d}x\right)^{\frac{1}{2}}\\
=&\left(\int_0^L \left(\left(|u-\tilde{u}|^2+|\phi-\tilde{\phi}|^2\right)^{\frac{1}{2}}\right)^4 \mathrm{d}x\right)^{\frac{1}{2}}\\
=&\left(\int_0^L \left(|u-\tilde{u}|^4+|\phi-\tilde{\phi}|^4+2|u-\tilde{u}|^2|\phi-\tilde{\phi}|^2\right)\mathrm{d}x\right)^{\frac{1}{2}}\\
=&\left(\int_0^L \left(|u-\tilde{u}|^4+|\phi-\tilde{\phi}|^4\right)\mathrm{d}x+\int_0^L 2|u-\tilde{u}|^2|\phi-\tilde{\phi}|^2\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.50}$$

By using Hölder inequality and Young inequality, we know (3.50) turns to

$$\begin{aligned}
&\left(\int_0^L |z-\tilde{z}|^4 \mathrm{d}x\right)^{\frac{1}{2}}\\
\leq&\left(\int_0^L \left(|u-\tilde{u}|^4+|\phi-\tilde{\phi}|^4\right)\mathrm{d}x+2\|u-\tilde{u}\|_4^2\|\phi-\tilde{\phi}\|_4^2\right)^{\frac{1}{2}}\\
\leq&\left(2\|u-\tilde{u}\|_4^4+2\|\phi-\tilde{\phi}\|_4^4\right)^{\frac{1}{2}}\\
=&2^{\frac{1}{2}}\|z-\tilde{z}\|_4^2.
\end{aligned} \tag{3.51}$$

Next, we deal with $\int_0^L(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1)^4\mathrm{d}x$ in (3.49). For $k_1,k_2,k_3,k_4,k_5\geq 0$, we have

$$\begin{aligned}
&(k_1+k_2+k_3+k_4+k_5)^4\\
\leq&(5\max\{k_1,k_2,k_3,k_4,k_5\})^4\\
=&5^4\max\left\{k_1^4,k_2^4,k_3^4,k_4^4,k_5^4\right\}\\
\leq&5^4\left(k_1^4+k_2^4+k_3^4+k_4^4+k_5^4\right).
\end{aligned}$$

From above observation, we have

$$\begin{aligned}
&\int_0^L \left(|u|^{p-1}+|\tilde{u}|^{p-1}+|\phi|^{p-1}+|\tilde{\phi}|^{p-1}+1\right)^4 \mathrm{d}x\\
\leq&5^4\int_0^L \left(|u|^{4(p-1)}+|\tilde{u}|^{4(p-1)}+|\phi|^{4(p-1)}+|\tilde{\phi}|^{4(p-1)}+1\right)\mathrm{d}x.
\end{aligned} \tag{3.52}$$

According to (3.51) and (3.52), (3.49) turns to

$$\int_0^L (\mathcal{F}(z)-\mathcal{F}(\tilde{z}))\cdot w_t \mathrm{d}x$$

$$
\begin{aligned}
&\leq C2^{\frac{1}{2}}\|z-\tilde{z}\|_4^2\left(5^4\int_0^L\left(|u|^{4(p-1)}+|\tilde{u}|^{4(p-1)}+|\phi|^{4(p-1)}+|\tilde{\phi}|^{4(p-1)}+1\right)\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad+\frac{C}{2}\|w_t\|_2^2\\
&=C2^{\frac{1}{2}}5^2\|z-\tilde{z}\|_4^2\left(\int_0^L\left(|u|^{4(p-1)}+|\phi|^{4(p-1)}\right)\mathrm{d}x\right.\\
&\quad\left.+\int_0^L\left(|\tilde{u}|^{4(p-1)}+|\tilde{\varphi}|^{4(p-1)}\right)\mathrm{d}x+L\right)^{\frac{1}{2}}+\frac{C}{2}\|w_t\|_2^2\\
&=C2^{\frac{1}{2}}5^2\|z-\tilde{z}\|_4^2\left(\|z\|_{4(p-1)}^{4(p-1)}+\|\tilde{z}\|_{4(p-1)}^{4(p-1)}+L\right)^{\frac{1}{2}}+\frac{C}{2}\|w_t\|_2^2. \qquad (3.53)
\end{aligned}
$$

By using (2.5), we know that (3.53) turns to

$$
\begin{aligned}
&\int_0^L(\mathcal{F}(z)-\mathcal{F}(\tilde{z}))\cdot w_t\mathrm{d}x\\
&\leq C2^{\frac{1}{2}}5^2R_4^{\frac{1}{2}}\|z-\tilde{z}\|_V^2\left(R_{4(p-1)}\|z\|_V^{4(p-1)}+R_{4(p-1)}\|\tilde{z}\|_V^{4(p-1)}+L\right)^{\frac{1}{2}}+\frac{C}{2}\|w_t\|_2^2. \qquad (3.54)
\end{aligned}
$$

According to (3.27) and the assumptions $\mathcal{E}(z_0,z_1)<\sigma\hat{d}$ and $\mathcal{E}(\tilde{z}_0,\tilde{z}_1)<\sigma\hat{d}$, we have

$$\|z\|_V^2<\frac{2(p+1)\sigma\hat{d}}{p-1} \qquad (3.55)$$

and

$$\|\tilde{z}\|_V^2<\frac{2(p+1)\sigma\hat{d}}{p-1}. \qquad (3.56)$$

Substituting (3.55) and (3.56) into (3.54), we have

$$
\begin{aligned}
&\int_0^L(\mathcal{F}(z)-\mathcal{F}(\tilde{z}))\cdot w_t\mathrm{d}x\\
&\leq C2^{\frac{1}{2}}5^2R_4^{\frac{1}{2}}\left(2R_{4(p-1)}\left(\frac{2(p+1)\sigma\hat{d}}{p-1}\right)^{2(p-1)}+L\right)^{\frac{1}{2}}\|w\|_V^2+\frac{C}{2}\|w_t\|_2^2\\
&\leq\bar{M}C\left(\frac{1}{2}\|w_t\|_2^2+\frac{1}{2}\|w\|_V^2\right)\\
&=\bar{M}C\widehat{E}(t). \qquad (3.57)
\end{aligned}
$$

Due to (3.57), we know

$$\int_0^t\int_0^L(\mathcal{F}(z(\tau))-\mathcal{F}(\tilde{z}(\tau)))\cdot w_t\mathrm{d}x\mathrm{d}\tau\leq\bar{M}C\int_0^t\widehat{E}(\tau)\mathrm{d}\tau. \qquad (3.58)$$

Substituting (3.58) into (3.43), we have

$$\widehat{E}(t)\leq\widehat{E}(0)+\bar{M}C\int_0^t\widehat{E}(\tau)\mathrm{d}\tau. \qquad (3.59)$$

Then, by a variation of Gronwall's inequality (see Appendix), we have

$$\widehat{E}(t) \leq \widehat{E}(0)e^{\tilde{M}Ct}. \tag{3.60}$$

As the growth estimate (3.60) we derived in Step I does not reflect the decay of the solution, we shall deal with the decay terms and the non-decay terms separately in Step II to upgrade the results obtained in Step I, i.e., (3.60), by giving an improved estimate to reflect the dissipative property of the system (1.1).

