Communications in Analysis and Mechanics

Communications in Analysis and Mechanics, 15(2): 189–213.
DOI: 10.3934/cam.2023011
Received: 31 March 2023
Revised: 05 May 2023
Accepted: 05 May 2023
Published: 12 May 2023

https://www.aimspress.com/journal/cam

Research article

Laguerre BV spaces, Laguerre perimeter and their applications

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Abstract: In this paper, we introduce the Laguerre bounded variation space and the Laguerre perimeter, thereby investigating their properties. Moreover, we prove the isoperimetric inequality and the Sobolev inequality in the Laguerre setting. As applications, we derive the mean curvature for the Laguerre perimeter.

Keywords: BV space; perimeter; isoperimetric inequality; Sobolev inequality; Laguerre operator
Mathematics Subject Classification: 26A45, 46E35, 33C45

1. Introduction

The spaces BV of functions of bounded variation in Euclidean spaces have been a class of function space which can be used in the geometric measure theory. For example, when working with minimization problems, reflexivity or the weak compactness property involving the function space $W^{1,p}(\mathbb{R}^d)$ for $p > 1$, in such cases, the space BV usually plays a crucial role. However, for the case of the space $W^{1,1}(\mathbb{R}^d)$, one possible approach to address its lack of reflexivity is to consider the space $BV(\mathbb{R}^d)$. The importance of generalizing the classical notion of variation has been pointed out in several occasions by E. De. Giorgi in [1]. Recently, Huang, Li and Liu in [2] investigate the capacity and perimeters derived from α-Hermite bounded variation. In a general framework of strictly local Dirichlet spaces with doubling measure, Alonso-Ruiz, Baudoin and Chen et al. in [3] introduce the class of bounded variation functions and proved the Sobolev inequality under the Bakry-Émery curvature type condition. For further information on this topic, we refer the reader to [4–6] and the references therein.

One of the aims of this paper is intended to explore and analyze a number of fundamental inquiries in geometric measure theory that are associated with the Laguerre operator in Laguerre BV spaces. To begin with, we will provide a brief introduction to the Laguerre operator.

Given a multiindex $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\alpha \in (-1, \infty)^d$, the Laguerre differential operator is defined by:

$$\mathcal{L}^{\alpha} = -\sum_{i=1}^{d}\Big[x_i\frac{\partial^2}{\partial {x_i}^2} + (\alpha_i + 1 - x_i)\frac{\partial}{\partial x_i}\Big].$$

Let the probabilistic gamma measure μ_α in $\mathbb{R}_+^d = (0,\infty)^d$ be defined as

$$d\mu_\alpha(x) = \prod_{i=1}^{d} \frac{{x_i}^{\alpha_i} e^{-x_i}}{\Gamma(\alpha_i+1)} dx := \omega(x)dx.$$

As we know that $\mathcal{L}^\alpha$ is positive and symmetric in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$, and it has a closure which is selfadjoint in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$ and will be denoted by $\mathcal{L}^\alpha$. The i-th partial derivative associated with $\mathcal{L}^\alpha$ is defined as

$$\delta_i = \sqrt{x_i}\frac{\partial}{\partial x_i},$$

see [7] or [8]. The operator $\mathcal{L}^\alpha$ has the following decomposition:

$$\mathcal{L}^\alpha = \sum_{i=1}^{d} \delta_i^* \delta_i,$$

where

$$\delta_i^* = -\sqrt{x_i}\Big(\partial x_i + \frac{\alpha_i + \frac{1}{2} - x_i}{x_i}\Big)$$

is the formal adjoint of δ_i in $L^2(\mathbb{R}_+^d, \mathrm{d}\mu_\alpha)$. Throughout this paper, suppose that $\Omega \subset \mathbb{R}_+^d$ be an open set. For $u \in C^1(\mathbb{R}_+^d)$ and $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in C^1(\mathbb{R}_+^d, \mathbb{R}^d)$, define the $\mathcal{L}^\alpha$-gradient and $\mathcal{L}^\alpha$-divergence operators that are associated with $\mathcal{L}^\alpha$:

$$\begin{cases} \nabla_{\mathcal{L}^\alpha} u := (\delta_1 u, \ldots, \delta_d u), \\ \mathrm{div}_{\mathcal{L}^\alpha}\varphi := \delta_1^*\varphi_1 + \delta_2^*\varphi_2 + \cdots + \delta_d^*\varphi_d, \end{cases}$$

which also gives

$$\mathcal{L}^\alpha u = \mathrm{div}_{\mathcal{L}^\alpha}(\nabla_{\mathcal{L}^\alpha} u) = -\sum_{i=1}^{d} \Big[x_i \frac{\partial^2}{\partial {x_i}^2} + (\alpha_i + 1 - x_i)\frac{\partial}{\partial x_i}\Big].$$

Naturally, we denote by $\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)$ the set of all functions possessing Laguerre bounded variation ($\mathcal{L}^\alpha$-BV in short) on Ω. Based on the results of [2], we investigate some related topics for the Laguerre setting, and the plan of the notes is given as follows. Section 2.1 collects some basic facts and notations used later, the lower semicontinuity (Lemma 2.1), the completeness (Lemma 2.2), the structure theorem (Theorem 2.3) and approximation via C_c^∞-functions (Theorem 2.4). Unlike Theorem 2 in [9, Section 5.2.2], we must utilize the mean value theorem for multivariate functions and the intrinsic nature of the Laguerre variation. Section 2.2 is focused on the perimeter $P_{\mathcal{L}^\alpha}(\cdot, \Omega)$ induced by $\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)$, as shown in equation (2.6) below.

Remember that the classical perimeter of $E \subseteq \mathbb{R}^d$ is defined as

$$P(E) = \sup_{\varphi \in \mathcal{F}(\mathbb{R}^d)} \left\{ \int_E \mathrm{div}\varphi(x) dx \right\},$$

here let $\mathcal{F}(\mathbb{R}^d)$ be the set that contains all functions

$$\varphi = (\varphi_1, \cdots, \varphi_d) \in C_c^1(\mathbb{R}^d, \mathbb{R}^d)$$

satisfying

$$\|\varphi\|_\infty = \sup_{x\in E}\left\{(|\varphi_1(x)|^2 + \cdots + |\varphi_d(x)|^2)^{\frac{1}{2}}\right\} \le 1.$$

As we all know that

$$P(E) = P(E^c),\ \forall E \subset \mathbb{R}^d \tag{1.1}$$

is an inherent property of $P(E)$ at the elementary level.