Step II: Decay estimate of $\widehat{E}(t)$.

According to (3.46), we have

$$\begin{aligned}
&\int_0^L (\mathcal{F}(z(t)) - \mathcal{F}(\tilde{z}(t))) \cdot w_t \mathrm{d}x \\
\leq& \underbrace{\int_0^L C|z-\tilde{z}|(|u|^{p-1} + |\tilde{u}|^{p-1} + |\phi|^{p-1} + |\tilde{\phi}|^{p-1})|u_t - \tilde{u}_t|\mathrm{d}x}_{:=A_3} \\
&+ \underbrace{\int_0^L C|z-\tilde{z}|(|u|^{p-1} + |\tilde{u}|^{p-1} + |\phi|^{p-1} + |\tilde{\phi}|^{p-1})|\phi_t - \tilde{\phi}_t|\mathrm{d}x}_{:=A_4} \\
&+ \int_0^L C|z-\tilde{z}||u_t - \tilde{u}_t|\mathrm{d}x + \int_0^L C|z-\tilde{z}||\phi_t - \tilde{\phi}_t|\mathrm{d}x.
\end{aligned} \tag{3.61}$$

By the similar process dealing with A_1 and A_2, we can treat A_3 and A_4 as

$$\begin{aligned}
A_3 \leq& \frac{C}{2}\int_0^L |z-\tilde{z}|^2(|u|^{p-1} + |\tilde{u}|^{p-1} + |\phi|^{p-1} + |\tilde{\phi}|^{p-1})^2\mathrm{d}x \\
&+ \frac{C}{2}\int_0^L |u_t - \tilde{u}_t|^2\mathrm{d}x
\end{aligned} \tag{3.62}$$

and

$$\begin{aligned}
A_4 \leq& \frac{C}{2}\int_0^L |z-\tilde{z}|^2(|u|^{p-1} + |\tilde{u}|^{p-1} + |\phi|^{p-1} + |\tilde{\phi}|^{p-1})^2\mathrm{d}x \\
&+ \frac{C}{2}\int_0^L |\phi_t - \tilde{\phi}_t|^2\mathrm{d}x.
\end{aligned} \tag{3.63}$$

By the similar process of obtaining (3.54), i.e.,

$$\begin{aligned}
A_1 + A_2 \leq& C2^{\frac{1}{2}}5^2R_4^{\frac{1}{2}}\|z-\tilde{z}\|_V^2\left(R_{4(p-1)}\|z\|_V^{4(p-1)} + R_{4(p-1)}\|\tilde{z}\|_V^{4(p-1)} + L\right)^{\frac{1}{2}} \\
&+ \frac{C}{2}\|w_t\|_2^2,
\end{aligned}$$

we can use (3.62) and (3.63) to give

$$A_3 + A_4 \leq C2^{\frac{1}{2}}4^2R_4^{\frac{1}{2}}R_{4(p-1)}^{\frac{1}{2}}\|z-\tilde{z}\|_V^2\left(\|z\|_V^{4(p-1)} + \|\tilde{z}\|_V^{4(p-1)}\right)^{\frac{1}{2}} + \frac{C}{2}\|w_t\|_2^2. \tag{3.64}$$

Due to (3.26), (2.16) and Lemma 3.1, we know

$$\frac{1}{2}\|z_t\|_2^2+\frac{p-1}{2(p+1)}\|z\|_V^2\leq\mathcal{E}(z(t),z_t(t))<K_0e^{-\lambda_0 t}\tag{3.65}$$

and

$$\frac{1}{2}\|\tilde{z}_t\|_2^2+\frac{p-1}{2(p+1)}\|\tilde{z}\|_V^2\leq\mathcal{E}(\tilde{z}(t),\tilde{z}_t(t))<K_0e^{-\lambda_0 t}.\tag{3.66}$$

According to (3.65) and (3.66), we have

$$\|z\|_V^{4(p-1)}<\left(\frac{2(p+1)}{p-1}K_0\right)^{2(p-1)}e^{-2\lambda_0(p-1)t}$$

and

$$\|\tilde{z}\|_V^{4(p-1)}<\left(\frac{2(p+1)}{p-1}K_0\right)^{2(p-1)}e^{-2\lambda_0(p-1)t},$$

which mean

$$\left(\|z\|_V^{4(p-1)}+\|\tilde{z}\|_V^{4(p-1)}\right)^{\frac{1}{2}}<2^{\frac{1}{2}}\left(\frac{2(p+1)}{p-1}K_0\right)^{p-1}e^{-\lambda_0(p-1)t}.\tag{3.67}$$

By substituting (3.67) into (3.64), we obtain

$$\begin{aligned}A_3+A_4\leq&C_4e^{-\lambda_0(p-1)t}\frac{1}{2}\|z-\tilde{z}\|_V^2+\frac{C}{2}\|z_t-\tilde{z}_t\|_2^2\\ \leq&C_4e^{-\lambda_0(p-1)t}\left(\frac{1}{2}\|z-\tilde{z}\|_V^2+\frac{1}{2}\|z_t-\tilde{z}_t\|_2^2\right)\\ &+\frac{C}{2}\|z_t-\tilde{z}_t\|_2^2.\end{aligned}\tag{3.68}$$

Substituting (3.68) into (3.61) and using Hölder inequality and (2.5), we have

$$\begin{aligned}&\int_0^L(\mathcal{F}(z(t))-\mathcal{F}(\tilde{z}(t)))\cdot w_t\mathrm{d}x\\ \leq&C_4e^{-\lambda_0(p-1)t}\widehat{E}(t)+\frac{C}{2}\|z_t-\tilde{z}_t\|_2^2+\int_0^L C|z-\tilde{z}||u_t-\tilde{u}_t|\mathrm{d}x\\ &+\int_0^L C|z-\tilde{z}||\phi_t-\tilde{\phi}_t|\mathrm{d}x\\ \leq&C_4e^{-\lambda_0(p-1)t}\widehat{E}(t)+\frac{C}{2}\|z_t-\tilde{z}_t\|_2^2+C\|z-\tilde{z}\|_2\|u_t-\tilde{u}_t\|_2\\ &+C\|z-\tilde{z}\|_2\|\phi_t-\tilde{\phi}_t\|_2\\ \leq&C_4e^{-\lambda_0(p-1)t}\widehat{E}(t)+\frac{C}{2}\|z_t-\tilde{z}_t\|_2^2+2C\|z-\tilde{z}\|_2\|z_t-\tilde{z}_t\|_2\\ \leq&C_4e^{-\lambda_0(p-1)t}\widehat{E}(t)+\frac{C}{2}\|z_t-\tilde{z}_t\|_2^2+2CR_2^{\frac{1}{2}}\|z-\tilde{z}\|_V\|z_t-\tilde{z}_t\|_2.\end{aligned}\tag{3.69}$$