In Lemma 2.10, we proved that (1.1) is valid for the Laguerre perimeter $P_{\mathcal{L}^\alpha}(\cdot)$. In Section 2.3, a coarea formula for $\mathcal{L}^\alpha$-BV functions is derived. In Theorem 2.12, we conclude that the isoperimetric inequality

$$\|f\|_{L^{\frac{d}{d-1}}(\Omega_1,d\mu_\alpha)} \lesssim |\nabla_{\mathcal{L}^\alpha} f|(\Omega_1) \tag{1.2}$$

shares equivalence with the Sobolev type inequality

$$\mu_\alpha(E)^{\frac{d-1}{d}} \lesssim P_{\mathcal{L}^\alpha}(E,\Omega_1)$$

as an application. We point out that, in the proof of (1.2), the inequality $|\nabla f(x)| \lesssim |\nabla_{\mathcal{L}^\alpha} f(x)|$ on Ω_1 holds true. With this in mind, we consider the subset

$$\Omega_1 = \Omega \setminus \{x \in \mathbb{R}^d_+ : \exists i \in 1, \cdots, d \text{ such that } \sqrt{x_i} < 1\} \tag{1.3}$$

of Ω which is a reasonable substitute of Ω and whose figure is given as follows:

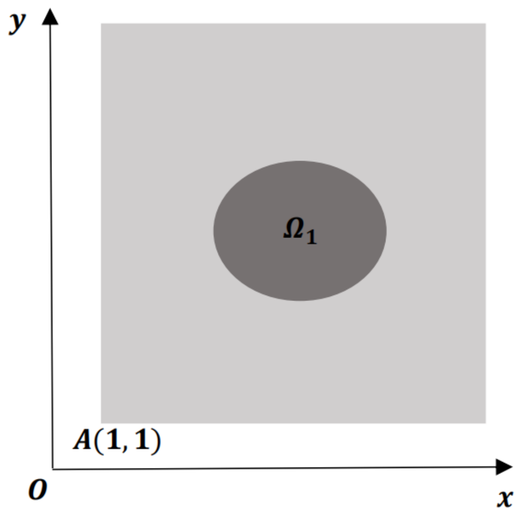

Figure 1. Set for the Sobolev inequality in the Laguerre setting on $\mathbb{R}^2_+$.

Our motivation comes not only from the fact that these objects are interesting on their own, but also from the possibility of their potential applications in further research concerning the Laguerre operator. Consequently, our aim in Section 3 is to examine the Laguerre mean curvature of a set that has a finite Laguerre perimeter. It is interesting to note that the sets of finite perimeter introduced by E. De Giorgi for the Laplace operator Δ have found applications in classical problems of the calculus of variations, such as the Plateau problem and the isoperimetric problem, see [10–12]. Barozzi, Gonzalez, and

Tamanini [13] demonstrated that for any finite classical perimeter set E within $\mathbb{R}^d$, its mean curvature is included in $L^1(\mathbb{R}^d)$. One might naturally wonder whether $P_{\mathcal{L}^\alpha}(E,\Omega)$, $\alpha \in (-1,\infty)^d$ holds similarly as [13]. Note that it is necessary to use identity (1.1) in the proof of the main theorem of [13]. In Theorem 3.1, we generalize the result of [13] to $P_{\mathcal{L}^\alpha}(\cdot,\Omega_1)$ and show that if a set E is a subset of Ω_1 such that $P_{\mathcal{L}^\alpha}(E,\Omega_1) < \infty$, then the mean curvature of E is in $L^1(\Omega_1, d\mu_\alpha)$.

Throughout this paper by C we always denote a positive constant that may vary at each occurrence; $A \approx B$ means that $\frac{1}{C}A \le B \le CA$ and the notation $X \lesssim Y$ is used to indicate that $X \le CY$ with a positive constant C independent of significant quantities. Similarly, one writes $X \gtrsim Y$ for $X \ge CY$.

2. $\mathcal{L}^\alpha$-BV functions

2.1. Fundamentals of $\mathcal{L}^\alpha$-BV Space

This section presents the $\mathcal{L}^\alpha$-BV space, which is defined as the set of all functions that exhibit Laguerre bounded variation and investigates its properties. The Laguerre variation ($\mathcal{L}^\alpha$-variation in short) of $f \in L^1(\Omega, d\mu_\alpha)$ is defined by

$$|\nabla_{\mathcal{L}^\alpha} f|(\Omega) = \sup_{\varphi\in\mathcal{F}(\Omega)} \left\{ \int_\Omega f(x)\mathrm{div}_{\mathcal{L}^\alpha}\varphi(x) d\mu_\alpha(x) \right\},$$

where $\mathcal{F}(\Omega)$ denotes the class of all functions

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in C_c^1(\Omega, \mathbb{R}^d)$$

satisfying

$$\|\varphi\|_{L^\infty} = \sup_{x\in\Omega} \left\{ (|\varphi_1(x)|^2 + \ldots + |\varphi_d(x)|^2)^{\frac{1}{2}} \right\} \le 1.$$

We say that an function $f \in L^1(\Omega, d\mu_\alpha)$ has the $\mathcal{L}^\alpha$-bounded variation on Ω if

$$|\nabla_{\mathcal{L}^\alpha} f|(\Omega) < \infty,$$

and denote by $\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)$ the class of all such functions, and it is a Banach space with the norm

$$\|f\|_{\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)} = \|f\|_{L^1(\Omega, d\mu_\alpha)} + |\nabla_{\mathcal{L}^\alpha} f|(\Omega).$$

Definition 2.1. Suppose Ω is an open set in $\mathbb{R}^d_+$. Let $1 \le p \le \infty$. The Sobolev space $W^{k,p}_{\mathcal{L}^\alpha}(\Omega)$ associated with $\mathcal{L}^\alpha$ is defined as the set of all functions $f \in L^p(\Omega, d\mu_\alpha)$ such that

$$\delta_{j_1} \ldots \delta_{j_m} f \in L^p(\Omega, d\mu_\alpha),\ 1 \le j_1, \ldots, j_m \le d,\ 1 \le m \le k.$$

The norm of $f \in W^{k,p}_{\mathcal{L}^\alpha}(\Omega)$ is given by

$$\|f\|_{W^{k,p}_{\mathcal{L}^\alpha}} := \sum_{1\le j_1\ldots j_m\le d,\ 1\le m\le k} \|\delta_{j_1} \ldots \delta_{j_m} f\|_{L^p(\Omega, d\mu_\alpha)} + \|f\|_{L^p(\Omega, d\mu_\alpha)}.$$

The upcoming results will gather certain properties of the space $\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)$. We omit the details of their proofs, since we can use the similar arguments as [2] to prove them.

Lemma 2.1.
(i) *Suppose* $f \in W^{1,1}_{\mathcal{L}^\alpha}(\Omega)$, *then*

$$|\nabla_{\mathcal{L}^\alpha} f|(\Omega) = \int_\Omega |\nabla_{\mathcal{L}^\alpha} f(x)| d\mu_\alpha(x),$$

which implies $W^{1,1}_{\mathcal{L}^\alpha}(\Omega) \subseteq \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)$.
(ii) *(Lower semicontinuity). Suppose* $f_k \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega),\ k \in \mathbb{N}$ *and* $f_k \to f$ *in* $L^1_{loc}(\Omega, d\mu_\alpha)$, *then*

$$|\nabla_{\mathcal{L}^\alpha} f|(\Omega) \leq \liminf_{k\to\infty} |\nabla_{\mathcal{L}^\alpha} f_k|(\Omega).$$

Lemma 2.2. *The space* $(\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega), \|\cdot\|_{\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)})$ *is a Banach space.*

The Hahn-Banach theorem and the Riesz representation theorem can be used to prove the structure theorem for $\mathcal{L}^\alpha$-BV functions, as presented in the following lemma.