According to (3.60), we know

$$\|z_t - \tilde{z}_t\|_2 \leq \left(2\widehat{E}(0)e^{\bar{M}Ct}\right)^{\frac{1}{2}},$$

i.e.,

$$\|z_t - \tilde{z}_t\|_2^a \leq \left(2\widehat{E}(0)\right)^{\frac{a}{2}} e^{\frac{a\bar{M}Ct}{2}} \tag{3.70}$$

for $0 < a < 1$. Meanwhile, combining (3.65) and (3.66), we also have

$$\|z_t - \tilde{z}_t\|_2 \leq \|z_t\|_2 + \|\tilde{z}_t\|_2 \leq 2\left(2K_0\right)^{\frac{1}{2}} e^{-\frac{\lambda_0}{2}t}, \tag{3.71}$$

i.e.,

$$\|z_t - \tilde{z}_t\|_2^{1-a} \leq 2^{1-a}\left(2K_0\right)^{\frac{1-a}{2}} e^{-\frac{\lambda_0(1-a)}{2}t}, \tag{3.72}$$

and

$$\|z - \tilde{z}\|_V \leq \|z\|_V + \|\tilde{z}\|_V \leq 2\left(\frac{2(p+1)K_0}{p-1}\right)^{\frac{1}{2}} e^{-\frac{\lambda_0}{2}t}. \tag{3.73}$$

Combining (3.70), (3.71) and (3.72), we have

$$\begin{aligned}\|z_t - \tilde{z}_t\|_2^2 =& \|z_t - \tilde{z}_t\|_2 \|z_t - \tilde{z}_t\|_2^a \|z_t - \tilde{z}_t\|_2^{1-a} \\ \leq& 2^{2-a}\left(2K_0\right)^{\frac{2-a}{2}} \left(2\widehat{E}(0)\right)^{\frac{a}{2}} e^{-\frac{\lambda_0(2-a)-a\bar{M}C}{2}t}.\end{aligned} \tag{3.74}$$

We choose $0 < a < \min\left\{\frac{2\lambda_0}{\bar{M}C+\lambda_0}, 1\right\}$ such that

$$\lambda_0(2-a) - a\bar{M}C > 0 \tag{3.75}$$

in (3.74). Meanwhile, according to (3.70), (3.72) and (3.73), we notice that

$$\begin{aligned}\|z - \tilde{z}\|_V \|z_t - \tilde{z}_t\|_2 =& \|z - \tilde{z}\|_V \|z_t - \tilde{z}_t\|_2^a \|z_t - \tilde{z}_t\|_2^{1-a} \\ \leq& 2^{2-a}\left(\frac{2(p+1)K_0}{p-1}\right)^{\frac{1}{2}} \left(2K_0\right)^{\frac{1-a}{2}} \left(2\widehat{E}(0)\right)^{\frac{a}{2}} e^{-\frac{\lambda_0(2-a)-a\bar{M}C}{2}t}.\end{aligned} \tag{3.76}$$

Due to (3.74) and (3.76), we see that (3.69) turns to

$$\begin{aligned}&\int_0^L (\mathcal{F}(z(t)) - \mathcal{F}(\tilde{z}(t))) \cdot w_t \mathrm{d}x \\ \leq& C_4 e^{-\lambda_0(p-1)t}\widehat{E}(t) \\ &+ 2^{1-a}C\left(2K_0\right)^{\frac{2-a}{2}} \left(2\widehat{E}(0)\right)^{\frac{a}{2}} e^{-\frac{\lambda_0(2-a)-a\bar{M}C}{2}t} \\ &+ 2^{3-a}CR_2^{\frac{1}{2}}\left(\frac{2(p+1)K_0}{p-1}\right)^{\frac{1}{2}} \left(2K_0\right)^{\frac{1-a}{2}} \left(2\widehat{E}(0)\right)^{\frac{a}{2}} e^{-\frac{\lambda_0(2-a)-a\bar{M}C}{2}t}.\end{aligned} \tag{3.77}$$

By substituting (3.77) into (3.43), we obtain

$$\begin{aligned}\widehat{E}(t) \leq & \widehat{E}(0) + C_4 \int_0^t e^{-\lambda_0(p-1)\tau}\widehat{E}(\tau)\mathrm{d}\tau \\ & + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} D,\end{aligned} \tag{3.78}$$

where

$$D := N\int_0^t e^{-\lambda_1\tau}\mathrm{d}\tau = \frac{N}{\lambda_1} - \frac{N}{\lambda_1}e^{-\lambda_1 t}. \tag{3.79}$$

Here, according to (3.79), we notice that $D \leq \frac{N}{\lambda_1}$, which means that (3.78) turns to

$$\widehat{E}(t) \leq \widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} \frac{N}{\lambda_1} + C_4 \int_0^t e^{-\lambda_0(p-1)\tau}\widehat{E}(\tau)\mathrm{d}\tau, \tag{3.80}$$

i.e.,

$$\begin{aligned}e^{-\lambda_0(p-1)t}\widehat{E}(t) \leq & e^{-\lambda_0(p-1)t}\left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} \frac{N}{\lambda_1}\right) \\ & + C_4 e^{-\lambda_0(p-1)t}\int_0^t e^{-\lambda_0(p-1)\tau}\widehat{E}(\tau)\mathrm{d}\tau.\end{aligned} \tag{3.81}$$

We define

$$F(t) := \int_0^t e^{-\lambda_0(p-1)\tau}\widehat{E}(\tau)\mathrm{d}\tau. \tag{3.82}$$

Thus, we can rewrite (3.81) as

$$\begin{aligned}F'(t) \leq & e^{-\lambda_0(p-1)t}\left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} \frac{N}{\lambda_1}\right) \\ & + C_4 e^{-\lambda_0(p-1)t}F(t).\end{aligned} \tag{3.83}$$

By applying Gronwall's inequality, (3.83) gives

$$\begin{aligned}F(t) \leq & \left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} \frac{N}{\lambda_1}\right) e^{C_4\int_0^t e^{-\lambda_0(p-1)\tau}\mathrm{d}\tau}\int_0^t e^{-\lambda_0(p-1)\tau}\mathrm{d}\tau \\ = & \left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} \frac{N}{\lambda_1}\right) e^{\frac{C_4}{\lambda_0(p-1)}\left(1-e^{-\lambda_0(p-1)t}\right)}\frac{1-e^{-\lambda_0(p-1)t}}{\lambda_0(p-1)} \\ \leq & \left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} \frac{N}{\lambda_1}\right) \frac{e^{\frac{C_4}{\lambda_0(p-1)}}}{\lambda_0(p-1)},\end{aligned}$$

which means (3.80) turns to

$$\widehat{E}(t) \leq \left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{a}{2}} \frac{N}{\lambda_1}\right)\left(1 + \frac{C_4 e^{\frac{C_4}{\lambda_0(p-1)}}}{\lambda_0(p-1)}\right). \tag{3.84}$$

For $0 < \rho < 1$, according to (3.84), we have

$$\begin{aligned}\widehat{E}(t) =&\widehat{E}(t)^{\rho}\widehat{E}(t)^{1-\rho}\\ \leq&\left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{\alpha}{2}}\frac{N}{\lambda_1}\right)^{\rho}\left(1 + \frac{C_4 e^{\frac{C_4}{\lambda_0(p-1)}}}{\lambda_0(p-1)}\right)^{\rho}\widehat{E}(t)^{1-\rho}. \end{aligned} \tag{3.85}$$

Here, by using Young inequality, we know

$$\begin{aligned}\widehat{E}(t)^{1-\rho} =&\left(\frac{1}{2}\|z_t - \tilde{z}_t\|_2^2 + \frac{1}{2}\|z - \tilde{z}\|_V^2\right)^{1-\rho}\\ \leq&\left(\frac{1}{2}\left(\|z_t\|_2 + \|\tilde{z}_t\|_2\right)^2 + \frac{1}{2}\left(\|z\|_V + \|\tilde{z}\|_V\right)^2\right)^{1-\rho}\\ =&\left(\frac{1}{2}\|z_t\|_2^2 + \|z_t\|_2\|\tilde{z}_t\|_2 + \frac{1}{2}\|\tilde{z}_t\|_2^2 + \frac{1}{2}\|z\|_V^2 + \|z\|_V\|\tilde{z}\|_V\right.\\ &\left.+\frac{1}{2}\|\tilde{z}\|_V^2\right)^{1-\rho}\\ \leq&\left(\|z_t\|_2^2 + \|\tilde{z}_t\|_2^2 + \|z\|_V^2 + \|\tilde{z}\|_V^2\right)^{1-\rho} \end{aligned} \tag{3.86}$$

According to (3.65) and (3.66), we know

$$\frac{p-1}{2(p+1)}\left(\|z_t\|_2^2 + \|z\|_V^2\right) < K_0 e^{-\lambda_0 t} \tag{3.87}$$

and

$$\frac{p-1}{2(p+1)}\left(\|\tilde{z}_t\|_2^2 + \|\tilde{z}\|_V^2\right) < K_0 e^{-\lambda_0 t}. \tag{3.88}$$

By substituting (3.87) and (3.88) into (3.86), we have

$$\widehat{E}(t)^{1-\rho} \leq \left(\frac{4(p+1)K_0}{p-1}\right)^{1-\rho} e^{-\lambda_0(1-\rho)t},$$

which means that (3.85) turns to (3.40).

4. Continuous dependence on initial data of the global solution for nonlinear weak damping case

In this section, we consider the continuous dependence of the global solution on the initial data for the nonlinear weak damping case of the model equations in problem (1.1) by supposing that $m \geq 1$, $r \geq 1$, and $\hat{m} = \hat{r} = 1$ in Assumption 2.1, which means that the weak damping terms $g_j(s)$, $j = 1, 2$, take the nonlinear form for $|s| \geq 1$ and linear form for $|s| < 1$. These conditions are applied to improve the estimate (1.2) and reflect the decay property of (1.1), which was clearly clarified in Corollary 2.14 in [5], that is, the condition $\hat{m} = \hat{r} = 1$ is necessary to obtain the exponential decay of the energy, which helps to get the exponential decay, and the absence of such linear condition can only lead to the polynomial decay of the energy. Hence although we discuss the nonlinear weak damping case here, we still need to assume that the terms $g_j(s)$, $j = 1, 2$, take the linear form for $|s| < 1$.

Theorem 4.1. *(Continuous dependence on initial data for nonlinear weak damping case) Let Assumption* 2.1 *and Assumption* 2.2 *hold with* $\hat{r} = \hat{m} = 1$, $\mathcal{E}(z_0, z_1) < \mathcal{M}(s_0 - \nu)$, $\mathcal{E}(\tilde{z}_0, \tilde{z}_1) < \mathcal{M}(s_0 - \nu)$, $\|z_0\|_V \le s_0 - \nu$, *and* $\|\tilde{z}_0\|_V \le s_0 - \nu$ *for some* $\nu > 0$. *Let* $z = (u, \phi)$ *and* $\tilde{z} = (\tilde{u}, \tilde{\phi})$ *are the global solutions to the problem (1.1) with the initial data* z_0, z_1, *and* $\tilde{z}_0$, $\tilde{z}_1$, *respectively, where* $\mathcal{M}$ *and* s_0 *are defined in (2.18) and (2.19), respectively. Then one has*

$$\widehat{E}(t) \le C_5 \left(\widehat{E}(0) + C_6 \left(\widehat{E}(0) \right)^{\frac{b_0}{2}} \right)^{\kappa} e^{-C_7 t}, \tag{4.1}$$

where

$$0 < \kappa < 1,$$

$$C_5 := \left(1 + \frac{C_8 T e^{\frac{C_8 T}{\theta_0 (p-1)}}}{\theta_0 (p-1)} \right)^{\kappa} \left(\frac{4(p+1) e^{\theta + \tilde{\theta}} \hat{d}}{p-1} \right)^{1-\kappa},$$

$$C_6 := 2^{\frac{b_0}{2}} \frac{N_1}{\lambda_2},$$

$$C_7 := \frac{\theta_0 (1-\kappa)}{T},$$

$$C_8 := 4^3 R_4^{\frac{1}{2}} R_{4(p-1)}^{\frac{1}{2}} C \left(\frac{2(p+1)\hat{d}}{p-1} e^{\theta + \tilde{\theta}} \right)^{p-1},$$

$$\theta_0 := \frac{\theta + \tilde{\theta} - |\theta - \tilde{\theta}|}{2} = \min\left\{ \theta, \tilde{\theta} \right\}, \tag{4.2}$$

and $\theta > 0$, $\tilde{\theta} > 0$, *and* $T > 0$ *satisfy*

$$\mathcal{E}(z(t), z_t(t)) \le e^{\theta} \mathcal{E}(z_0, z_1) e^{-\frac{\theta}{T} t} \tag{4.3}$$

and

$$\mathcal{E}(\tilde{z}(t), \tilde{z}_t(t)) \le e^{\tilde{\theta}} \mathcal{E}(\tilde{z}_0, \tilde{z}_1) e^{-\frac{\tilde{\theta}}{T} t}, \tag{4.4}$$

$$b_0 := \frac{\theta_0}{\theta_0 + \bar{M} C T}, \tag{4.5}$$

$\bar{M}$ *is defined in (3.42),*

$$N_1 := \left(8 e^{\theta + \tilde{\theta}} \hat{d} \right)^{\frac{2-b_0}{2}} \frac{C}{2} + 2 C R_2^{\frac{1}{2}} \left(8 e^{\theta + \tilde{\theta}} \hat{d} \right)^{\frac{1-b_0}{2}} \left(\frac{8(p+1)}{p-1} e^{\theta + \tilde{\theta}} \hat{d} \right)^{\frac{1}{2}},$$

and

$$\begin{aligned} \lambda_2 :=& \frac{\theta_0 (2 - b_0)}{2T} - \frac{b_0 \bar{M} C}{2} \\ =& \frac{\theta_0 (2 - b_0) - b_0 \bar{M} C T}{2T}. \end{aligned}$$