Lemma 2.3. *(Structure theorem for* $\mathcal{BV}_{\mathcal{L}^\alpha}$ *functions). Let* $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)$. *Then there exists a Radon measure* $\mu_{\mathcal{L}^\alpha}$ *on* Ω *such that*

$$\int_\Omega f(x) \mathrm{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_\alpha(x) = \int_\Omega \varphi(x) \cdot d\mu_{\mathcal{L}^\alpha}(x)$$

for every $\varphi \in C^\infty_c(\Omega, \mathbb{R}^d)$ *and*

$$|\nabla_{\mathcal{L}^\alpha} f|(\Omega) = |\mu_{\mathcal{L}^\alpha}|(\Omega),$$

where $|\mu_{\mathcal{L}^\alpha}|$ *represents the total variation of the measure* $\mu_{\mathcal{L}^\alpha}$.

We can obtain an approximation result for the $\mathcal{L}^\alpha$-variation in the following theorem.

Theorem 2.4. *Let* Ω_1 *be an open set defined in (1.3). Assume that* $u \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)$, *then there exists a sequence* $\{u_h\}_{h\in\mathbb{N}} \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1) \cap C^\infty(\Omega_1)$ *such that*

$$\lim_{h\to\infty} \|u_h - u\|_{L^1(\Omega_1, d\mu_\alpha)} = 0$$

and

$$\lim_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} u_h(x)| d\mu_\alpha(x) = |\nabla_{\mathcal{L}^\alpha} u|(\Omega_1).$$

Proof. The approach we take differs from the proof presented in [9, Section 5.2.2, Theorem 2] as we utilize the mean value theorem of multivariate functions and the intrinsic nature of the $\mathcal{L}^\alpha$-variation. Via the lower semicontinuity of $\mathcal{L}^\alpha$-BV functions, it suffices to demonstrate that for $\varepsilon > 0$ there exists a function $u_\varepsilon \in C^\infty(\Omega_1)$ such that

$$\int_{\Omega_1} |u_\varepsilon(x) - u(x)| d\mu_\alpha(x) < \varepsilon$$

and

$$|\nabla_{\mathcal{L}^\alpha} u_\varepsilon|(\Omega_1) \leq |\nabla_{\mathcal{L}^\alpha} u|(\Omega_1) + \varepsilon.$$

Fix $\varepsilon > 0$. If m is a given positive integer, then construct a series of open sets,

$$\Omega_{1,j} := \left\{x \in \Omega_1 : \mathrm{dist}(x, \partial\Omega_1) > \frac{1}{m+j}\right\} \cap B(0, m+j),\ \ j \in \mathbb{N},$$

where $\text{dist}(x, \partial\Omega_1) = \inf\{|x-y| : y \in \partial\Omega_1\}$. Note that $\Omega_{1,j} \subset \Omega_{1,j+1} \subset \Omega_1,\ j \in \mathbb{N}$ and $\bigcup_{j=0}^{\infty} \Omega_{1,j} = \Omega_1$. Since $|\nabla_{\mathcal{L}^\alpha} u|(\cdot)$ is a measure, then choose a value $m \in \mathbb{N}$ to be sufficiently large such that

$$|\nabla_{\mathcal{L}^\alpha} u|(\Omega_1 \backslash \Omega_{1,0}) < \varepsilon. \tag{2.1}$$

Set $U_0 := \Omega_{1,0}$ and $U_j := \Omega_{1,j+1} \backslash \overline{\Omega}_{1,j-1}$ for $j \geq 1$. Based on the standard outcomes from [9, Section 5.2.2, Theorem 2], our inference is that there exists a partition of unity connected to the covering $\{U_j\}_{j\in\mathbb{N}}$. Namely, there exist functions $\{f_j\}_{j\in\mathbb{N}} \in C_c^\infty(U_j)$ such that $0 \leq f_j \leq 1,\ j \geq 0$ and $\sum_{j=0}^{\infty} f_j = 1$ on Ω_1. Thus we have the fact that

$$\begin{aligned}\sum_{j=0}^{\infty} \nabla_{\mathcal{L}^\alpha} f_j &= \Big(\sqrt{x_1} \frac{\partial}{\partial x_1} \Big(\sum_{j=0}^{\infty} f_j \Big), \sqrt{x_2} \frac{\partial}{\partial x_2} \Big(\sum_{j=0}^{\infty} f_j \Big), \cdots, \sqrt{x_d} \frac{\partial}{\partial x_d} \Big(\sum_{j=0}^{\infty} f_j \Big) \Big) \\ &= 0 \end{aligned} \tag{2.2}$$

on Ω_1. Given $\varepsilon > 0$ and $u \in L^1(\Omega_1, \mathbb{R})$, extended to zero out of Ω_1, the regularization can be defined as

$$u_\varepsilon(x) := \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \eta\Big(\frac{x-y}{\varepsilon}\Big) u(y) d\mu_\alpha(y),$$

where $\eta \in C_c^\infty(\mathbb{R}_+^d)$ is a nonnegative radial function satisfying

$$\frac{1}{\epsilon_j^d} \int_{\mathbb{R}_+^d} \eta(\frac{x-y}{\epsilon_j}) d\mu_\alpha(x) = 1,\ \forall\ j \in \mathbb{N},$$

and $\text{supp}\, \eta \subset B(0,1) \cap \mathbb{R}_+^d$. Then for each j, there exists $0 < \varepsilon_j < \varepsilon$ so small such that

$$\begin{cases} \text{supp}((f_j u)_{\varepsilon_j}) \subseteq U_j, \\ \int_{\Omega_1} |(f_j u)_{\varepsilon_j}(x) - f_j u(x)| d\mu_\alpha(x) < \varepsilon 2^{-(j+1)}, \\ \int_{\Omega_1} |(u \nabla_{\mathcal{L}^\alpha} f_j)_{\varepsilon_j}(x) - u \nabla_{\mathcal{L}^\alpha} f_j(x)| d\mu_\alpha(x) < \varepsilon 2^{-(j+1)}. \end{cases} \tag{2.3}$$

Construct

$$v_\varepsilon(x) := \sum_{j=0}^{\infty} (u f_j)_{\varepsilon_j}(x).$$

In some neighborhood of each point $x \in \Omega_1$, there are only finitely many nonzero terms in this sum, hence $v_\varepsilon \in C^\infty(\Omega_1)$ and $u = \sum_{j=0}^{\infty} u f_j$. Therefore, by a simple computation, we obtain

$$\|v_\varepsilon - u\|_{L^1(\Omega_1, d\mu_\alpha)} \leq \sum_{j=0}^{\infty} \int_{\Omega_1} |(f_j u)_{\varepsilon_j}(x) - f_j(x) u(x)| d\mu_\alpha(x) < \varepsilon.$$

Consequently,

$$v_\varepsilon \to u \ \text{ in } \ L^1(\Omega_1, d\mu_\alpha) \ \text{ as } \ \varepsilon \to 0.$$

Now, assume $\varphi \in C_c^1(\Omega_1, \mathbb{R}^d)$ and $|\varphi| \le 1$. We decompose the integral as follows:

$$\begin{aligned}
&\int_{\Omega_1} v_\varepsilon(x)\mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)\\
&= \int_{\Omega_1}\Big(\sum_{j=0}^{\infty}(uf_j)_{\varepsilon_j}(x)\Big)\mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)\\
&= \sum_{j=0}^{\infty}\int_{\Omega_1}(uf_j)_{\varepsilon_j}(x)\Big(\delta_1^*\varphi_1(x) + \delta_2^*\varphi_2(x) + \cdots + \delta_d^*\varphi_d(x)\Big)d\mu_\alpha(x)\\
&:= I + II,
\end{aligned}$$

where

$$\begin{cases}
I := -\displaystyle\sum_{j=0}^{\infty}\int_{\Omega_1}(uf_j)_{\varepsilon_j}(x)\Big(\sqrt{x_1}\frac{\partial}{\partial x_1}\varphi_1(x) + \cdots + \sqrt{x_d}\frac{\partial}{\partial x_d}\varphi_d(x)\Big)d\mu_\alpha(x),\\
II := -\displaystyle\sum_{j=0}^{\infty}\int_{\Omega_1}(uf_j)_{\varepsilon_j}(x)\Big(\frac{\alpha_1 + \frac{1}{2} - x_1}{\sqrt{x_1}}\varphi_1(x) + \cdots + \frac{\alpha_d + \frac{1}{2} - x_d}{\sqrt{x_d}}\varphi_d(x)\Big)d\mu_\alpha(x).
\end{cases}$$

For the sake of research, let

$$\widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}\varphi = \delta_1\varphi_1 + \delta_2\varphi_2 + \cdots + \delta_d\varphi_d. \tag{2.4}$$

As for I, we obtain

$$\begin{aligned}
I &= -\sum_{j=0}^{\infty}\int_{\Omega_1}(uf_j)_{\varepsilon_j}(x)\widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)\\
&= -\sum_{j=0}^{\infty}\int_{\Omega_1}(uf_j)(y)\widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}(\eta_{\varepsilon_j} * \varphi(y))d\mu_\alpha(y)\\
&= -\sum_{j=0}^{\infty}\int_{\Omega_1}u(y)\widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}(f_j(\eta_{\varepsilon_j} * \varphi))(y)d\mu_\alpha(y)\\
&\quad + \sum_{j=0}^{\infty}\int_{\Omega_1}u(y)\nabla_{\mathcal{L}^\alpha}f_j \cdot (\eta_{\varepsilon_j} * \varphi)(y)d\mu_\alpha(y)\\
&= -\sum_{j=0}^{\infty}\int_{\Omega_1}u(y)\widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}(f_j(\eta_{\varepsilon_j} * \varphi))(y)d\mu_\alpha(y)\\
&\quad - \sum_{j=0}^{\infty}\int_{\Omega_1}\varphi(y)\Big(\eta_{\varepsilon_j} * (u\nabla_{\mathcal{L}^\alpha}f_j)(y) - u\nabla_{\mathcal{L}^\alpha}f_j(y)\Big)d\mu_\alpha(y)\\
&:= I_1 + I_2,
\end{aligned}$$

where in the last equality we have used the fact (2.2). In fact, when $\|\varphi\|_{L^\infty} \le 1$, then $|f_j(\eta_{\varepsilon_j} * \varphi)(x)| \le 1$, $j \in \mathbb{N}$, and each point in Ω is contained in at most three of the sets $\{U_j\}_{j=0}^{\infty}$. Furthemore, (2.3) implies that $|I_2| < \varepsilon$.

On the other hand, we modify the integration order to obtain

$$
\begin{aligned}
II =& -\sum_{j=0}^{\infty}\int_{\Omega_1}(uf_j)_{\varepsilon_j}(x)\left(\frac{\alpha_1+\frac{1}{2}-x_1}{\sqrt{x_1}}\varphi_1(x)+\cdots+\frac{\alpha_d+\frac{1}{2}-x_d}{\sqrt{x_d}}\varphi_d(x)\right)d\mu_\alpha(x)\\
=& -\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\Big(\sum_{k=1}^{d}\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\varphi_k(x)\Big)d\mu_\alpha(y)d\mu_\alpha(x)\\
&-\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\\
&\times\left(\sum_{k=1}^{d}\Big(\frac{\alpha_k+\frac{1}{2}-x_k}{\sqrt{x_k}}-\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\Big)\varphi_k(x)\right)d\mu_\alpha(y)d\mu_\alpha(x)\\
=& -\sum_{j=0}^{\infty}\int_{\Omega_1}u(y)f_j(y)\Big(\Big(\sum_{k=1}^{d}\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\varphi_k\Big)*\eta_{\varepsilon_j}(y)\Big)d\mu_\alpha(y)\\
&-\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\\
&\times\left(\sum_{k=1}^{d}\Big(\frac{\alpha_k+\frac{1}{2}-x_k}{\sqrt{x_k}}-\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\Big)\varphi_k(x)\right)d\mu_\alpha(y)d\mu_\alpha(x).
\end{aligned}
$$

Therefore, the estimation presented for term I_2 above indicates that

$$
\left|\int_{\Omega_1}v_\varepsilon(x)\mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)\right|=|I_1+I_2+II|\le J_1+J_2+\varepsilon,
$$

where

$$
\begin{aligned}
J_1:=\Bigg|&-\sum_{j=0}^{\infty}\int_{\Omega_1}u(y)\widetilde{\mathrm{div}}(f_j(\eta_{\varepsilon_j}*\varphi))(y)d\mu_\alpha(y)\\
&-\sum_{j=0}^{\infty}\int_{\Omega_1}u(y)f_j(y)\left(\sum_{k=1}^{d}\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}(\varphi_k*\eta_{\varepsilon_j}(y))\right)d\mu_\alpha(y)\Bigg|
\end{aligned}
$$

and

$$
\begin{aligned}
J_2:=\Bigg|&-\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\\
&\times\left(\sum_{k=1}^{d}\Big(\frac{\alpha_k+\frac{1}{2}-x_k}{\sqrt{x_k}}-\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\Big)\varphi_k(x)\right)d\mu_\alpha(y)d\mu_\alpha(x)\Bigg|.
\end{aligned}
$$