Proof. Due to Corollary 2.14 in [5], for any $T > 0$, we know that there exist θ and $\tilde{\theta}$ to make (4.3) and (4.4) hold, where θ is dependent on $\mathcal{E}(z_0, z_1)$ and T, and $\tilde{\theta}$ is dependent on $\mathcal{E}(\tilde{z}_0, \tilde{z}_1)$ and T. According to Proposition 2.11 in [5], the assumptions $\mathcal{E}(z_0, z_1) < \mathcal{M}(s_0 - \nu)$, $\|z_0\|_V \le s_0 - \nu$, and $\mathcal{E}(\tilde{z}_0, \tilde{z}_1) < \mathcal{M}(s_0 - \nu)$, $\|\tilde{z}_0\|_V \le s_0 - \nu$ give $z_0 \in W$ and $\tilde{z}_0 \in W$, respectively. Here $\mathcal{M}(s_0 - \nu) < \hat{d}$ can be observed according to (2.17). Thus, we know (3.26) also holds. According to these facts and (2.16), we have

$$\frac{1}{2}\|z_t\|_2^2 + \frac{p-1}{2(p+1)}\|z\|_V^2 < \mathcal{E}(z(t), z_t(t)) \le e^{\theta}\mathcal{E}(z_0, z_1)e^{-\frac{\theta}{T}t} < e^{\theta}\hat{d}e^{-\frac{\theta}{T}t}, \tag{4.6}$$

and

$$\frac{1}{2}\|\tilde{z}_t\|_2^2 + \frac{p-1}{2(p+1)}\|\tilde{z}\|_V^2 < \mathcal{E}(\tilde{z}(t), \tilde{z}_t(t)) \le e^{\tilde{\theta}}\mathcal{E}(\tilde{z}_0, \tilde{z}_1)e^{-\frac{\tilde{\theta}}{T}t} < e^{\tilde{\theta}}\hat{d}e^{-\frac{\tilde{\theta}}{T}t}. \tag{4.7}$$

Due to (4.2), we know that (4.6) and (4.7) turn to

$$\frac{1}{2}\|z_t\|_2^2 + \frac{p-1}{2(p+1)}\|z\|_V^2 < e^{\theta}\hat{d}e^{-\frac{\theta}{T}t} < e^{\theta+\tilde{\theta}}\hat{d}e^{-\frac{\theta_0}{T}t} \tag{4.8}$$

and

$$\frac{1}{2}\|\tilde{z}_t\|_2^2 + \frac{p-1}{2(p+1)}\|\tilde{z}\|_V^2 < e^{\tilde{\theta}}\hat{d}e^{-\frac{\tilde{\theta}}{T}t} < e^{\theta+\tilde{\theta}}\hat{d}e^{-\frac{\theta_0}{T}t}, \tag{4.9}$$

respectively. According to (4.8) and (4.9), we have

$$\|z\|_V^{4(p-1)} < \left(\frac{2(p+1)\hat{d}}{p-1}e^{\theta+\tilde{\theta}}\right)^{2(p-1)} e^{-\frac{2\theta_0(p-1)}{T}t}$$

and

$$\|\tilde{z}\|_V^{4(p-1)} < \left(\frac{2(p+1)\hat{d}}{p-1}e^{\theta+\tilde{\theta}}\right)^{2(p-1)} e^{-\frac{2\theta_0(p-1)}{T}t},$$

which mean

$$\left(\|z\|_V^{4(p-1)} + \|\tilde{z}\|_V^{4(p-1)}\right)^{\frac{1}{2}} < \left(\frac{2(p+1)\hat{d}}{p-1}e^{\theta+\tilde{\theta}}\right)^{p-1} 2^{\frac{1}{2}}e^{-\frac{\theta_0(p-1)}{T}t}. \tag{4.10}$$

Next, we need to use the estimate (3.64) to continue this proof. More precisely, by substituting (4.10) into (3.64), we obtain

$$\begin{aligned} A_3 + A_4 \le& C_8 e^{-\frac{\theta_0(p-1)}{T}t}\frac{1}{2}\|z-\tilde{z}\|_V^2 + \frac{C}{2}\|w_t\|_2^2 \\ \le& C_8 e^{-\frac{\theta_0(p-1)}{T}t}\left(\frac{1}{2}\|z-\tilde{z}\|_V^2 + \frac{1}{2}\|z_t-\tilde{z}_t\|_2^2\right) + \frac{C}{2}\|w_t\|_2^2. \end{aligned} \tag{4.11}$$

By substituting (4.11) into (3.61) and the similar process of obtaining (3.69), we have

$$\int_0^L (\mathcal{F}(z(t)) - \mathcal{F}(\tilde{z}(t))) \cdot w_t \mathrm{d}x$$

$$\begin{aligned}&\le C_8 e^{-\frac{\theta_0(p-1)}{T}t}\widehat{E}(t) + \frac{C}{2}\|w_t\|_2^2\\&\quad + 2C\|z-\tilde{z}\|_2\|z_t-\tilde{z}_t\|_2\\&\le C_8 e^{-\frac{\theta_0(p-1)}{T}t}\widehat{E}(t) + \frac{C}{2}\|w_t\|_2^2\\&\quad + 2CR_2^{\frac{1}{2}}\|z-\tilde{z}\|_V\|z_t-\tilde{z}_t\|_2. \qquad (4.12)\end{aligned}$$

According to (3.60), we know

$$\|z_t-\tilde{z}_t\|_2 \le \left(2\widehat{E}(0)e^{\bar{M}Ct}\right)^{\frac{1}{2}},$$

i.e.,

$$\|z_t-\tilde{z}_t\|_2^{b_0} \le \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} e^{\frac{b_0\bar{M}Ct}{2}}, \qquad (4.13)$$

where b_0 is defined by (4.5). Meanwhile, combining (4.8) and (4.9), we know

$$\|z_t-\tilde{z}_t\|_2 \le \|z_t\|_2 + \|\tilde{z}_t\|_2 \le \left(8e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{1}{2}} e^{-\frac{\theta_0}{2T}t}, \qquad (4.14)$$

i.e.,

$$\|z_t-\tilde{z}_t\|_2^{1-b_0} \le \left(8e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{1-b_0}{2}} e^{-\frac{\theta_0(1-b_0)}{2T}t}. \qquad (4.15)$$

and

$$\|z-\tilde{z}\|_V \le \|z\|_V + \|\tilde{z}\|_V \le \left(\frac{8(p+1)}{p-1}e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{1}{2}} e^{-\frac{\theta_0}{2T}t}, \qquad (4.16)$$

where $b_0 > 0$ and $1 - b_0 > 0$ are ensured by (4.5). According to (4.13), (4.14) and (4.15), we have

$$\begin{aligned}\|z_t-\tilde{z}_t\|_2^2 =&\|z_t-\tilde{z}_t\|_2\|z_t-\tilde{z}_t\|_2^{b_0}\|z_t-\tilde{z}_t\|_2^{1-b_0}\\ \le&\left(2\widehat{E}(0)\right)^{\frac{b_0}{2}}\left(8e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{2-b_0}{2}} e^{-\left(\frac{\theta_0(2-b_0)}{2T}-\frac{b_0\bar{M}C}{2}\right)t}. \qquad (4.17)\end{aligned}$$