Furthermore,

$$
J_1=\Bigg|-\sum_{j=0}^{\infty}\int_{\Omega_1}u(y)\widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}(f_j(\eta_{\varepsilon_j}*\varphi))(y)d\mu_\alpha(y)
$$

$$
\begin{aligned}
&-\sum_{j=0}^{\infty}\int_{\Omega_1} u(y)f_j(y)\left(\sum_{k=1}^{d}\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\varphi_k*\eta_{\varepsilon_j}(y)\right)d\mu_\alpha(y)\Bigg|\\
&\le \Bigg|-\int_{\Omega_1} u(y)\widetilde{\operatorname{div}}_{\mathcal{L}^\alpha}(f_0(\eta_{\varepsilon_0}*\varphi))(y)d\mu_\alpha(y)\\
&\quad -\int_{\Omega_1} u(y)f_0(y)\left(\sum_{k=1}^{d}\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\varphi_k*\eta_{\varepsilon_0}(y)\right)d\mu_\alpha(y)\Bigg|\\
&\quad +\Bigg|-\sum_{j=1}^{\infty}\int_{\Omega_1} u(y)\widetilde{\operatorname{div}}_{\mathcal{L}^\alpha}(f_j(\eta_{\varepsilon_j}*\varphi))(y)d\mu_\alpha(y)\\
&\quad -\sum_{j=1}^{\infty}\int_{\Omega_1} u(y)f_j(y)\left(\sum_{k=1}^{d}\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\varphi_k*\eta_{\varepsilon_j}(y)\right)d\mu_\alpha(y)\Bigg|\\
&\le |\nabla_{\mathcal{L}^\alpha}u|(\Omega_1)+\sum_{j=1}^{\infty}|\nabla_{\mathcal{L}^\alpha}u|(U_j)\\
&\le |\nabla_{\mathcal{L}^\alpha}u|(\Omega_1)+|\nabla_{\mathcal{L}^\alpha}u|(\Omega_1\backslash\Omega_{1,0})\\
&\le |\nabla_{\mathcal{L}^\alpha}u|(\Omega_1)+3\varepsilon,
\end{aligned}
$$

where we applied the fact (2.1) in the final inequality. Note that $\psi(x_k)=\frac{\alpha_k+\frac{1}{2}-x_k}{\sqrt{x_k}}$, $\|\varphi\|_{L^\infty}\le 1$ and supp $\eta\subseteq B(0,1)\cap\mathbb{R}^d_+$. Assuming $|x_k-y_k|<\varepsilon_j<|y_k|/2$, the mean value theorem of multivariate functions guarantees the existence of $\theta\in(0,1)$ such that

$$
\begin{aligned}
|\psi(x_k)-\psi(y_k)| &= \left|\frac{\alpha_k+\frac{1}{2}}{2}(y_k+\theta(x_k-y_k))^{-\frac{3}{2}}+\frac{1}{2}(y_k+\theta(x_k-y_k))^{-\frac{1}{2}}\right||x_k-y_k|\\
&\le \Big(\frac{|\alpha_k+\frac{1}{2}|}{2}|y_k+\theta(x_k-y_k)|^{-\frac{3}{2}}+\frac{1}{2}|y_k+\theta(x_k-y_k)|^{-\frac{1}{2}}\Big)|x_k-y_k|.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
J_2 &= \Bigg|\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\bigg(\sum_{k=1}^{d}\Big(\frac{\alpha_k+\frac{1}{2}-x_k}{\sqrt{x_k}}-\frac{\alpha_k+\frac{1}{2}-y_k}{\sqrt{y_k}}\Big)\varphi_k(x)\bigg)\\
&\quad\times d\mu_\alpha(y)d\mu_\alpha(x)\Bigg|\\
&\le \frac{\varepsilon_j}{2}\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\left|\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\right|\sum_{k=1}^{d}\left|\alpha_k+\frac{1}{2}\right||y_k+\theta(x-y_k)|^{-\frac{3}{2}}\\
&\quad\times|\varphi_k(x)|d\mu_\alpha(y)d\mu_\alpha(x)\\
&\quad+\frac{\varepsilon_j}{2}\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\left|\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\right|\sum_{k=1}^{d}|y_k+\theta(x_k-y_k)|^{-\frac{1}{2}}\\
&\quad\times|\varphi_k(x)|d\mu_\alpha(y)d\mu_\alpha(x)\\
&\le C\varepsilon_j\sum_{j=0}^{\infty}\int_{\Omega_1}\int_{\Omega_1}\left|\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\right|\sum_{k=1}^{d}\left|\alpha_k+\frac{1}{2}\right||y_k|^{-\frac{3}{2}}d\mu_\alpha(y)d\mu_\alpha(x)
\end{aligned}
$$

$$+ C\varepsilon_j \sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \Big|\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)u(y)f_j(y)\Big| \sum_{k=1}^{d} |y_k|^{-\frac{1}{2}} d\mu_\alpha(y) d\mu_\alpha(x)$$

$$\le C\varepsilon_j \sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \Big|\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)\Big| d\mu_\alpha(x) \sum_{k=1}^{d} \Big|\alpha_k + \frac{1}{2}\Big| |u(y)||f_j(y)||y_k|^{-\frac{3}{2}} d\mu_\alpha(y)$$

$$+ C\varepsilon_j \sum_{j=0}^{\infty} \int_{\Omega_1} \int_{\Omega_1} \Big|\frac{1}{\varepsilon_j^d}\eta\Big(\frac{x-y}{\varepsilon_j}\Big)\Big| d\mu_\alpha(x) \sum_{k=1}^{d} |u(y)||f_j(y)||y_k|^{-\frac{1}{2}} d\mu_\alpha(y)$$

$$\le C\varepsilon_j \int_{\mathbb{R}_+^d} \Big|\frac{1}{\epsilon_j^d}\eta\Big(\frac{x-y}{\epsilon_j}\Big)\Big| d\mu_\alpha(x) \sum_{k=1}^{d} \int_{\Omega_1} \Big|\alpha_k + \frac{1}{2}\Big| |u(y)| \Big| \sum_{j=0}^{\infty} f_j(y) \Big| |y_k|^{-\frac{3}{2}} d\mu_\alpha(y)$$

$$+ C\varepsilon_j \int_{\mathbb{R}_+^d} \Big|\frac{1}{\epsilon_j^d}\eta\Big(\frac{x-y}{\epsilon_j}\Big)\Big| d\mu_\alpha(x) \sum_{k=1}^{d} \int_{\Omega_1} |u(y)| \Big| \sum_{j=0}^{\infty} f_j(y) \Big| |y_k|^{-\frac{1}{2}} d\mu_\alpha(y)$$

$$= C\varepsilon_j \int_{\Omega_1} |u(y)| \sum_{k=1}^{d} \Big|\alpha_k + \frac{1}{2}\Big| |y_k|^{-\frac{3}{2}} d\mu_\alpha(y) + C\varepsilon_j \int_{\Omega_1} |u(y)| \sum_{k=1}^{d} |y_k|^{-\frac{1}{2}} d\mu_\alpha(y)$$

$$\lesssim \varepsilon,$$

where we have used the facts that

$$\begin{cases} \displaystyle\int_{\Omega_1} |u(y)| \sum_{k=1}^{d} |y_k|^{-\frac{1}{2}} d\mu_\alpha(y) < \infty, \\ \displaystyle\int_{\Omega_1} |u(y)| \sum_{k=1}^{d} \Big|\alpha_k + \frac{1}{2}\Big| |y_k|^{-\frac{3}{2}} d\mu_\alpha(y) < \infty, \end{cases} \tag{2.5}$$

and in the third inequality we have used the fact that

$$|y_k + \theta(x_k - y_k)| \ge |y_k| - \theta|x_k - y_k| = \Big(1 - \frac{\theta}{2}\Big)|y_k|.$$

Through taking the supremum over φ and considering the arbitrariness of $\varepsilon > 0$, we prove the theorem.

Remark 2.5. By computation, we conclude that the function $u \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega)$ satisfies (2.5) in Theorem 2.4 when $d \ge 3$, at this time, Theorem 2.4 is valid for any open set $\Omega \subseteq \mathbb{R}^d$.

Additionally, the max-min property of the $\mathcal{L}^\alpha$-variation can be observed from Lemma 2.1 and Theorem 2.4.

Theorem 2.6. *Let Ω_1 be an open set defined in (1.3). Suppose $u, v \in L^1(\Omega_1, d\mu_\alpha)$, then*

$$|\nabla_{\mathcal{L}^\alpha} \max\{u, v\}|(\Omega_1) + |\nabla_{\mathcal{L}^\alpha} \min\{u, v\}|(\Omega_1) \le |\nabla_{\mathcal{L}^\alpha} u|(\Omega_1) + |\nabla_{\mathcal{L}^\alpha} v|(\Omega_1).$$

Proof. One may assume, without any loss of generality,

$$|\nabla_{\mathcal{L}^\alpha} u|(\Omega_1) + |\nabla_{\mathcal{L}^\alpha} v|(\Omega_1) < \infty.$$

By Theorem 2.4, we take two functions

$$u_h, v_h \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1) \cap C_c^\infty(\Omega_1), \quad h = 1, 2, ...,$$

such that

$$\begin{cases} u_h \to u, v_h \to v \ \text{ in } \ L^1(\Omega_1, d\mu_\alpha), \\ \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} u_h(x)| d\mu_\alpha(x) \to |\nabla_{\mathcal{L}^\alpha} u|(\Omega_1), \\ \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} v_h(x)| d\mu_\alpha(x) \to |\nabla_{\mathcal{L}^\alpha} v|(\Omega_1). \end{cases}$$

Since

$$\max\{u_h, v_h\} \to \max\{u, v\} \ \ \& \ \ \min\{u_h, v_h\} \to \min\{u, v\} \ \text{ in } \ L^1(\Omega_1, d\mu_\alpha).$$

Via Lemma 2.1, it follows that

$$\begin{aligned}
&|\nabla_{\mathcal{L}^\alpha} \max\{u, v\}|(\Omega_1) + |\nabla_{\mathcal{L}^\alpha} \min\{u, v\}|(\Omega_1) \\
&\le \liminf_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} \max\{u_h, v_h\}| d\mu_\alpha(x) + \liminf_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} \min\{u_h, v_h\}| d\mu_\alpha(x) \\
&\le \liminf_{h\to\infty} \Big(\int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} \max\{u_h, v_h\}| d\mu_\alpha(x) + \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} \min\{u_h, v_h\}| d\mu_\alpha(x) \Big) \\
&\le \liminf_{h\to\infty} \Big(\int_{\{x\in\Omega_1 : u_h \le v_h\}} |\nabla_{\mathcal{L}^\alpha} v_h| d\mu_\alpha(x) + \int_{\{x\in\Omega_1 : u_h > v_h\}} |\nabla_{\mathcal{L}^\alpha} u_h| d\mu_\alpha(x) \\
&\quad + \int_{\{x\in\Omega_1 : u_h \le v_h\}} |\nabla_{\mathcal{L}^\alpha} u_h| d\mu_\alpha(x) + \int_{\{x\in\Omega_1 : u_h > v_h\}} |\nabla_{\mathcal{L}^\alpha} v_h| d\mu_\alpha(x) \Big) \\
&= \liminf_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} u_h(x)| d\mu_\alpha(x) + \liminf_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} v_h(x)| d\mu_\alpha(x) \\
&\le \lim_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} u_h(x)| d\mu_\alpha(x) + \lim_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} v_h(x)| d\mu_\alpha(x) \\
&= |\nabla_{\mathcal{L}^\alpha} u|(\Omega_1) + |\nabla_{\mathcal{L}^\alpha} v|(\Omega_1).
\end{aligned}$$

2.2. *Basic properties of Laguerre perimeter*

This subsection presents a new type of perimeter: the Laguerre perimeter ($\mathcal{L}^\alpha$-perimeter in short). Moreover, we establish the related results for it.

We define the $\mathcal{L}^\alpha$-perimeter of $E \subset \Omega$ as follows:

$$P_{\mathcal{L}^\alpha}(E, \Omega) = |\nabla_{\mathcal{L}^\alpha} 1_E|(\Omega) = \sup_{\varphi\in\mathcal{F}(\Omega)} \Big\{ \int_E \operatorname{div}_{\mathcal{L}^\alpha} \varphi(x) d\mu_\alpha(x) \Big\}, \tag{2.6}$$

where $\mathcal{F}(\Omega)$ is defined in Section 2.1. Specifically, we will also use the notation

$$P_{\mathcal{L}^\alpha}(E, \mathbb{R}_+^d) = P_{\mathcal{L}^\alpha}(E).$$

We immediately deduce Lemma 2.1 by replacing f with 1_E.

Corollary 2.7. *(Lower semicontinuity of $P_{\mathcal{L}^\alpha}$). Assume $1_{E_k} \to 1_E$ in $L^1_{loc}(\Omega, d\mu_\alpha)$, where E and E_k, $k \in \mathbb{N}$, are subsets of Ω, then*

$$P_{\mathcal{L}^\alpha}(E, \Omega) \leq \liminf_{k\to\infty} P_{\mathcal{L}^\alpha}(E_k, \Omega).$$

Additionally, utilizing Theorem 2.6 and selecting $u = 1_E$ and $v = 1_F$ for every compact subsets E, F in Ω_1, we can promptly acquire the subsequent corollary. According to Xiao and Zhang's result in [14, Section 1.1 (iii)], the equality condition of (2.7) is also provided by us.

Corollary 2.8. *For all compact subsets E, F within Ω_1, we get*

$$P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) \leq P_{\mathcal{L}^\alpha}(E, \Omega_1) + P_{\mathcal{L}^\alpha}(F, \Omega_1), \tag{2.7}$$

where Ω_1 is an open set defined in (1.3). Especially, if $P_{\mathcal{L}^\alpha}(E \setminus (E \cap F), \Omega_1) \cdot P_{\mathcal{L}^\alpha}(F \setminus (F \cap E), \Omega_1) = 0$, the equality of (2.7) holds true.