According to (4.5), we have

$$0 < b_0 < \frac{2\theta_0}{\theta_0 + \bar{M}CT}, \qquad (4.18)$$

i.e.,

$$\frac{\theta_0(2-b_0)}{2T} - \frac{b_0\bar{M}C}{2} > 0$$

in (4.17). Meanwhile, according to (4.13), (4.15) and (4.16), we notice that

$$\|z-\tilde{z}\|_V\|z_t-\tilde{z}_t\|_2 =\|z-\tilde{z}\|_V\|z_t-\tilde{z}_t\|_2^{b_0}\|z_t-\tilde{z}_t\|_2^{1-b_0}$$

$$\leq \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} \left(8e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{1-b_0}{2}} \left(\frac{8(p+1)}{p-1}e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{1}{2}}$$
$$e^{-\left(\frac{\theta_0(2-b_0)}{2T}-\frac{b_0\bar{M}C}{2}\right)t}. \tag{4.19}$$

Due to (4.17) and (4.19), we know that (4.12) turns to

$$\begin{aligned}
&\int_0^L (\mathcal{F}(z(t)) - \mathcal{F}(\tilde{z}(t))) \cdot w_t \mathrm{d}x \\
\leq &C_8 e^{-\frac{\theta_0(p-1)}{T}t}\widehat{E}(t) + \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} \left(8e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{2-b_0}{2}} \frac{Ce^{-\left(\frac{\theta_0(2-b_0)}{2T}-\frac{b_0\bar{M}C}{2}\right)t}}{2} \\
&+ 2CR_2^{\frac{1}{2}}\left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} \left(8e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{1-b_0}{2}} \left(\frac{8(p+1)}{p-1}e^{\theta+\tilde{\theta}}\hat{d}\right)^{\frac{1}{2}} e^{-\left(\frac{\theta_0(2-b_0)}{2T}-\frac{b_0\bar{M}C}{2}\right)t}.
\end{aligned} \tag{4.20}$$

By substituting (4.20) into (3.43), we obtain

$$\widehat{E}(t) \leq \widehat{E}(0) + C_8 \int_0^t e^{-\frac{\theta_0(p-1)}{T}\tau}\widehat{E}(\tau)\mathrm{d}\tau + \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} D_1, \tag{4.21}$$

where

$$D_1 := N_1 \int_0^t e^{-\lambda_2\tau}\mathrm{d}\tau = \frac{N_1}{\lambda_2} - \frac{N_1}{\lambda_2}e^{-\lambda_2 t}. \tag{4.22}$$

Here, according to (4.22), we notice that $D_1 \leq \frac{N_1}{\lambda_2}$, which means that (4.21) turns to

$$\widehat{E}(t) \leq \widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} \frac{N_1}{\lambda_2} + C_8 \int_0^t e^{-\frac{\theta_0(p-1)}{T}\tau}\widehat{E}(\tau)\mathrm{d}\tau, \tag{4.23}$$

i.e.,

$$\begin{aligned}
e^{-\frac{\theta_0(p-1)}{T}t}\widehat{E}(t) \leq &e^{-\frac{\theta_0(p-1)}{T}t}\left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} \frac{N_1}{\lambda_2}\right) \\
&+ C_8 e^{-\frac{\theta_0(p-1)}{T}t}\int_0^t e^{-\frac{\theta_0(p-1)}{T}\tau}\widehat{E}(\tau)\mathrm{d}\tau.
\end{aligned} \tag{4.24}$$

By similar process of obtaining (3.84), we have

$$\widehat{E}(t) \leq \left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} \frac{N_1}{\lambda_2}\right)\left(1 + \frac{C_8 T e^{\frac{C_8 T}{\theta_0(p-1)}}}{\theta_0(p-1)}\right). \tag{4.25}$$

For $0 < \kappa < 1$, according to (4.25), we know

$$\begin{aligned}
\widehat{E}(t) =&\widehat{E}(t)^{\kappa}\widehat{E}(t)^{1-\kappa} \\
\leq&\left(\widehat{E}(0) + \left(2\widehat{E}(0)\right)^{\frac{b_0}{2}} \frac{N_1}{\lambda_2}\right)^{\kappa}\left(1 + \frac{C_8 T e^{\frac{C_8 T}{\theta_0(p-1)}}}{\theta_0(p-1)}\right)^{\kappa}\widehat{E}(t)^{1-\kappa}.
\end{aligned} \tag{4.26}$$

By the similar process of obtaining (3.86), we have

$$\widehat{E}(t)^{1-\kappa} \leq \left(\|z_t\|_2^2 + \|\tilde{z}_t\|_2^2 + \|z\|_V^2 + \|\tilde{z}\|_V^2\right)^{1-\kappa}. \tag{4.27}$$

According to (4.8) and (4.9), we know

$$\frac{p-1}{2(p+1)}\left(\|z_t\|_2^2 + \|z\|_V^2\right) < e^{\theta+\tilde{\theta}}\hat{d}e^{-\frac{\theta_0}{T}t} \tag{4.28}$$

and

$$\frac{p-1}{2(p+1)}\left(\|\tilde{z}_t\|_2^2 + \|\tilde{z}\|_V^2\right) < e^{\theta+\tilde{\theta}}\hat{d}e^{-\frac{\theta_0}{T}t}. \tag{4.29}$$

By substituting (4.28) and (4.29) into (4.27), we have

$$\widehat{E}(t)^{1-\kappa} \leq \left(\frac{4(p+1)e^{\theta+\tilde{\theta}}\hat{d}}{p-1}\right)^{1-\kappa} e^{-\frac{\theta_0(1-\kappa)}{T}t},$$

which means that (4.26) turns to (4.1).

5. Lower bound estimate of blowup time for positive initial energy and nonlinear weak damping case

The finite time blowup at the positive initial energy level was established for the linear weak damping case and nonlinear weak damping case in [22], and for the linear weak damping case, the lower and upper bounds of the blowup time were also estimated there. Hence in this section, we shall estimate the lower bound of the blowup time at the positive initial energy level for the nonlinear weak damping case.

Theorem 5.1. *(Lower bound of blowup time for positive initial energy and nonlinear weak damping case) Let Assumption* 2.1 *and Assumption* 2.2 *hold, and* $\mathcal{E}(z_0, z_1) \geq 0$. *Suppose* $z(x,t)$ *is the solution to problem (1.1). If* $z(x,t)$ *blows up at a finite time* T_0, *then we have the estimate of blowup time*

$$T_0 \geq \int_{G(0)}^{\infty} \frac{1}{C_9 y^p + C_{10} y + C_{11}} \mathrm{d}y,$$

where

$$\begin{aligned} C_9 &:= (p+1)R_{2p}2^{2p-2}M^p, \\ C_{10} &:= (p+1)M, \\ C_{11} &:= (p+1)\mathcal{E}(z_0, z_1) + (p+1)R_{2p}2^{2p-2}\left(\mathcal{E}(z_0, z_1)\right)^p, \end{aligned}$$

and

$$G(0) := \|z_0\|_{p+1}^{p+1}.$$

Proof. Let $z = (u, \phi)$ be a weak solution to problem (1.1). We suppose that such solution blows up at a finite time T_0. Our goal is to obtain an estimate of the lower bound of T_0.

For $t \in [0, T_0)$, we define

$$G(t) := \|z(t)\|_{p+1}^{p+1} = \|u(t)\|_{p+1}^{p+1} + \|\phi(t)\|_{p+1}^{p+1}, \tag{5.1}$$

then, by Hölder inequality and Young inequality, we have

$$\begin{aligned} G'(t) =&(p+1)\int_0^L |u|^{p-1}uu_t\mathrm{d}x + (p+1)\int_0^L |\phi|^{p-1}\phi\phi_t\mathrm{d}x \\ \le&(p+1)\int_0^L |u|^{p}|u_t|\mathrm{d}x + (p+1)\int_0^L |\phi|^{p}|\phi_t|\mathrm{d}x \\ \le&(p+1)\|u\|_{2p}^p\|u_t\|_2 + (p+1)\|\phi\|_{2p}^p\|\phi_t\|_2 \\ \le&\frac{p+1}{2}\left(\|u\|_{2p}^{2p} + \|u_t\|_2^2 + \|\phi\|_{2p}^{2p} + \|\phi_t\|_2^2\right) \\ =&\frac{p+1}{2}\left(\|z\|_{2p}^{2p} + \|z_t\|_2^2\right). \end{aligned} \tag{5.2}$$

Next task is to estimate the terms in the last line of (5.2). By (2.14) and (2.16), we obtain

$$\begin{aligned} \mathcal{E}(z(t), z_t(t)) =&\frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\|z\|_V^2 - \int_0^L F(z(t))\mathrm{d}x \\ \ge&\frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\|z\|_V^2 - M\int_0^L \left(|u|^{p+1} + |\phi|^{p+1}\right)\mathrm{d}x \\ =&\frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\|z\|_V^2 - M\|z\|_{p+1}^{p+1}. \end{aligned} \tag{5.3}$$

According to (3.16), we know

$$\mathcal{E}(z(t), z_t(t)) \le \mathcal{E}(z_0, z_1),\ t \in [0, T_0), \tag{5.4}$$

where $\mathcal{E}(z_0, z_1) \ge 0$. We notice that (5.3) and (5.4) give

$$\|z_t\|_2^2 + \|z\|_V^2 \le 2\mathcal{E}(z_0, z_1) + 2MG(t), \tag{5.5}$$

which means

$$\|z\|_V^2 \le 2\mathcal{E}(z_0, z_1) + 2MG(t), \tag{5.6}$$

and

$$\|z_t\|_2^2 \le 2\mathcal{E}(z_0, z_1) + 2MG(t). \tag{5.7}$$

Combining (2.5) and (5.6), we see

$$\|z\|_{2p}^{2p} \le R_{2p}\left(2\mathcal{E}(z_0, z_1) + 2MG(t)\right)^p. \tag{5.8}$$

By substituting (5.7) and (5.8) into (5.2), we have

$$G'(t) \le \frac{(p+1)R_{2p}}{2}\left(2\mathcal{E}(z_0, z_1) + 2MG(t)\right)^p + (p+1)\left(\mathcal{E}(z_0, z_1) + MG(t)\right). \tag{5.9}$$

We consider the function $h(x) := x^p, x > 0, p > 1$. Since $h''(x) = p(p-1)x^{p-2} > 0$, $h(x)$ is a convex function. Thus it gives that

$$h\left(\frac{\tilde{k}_1 + \tilde{k}_2}{2}\right) \le \frac{1}{2}h(\tilde{k}_1) + \frac{1}{2}h(\tilde{k}_2),\ \ \tilde{k}_1, \tilde{k}_2 \ge 0,$$

that is to say

$$(\tilde{k}_1 + \tilde{k}_2)^p \leq 2^{p-1}(\tilde{k}_1^p + \tilde{k}_2^p).$$

Then, due to $\mathcal{E}(z_0, z_1) \geq 0$ and $G(t) \geq 0$, we can get

$$(2\mathcal{E}(z_0, z_1) + 2MG(t))^p \leq 2^{p-1}\left((2\mathcal{E}(z_0, z_1))^p + (2MG(t))^p\right), \tag{5.10}$$

which means that (5.9) turns to

$$\begin{aligned} G^{'}(t) \leq &(p+1)R_{2p}2^{2p-2}M^p\,(G(t))^p + (p+1)MG(t) + (p+1)\mathcal{E}(z_0, z_1) \\ &+ (p+1)R_{2p}2^{2p-2}\,(\mathcal{E}(z_0, z_1))^p\,, \end{aligned}$$

i.e.,

$$\frac{G^{'}(t)}{C_9\,(G(t))^p + C_{10}G(t) + C_{11}} \leq 1. \tag{5.11}$$

Recalling the assumption that the solution of problem (1.1) blows up in finite time T_0, we have

$$\lim_{t\to T_0} G(t) = \lim_{t\to T_0} \|z(t)\|_{p+1}^{p+1} = \infty. \tag{5.12}$$

Then, integrating both sides of (5.11) on $(0, T_0)$ and combining (5.12), we get

$$\int_{G(0)}^{\infty} \frac{1}{C_9 y^p + C_{10} y + C_{11}} \mathrm{d}y \leq T_0.$$

Thus, the proof of Theorem 5.1 is completed.

6. Appendix: a variation of Gronwall's inequality

In Sept I of the proofs of Theorem 3.2, by the classical form of Gronwall's inequality (integral form) shown in Appendix B.2 of [4], we know that (3.59) gives

$$\widehat{E}(t) \leq \widehat{E}(0)(1 + \bar{M}Cte^{\bar{M}Ct}). \tag{6.1}$$

In (6.1), the growth order of the distance of the solutions, i.e., $\widehat{E}(t)$, is controlled by the product of an exponential function and a polynomial function, which is higher than that in (1.2) established for the local solution. In Sept I of the proofs of Theorem 3.2, in order to build the growth estimate of $\widehat{E}(t)$ in the same form as (1.2) for the global solution, i.e., (3.60), we need the following variation of Gronwall's inequality.

Proposition 6.1. For a nonnegative, summable function $\zeta(t)$ on $[0, \bar{T}]$ with satisfying

$$\zeta(t) \leq \bar{C}_1 \int_0^t \zeta(\tau)\mathrm{d}\tau + \bar{C}_2 \tag{6.2}$$

for the constants $\bar{C}_1, \bar{C}_2 \geq 0$, one has

$$\zeta(t) \leq \bar{C}_2 e^{\bar{C}_1 t} \tag{6.3}$$

for a.e. $0 \leq t \leq \bar{T}$.