Proof. Given that (2.7) is true, we only need to demonstrate that its opposite inequality is also valid, provided that the above condition is satisfied. It is evident that the condition $P_{\mathcal{L}^\alpha}(E \setminus (E \cap F), \Omega_1) \cdot P_{\mathcal{L}^\alpha}(F \setminus (E \cap F), \Omega_1) = 0$ leads to $P_{\mathcal{L}^\alpha}(E \setminus (E \cap F), \Omega_1) = 0$ or $P_{\mathcal{L}^\alpha}(F \setminus (E \cap F), \Omega_1) = 0$. Suppose $P_{\mathcal{L}^\alpha}(E \setminus (E \cap F), \Omega_1) = 0$. By (2.7), we have

$$\begin{aligned} P_{\mathcal{L}^\alpha}(E, \Omega_1) &= P_{\mathcal{L}^\alpha}((E \setminus (E \cap F)) \cup (E \cap F), \Omega_1) \\ &\leq P_{\mathcal{L}^\alpha}(E \setminus (E \cap F), \Omega_1) + P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) \\ &= P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1). \end{aligned} \tag{2.8}$$

Via (2.6) and $E \cup F = F \cup (E \setminus (E \cap F))$, we have

$$\begin{aligned} P_{\mathcal{L}^\alpha}(F, \Omega_1) &= \sup_{\varphi\in\mathcal{F}(\Omega_1)} \Big\{ \int_F \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x) d\mu_\alpha(x) \Big\} \\ &= \sup_{\varphi\in\mathcal{F}(\Omega_1)} \Big\{ \int_{E\cup F} \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x) d\mu_\alpha(x) - \int_{E\setminus(E\cap F)} \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x) d\mu_\alpha(x) \Big\} \\ &\leq \sup_{\varphi\in\mathcal{F}(\Omega_1)} \Big\{ \int_{E\cup F} \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x) d\mu_\alpha(x) \Big\} \\ &\quad + \sup_{\varphi\in\mathcal{F}(\Omega_1)} \Big\{ \int_{E\setminus(E\cap F)} \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x) d\mu_\alpha(x) \Big\} \\ &= P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) + P_{\mathcal{L}^\alpha}(E \setminus (E \cap F), \Omega_1) \\ &= P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1). \end{aligned} \tag{2.9}$$

Combining (2.8) with (2.9) deduces that

$$P_{\mathcal{L}^\alpha}(E, \Omega_1) + P_{\mathcal{L}^\alpha}(F, \Omega_1) \leq P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) + P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1),$$

the desired result can be obtained from it. Another similar case can be proven as well, but the details are omitted.

We will now demonstrate that sets with finite $\mathcal{L}^\alpha$-perimeter satisfy the Gauss-Green formula.

Theorem 2.9. *(Gauss-Green formula). Let $E \subseteq \Omega$ be subset with finite $\mathcal{L}^\alpha$-perimeter. Then we have*

$$\begin{aligned}&\int_E \widetilde{\operatorname{div}}_{\mathcal{L}^\alpha}\varphi(x)\, d\mu_\alpha(x)\\ &= -\int_{\partial E^c} (\sqrt{x_1}\varphi_1(x), \cdots, \sqrt{x_d}\varphi_d(x)) \cdot \vec{n}\omega(x) d\mathcal{H}^{d-1}(x)\\ &\quad - \int_E \sum_{i=1}^d \frac{\alpha_i + \frac{1}{2} - x_i}{\sqrt{x_i}}\varphi_i(x)\omega(x)dx,\end{aligned}$$

where the outward normal to E is represented by the unit vector $\vec{n}(x)$ and $\widetilde{\operatorname{div}}_{\mathcal{L}^\alpha}(\cdot)$ is defined in (2.4).

Proof. By calculating, we have

$$\begin{aligned}&\int_E \widetilde{\operatorname{div}}_{\mathcal{L}^\alpha}\varphi(x)\, d\mu_\alpha(x)\\ &= \int_E \left(\sum_{i=1}^d \sqrt{x_i}\frac{\partial}{\partial x_i}\varphi_i(x)\right)\omega(x)dx\\ &= \int_E \operatorname{div}(\sqrt{x_1}\varphi_1(x)\omega(x), \cdots, \sqrt{x_d}\varphi_d(x)\omega(x))dx - \int_E \sum_{i=1}^d \sqrt{x_i}\varphi_i(x)\frac{\partial}{\partial x_i}\omega(x)dx\\ &\quad - \frac{1}{2}\int_E \sum_{i=1}^d \frac{1}{\sqrt{x_i}}\varphi_i(x)\omega(x)dx\\ &= -\int_{\partial E^c} (\sqrt{x_1}\varphi_1(x), \cdots, \sqrt{x_d}\varphi_d(x)) \cdot \vec{n}\omega(x) d\mathcal{H}^{d-1}(x)\\ &\quad - \int_E \sum_{i=1}^d \sqrt{x_i}\varphi_i(x)\frac{\partial}{\partial x_i}\omega(x)dx - \frac{1}{2}\int_E \sum_{i=1}^d \frac{1}{\sqrt{x_i}}\varphi_i(x)\omega(x)dx\\ &= -\int_{\partial E^c} (\sqrt{x_1}\varphi_1(x), \cdots, \sqrt{x_d}\varphi_d(x)) \cdot \vec{n}\omega(x) d\mathcal{H}^{d-1}(x)\\ &\quad - \int_E \sum_{i=1}^d \frac{\alpha_i + \frac{1}{2} - x_i}{\sqrt{x_i}}\varphi_i(x)\omega(x)dx,\end{aligned}$$

where we have used the classical Gauss-Green formula and the following facts regarding the derivatives of $\omega(x)$:

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(\prod_{j=1}^d \frac{x_j^{\alpha_j}e^{-x_j}}{\Gamma(\alpha_j+1)}\right) &= \prod_{j=1, j\neq i}^d \frac{x_j^{\alpha_j}e^{-x_j}}{\Gamma(\alpha_j+1)}\frac{1}{\Gamma(\alpha_i+1)}(-e^{-x_i}x_i^{\alpha_i} + \alpha_i e^{-x_i}x_i^{\alpha_i-1})\\ &= \Big(-1 + \frac{\alpha_i}{x_i}\Big)\omega(x)\end{aligned}$$

for $1 \le i \le d$. This completes the proof.