Proof. We use the similar idea of proving the classical form of Gronwall's inequality shown by Appendix B in [4] to give the proofs. We first define the auxiliary function

$$\chi(t) := e^{-\bar{C}_1 t} \int_0^t \zeta(\tau) \mathrm{d}\tau. \tag{6.4}$$

By direct calculation, we have

$$\chi'(t) = e^{-\bar{C}_1 t} \left(\zeta(t) - \bar{C}_1 \int_0^t \zeta(\tau) \mathrm{d}\tau \right). \tag{6.5}$$

Substituting (6.2) into (6.5), we have

$$\chi'(t) \le e^{-\bar{C}_1 t} \bar{C}_2,$$

which means

$$\int_0^t \chi'(\tau) \mathrm{d}\tau \le \int_0^t e^{-\bar{C}_1 \tau} \bar{C}_2 \mathrm{d}\tau,$$

i.e.,

$$\chi(t) \le \frac{\bar{C}_2}{\bar{C}_1} \left(1 - e^{-\bar{C}_1 t} \right). \tag{6.6}$$

According to (6.4) and (6.6), we have

$$\int_0^t \zeta(\tau) \mathrm{d}\tau \le \frac{\bar{C}_2}{\bar{C}_1} \left(e^{\bar{C}_1 t} - 1 \right). \tag{6.7}$$

Substituting (6.7) into (6.2), we obtain (6.3).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. T. A. Apalara, A general decay for a weakly nonlinearly damped porous system, *J. Dyn. Control Syst.,* **25** (2019), 311–322. https://doi.org/10.1007/s10883-018-9407-x
2. S. C. Cowin, M. A. Goodman, A variational principle for granular materials, *ZAMM Z. Angew. Math. Mech.*, **56** (1976), 281–286. https://doi.org/10.1002/zamm.19760560702
3. S. C. Cowin, J. W. Nunziato, Linear elastic materials with voids, *J. Elasticity*, **13** (1983), 125–147. https://doi.org/10.1007/BF00041230
4. L. C. Evans, Partial Differential Equations, 2nd edn, American Mathematical Society, Providence, 2010. https://doi.org/10.1090/gsm/019
5. M. M. Freitas, M. L. Santos, J. A. Langa, Porous elastic system with nonlinear damping and sources terms, *J. Differential Equations*, **264** (2018), 2970–3051. https://doi.org/10.1016/j.jde.2017.11.006
6. B. W. Feng, On the decay rates for a one-dimensional porous elasticity system with past history, *Commun. Pure Appl. Anal.*, **18** (2019), 2905–2921. https://doi.org/10.3934/cpaa.2019130
7. B. W. Feng, L. Yan, D. S. Almeida Júnior, Stabilization for an inhomogeneous porous-elastic system with temperature and microtemperature, *ZAMM Z. Angew. Math. Mech.*, **101** (2021), 202000058. https://doi.org/10.1002/zamm.202000058
8. M. A. Goodman, S. C. Cowin, A continuum theory for granular materials, *Arch. Ration. Mech. Anal.*, **44** (1972), 249–266. https://doi.org/10.1007/BF00284326
9. F. Gazzola, T. Weth, Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level, *Differential Integral Equations*, **18** (2005), 961–990. https://doi.org/10.57262/die/1356060117
10. F. Gazzola, M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **23** (2006), 185-207. https://doi.org/10.1016/j.anihpc.2005.02.007
11. J. B. Han, R. Z. Xu, Y. B. Yang, Asymptotic behavior and finite time blow up for damped fourth order nonlinear evolution equation, *Asymptot. Anal.*, **122** (2021), 349–369. https://doi.org/10.3233/ASY-201621
12. H. E. Khochemane, A. Djebabla, S. Zitouni, L. Bouzettouta, Well-posedness and general decay of a nonlinear damping porous-elastic system with infinite memory, *J. Math. Phys.,* **61** (2020), 021505. https://doi.org/10.1063/1.5131031
13. T. Louis, A localized nonstandard stabilizer for the Timoshenko beam, *C. R. Math. Acad. Sci. Paris*, **353** (2015), 247–253. https://doi.org/10.1016/j.crma.2015.01.004
14. W. Lian, R. Z. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), 613–632. https://doi.org/10.1515/anona-2020-0016
15. W. Lian, V. D. Rădulescu, R. Z. Xu, Y. B. Yang, N. Zhao, Global well-posedness for a class of fourth-order nonlinear strongly damped wave equations, *Adv. Calc. Var.*, **14** (2021), 589–611. https://doi.org/10.1515/acv-2019-0039

16. A. Magaña, R. Quintanilla, On the time decay of solutions in one-dimensional theories of porous materials, *Internat. J. Solids Structures*, **43** (2006), 3414–3427. https://doi.org/10.1016/j.ijsolstr.2005.06.077

17. J. Muñoz-Rivera, R. Quintanilla, On the time polynomial decay in elastic solids with voids, *J. Math. Anal. Appl.*, **338** (2008), 1296–1309. https://doi.org/10.1016/j.jmaa.2007.06.005

18. J. W. Nunziato, S. C. Cowin, A nonlinear theory of elastic materials with voids, *Arch. Ration. Mech. Anal.*, **72** (1979), 175–201. https://doi.org/10.1007/BF00249363

19. R. Quintanilla, Slow decay for one-dimensional porous dissipation elasticity, *Appl. Math. Lett.*, **16** (2003), 487–491. https://doi.org/10.1016/S0893-9659(03)00025-9

20. M. L. Santos, D. S. Almeida Júnior, On porous-elastic system with localized damping, *Z. Angew. Math. Phys.*, **67** (2016), 63. https://doi.org/10.1007/s00033-016-0622-6

21. M. L. Santos, D. S. Almeida Júnior, S. M. Cordeiro, Energy decay for a porous-elastic system with nonlinear localized damping, *Z. Angew. Math. Phys.*, **73** (2022), 7. https://doi.org/10.1007/s00033-021-01636-1

22. Q. M. Tran, T. T. Vu, M. M. Freitas, Blow-up of weak solutions for a porous elastic system with nonlinear damping and source terms, *J. Math. Anal. Appl.*, **512** (2022), 126132. https://doi.org/10.1016/j.jmaa.2022.126132

23. R. Z. Xu, J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763. https://doi.org/10.1016/j.jfa.2013.03.010

24. H. Y. Xu, Existence and blow-up of solutions for finitely degenerate semilinear parabolic equations with singular potentials, *Commun. Anal. Mech.*, **15** (2023), 132–161. https://doi.org/10.3934/cam.2023008

25. Y. B. Yang, R. Z. Xu, Nonlinear wave equation with both strongly and weakly damped terms: supercritical initial energy finite time blow up, *Commun. Pure Appl. Anal.*, **18** (2019), 1351–1358. https://doi.org/10.3934/cpaa.2019065