Lemma 2.10. *If a set E is in Ω and has finite $\mathcal{L}^\alpha$-perimeter, then*

$$P_{\mathcal{L}^\alpha}(E, \Omega) = P_{\mathcal{L}^\alpha}(\Omega\backslash E, \Omega).$$

Proof. For any $\varphi \in \mathcal{F}(\mathbb{R}_+^d)$, since $P_{\mathcal{L}^\alpha}(E,\Omega) < \infty$, then

$$\sup_{\varphi\in\mathcal{F}(\mathbb{R}_+^d)} \int_E \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x) < \infty.$$

Via the extended Gauss-Green formula (Theorem 2.9) and taking into consideration the fact that φ has a compact support, we obtain

$$\begin{aligned}
&\int_E \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)\\
&= -\int_E \widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}(\varphi_1(x),\ldots,\varphi_d(x))d\mu_\alpha(x) - \int_E \sum_{k=1}^d \frac{\alpha_i+\frac{1}{2}-x_i}{\sqrt{x_i}}\varphi_i(x)d\mu_\alpha(x)\\
&= \int_{\partial E^c} (\sqrt{x_1}\varphi_1(x),\cdots,\sqrt{x_d}\varphi_d(x))\cdot\vec{n}\omega(x)d\mathcal{H}^{d-1}(x)\\
&\quad + \int_E \sum_{k=1}^d \frac{\alpha_i+\frac{1}{2}-x_i}{\sqrt{x_i}}\varphi_i(x)d\mu_\alpha(x) - \int_E \sum_{k=1}^d \frac{\alpha_i+\frac{1}{2}-x_i}{\sqrt{x_i}}\varphi_i(x)d\mu_\alpha(x)\\
&= \int_{\partial E} (\sqrt{x_1}\varphi_1(x),\cdots,\sqrt{x_d}\varphi_d(x))\cdot\vec{n}\omega(x)d\mathcal{H}^{d-1}(x)\\
&= -\int_{E^c} \widetilde{\mathrm{div}}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x) - \int_{E^c} \sum_{k=1}^d \frac{\alpha_i+\frac{1}{2}-x_i}{\sqrt{x_i}}\varphi_i(x)d\mu_\alpha(x)\\
&= \int_{E^c} \mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x),
\end{aligned}$$

where the unit exterior normal vector to E at x is denoted by $\vec{n}(x)$. The arbitrary nature of φ results in the attainment of

$$P_{\mathcal{L}^\alpha}(E,\Omega) = P_{\mathcal{L}^\alpha}(\Omega\backslash E,\Omega)$$

through the use of supremum.

2.3. Coarea formula of $\mathcal{L}^\alpha$-BV functions and the Sobolev inequality

Below we prove the coarea formula and the Sobolev inequality for $\mathcal{L}^\alpha$-perimeter.

Theorem 2.11. *Let Ω_1 be an open set defined in (1.3). If $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)$, then*

$$|\nabla_{\mathcal{L}^\alpha} f|(\Omega_1) \approx \int_{-\infty}^{+\infty} P_{\mathcal{L}^\alpha}(E_t,\Omega_1)dt, \tag{2.10}$$

where $E_t = \{x \in \Omega_1 : f(x) > t\}$ for $t \in \mathbb{R}$.

Proof. At first, assume

$$\varphi = (\varphi_1,\varphi_2,\ldots,\varphi_d) \in C_c^1(\Omega_1,\mathbb{R}^d).$$

It is straightforward to prove that for $i = 1,2,\ldots,d$,

$$\int_{\Omega_1} f(x)\sqrt{x_i}\frac{\partial}{\partial x_i}\varphi_i(x)d\mu_\alpha(x) = \int_{-\infty}^{+\infty}\Big(\int_{E_t}\sqrt{x_i}\frac{\partial}{\partial x_i}\varphi_i(x)d\mu_\alpha(x)\Big)dt,$$

and

$$\int_{\Omega_1} f(x)\frac{\alpha_i+\frac{1}{2}-x_i}{\sqrt{x_i}}\varphi_i(x)d\mu_\alpha(x)=\int_{-\infty}^{+\infty}\Big(\int_{E_t}\frac{\alpha_i+\frac{1}{2}-x_i}{\sqrt{x_i}}\varphi_i(x)d\mu_\alpha(x)\Big)dt,$$

where the proof of [9, Section 5.5, Theorem 1] displays the latter. Therefore,

$$\int_{\Omega_1} f(x)\mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)=\int_{-\infty}^{+\infty}\Big(\int_{E_t}\mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)\Big)dt.$$

Therefore, we conclude that for all $\varphi\in\mathcal{F}(\Omega_1)$,

$$\int_{\Omega_1} f(x)\mathrm{div}_{\mathcal{L}^\alpha}\varphi(x)d\mu_\alpha(x)\le\int_{-\infty}^{+\infty}P_{\mathcal{L}^\alpha}(E_t,\Omega_1)dt.$$

Furthermore,

$$|\nabla_{\mathcal{L}^\alpha}f|(\Omega_1)\le\int_{-\infty}^{+\infty}P_{\mathcal{L}^\alpha}(E_t,\Omega_1)dt.$$

Secondly, it can be assumed without any loss of generality that we simply need to confirm that

$$|\nabla_{\mathcal{L}^\alpha}f|(\Omega_1)\ge\int_{-\infty}^{+\infty}P_{\mathcal{L}^\alpha}(E_t,\Omega_1)dt$$

holds for $f\in\mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)\bigcap C^\infty(\Omega_1)$. The idea of [15, Proposition 4.2] can be referenced in this proof. Denote by

$$m(t)=\int_{\{x\in\Omega_1:\ f(x)\le t\}}\Big|\sum_{i=1}^{d}\sqrt{x_i}\frac{\partial}{\partial x_i}f(x)\Big|d\mu_\alpha(x).$$

Obviously,

$$\int_{-\infty}^{+\infty}m'(t)dt=\int_{\Omega_1}\Big|\sum_{i=1}^{d}\sqrt{x_i}\frac{\partial}{\partial x_i}f(x)\Big|d\mu_\alpha(x).$$

Define the following function g_h as

$$g_h(s):=\begin{cases}0, & \text{if } s\le t,\\ h(s-t), & \text{if } t\le s\le t+1/h,\\ 1, & \text{if } s\ge t+1/h,\end{cases}$$

where $t\in\mathbb{R}$. Set the sequence $v_h(x):=g_h(f(x))$. At this time, $v_h\to 1_{E_t}$ in $L^1(\Omega_1,d\mu_\alpha)$. In fact,

$$\begin{aligned}\int_{\Omega_1}|v_h(x)-1_{E_t}|d\mu_\alpha(x)&=\int_{\{x\in\Omega_1:t<f(x)\le t+1/h\}}|g_h(f(x))-1|d\mu_\alpha(x)\\&\le\int_{\{x\in\Omega_1:t<f(x)\le t+1/h\}}d\mu_\alpha(x)\to 0.\end{aligned}$$

As $h\to\infty$, $\{x\in\Omega_1: t<f(x)\le t+1/h\}\to\emptyset$, we then obtain

$$\int_{\Omega_1}|\nabla_{\mathcal{L}^\alpha}v_h(x)|d\mu_\alpha(x)$$

$$
\begin{aligned}
&= \int_{\{x\in\Omega_1 : t<f(x)\le t+1/h\}} |\nabla_{\mathcal{L}^\alpha}(h(f(x)-t))| d\mu_\alpha(x) \\
&\quad + \int_{\{x\in\Omega_1 : f(x)\ge t+1/h\}} |\nabla_{\mathcal{L}^\alpha} 1| d\mu_\alpha(x) \\
&= h \int_{\{x\in\Omega_1 : t<f(x)\le t+1/h\}} \Big| \sum_{i=1}^{d} \sqrt{x_i}\frac{\partial}{\partial x_i} f(x) \Big| d\mu_\alpha(x).
\end{aligned}
$$

By utilizing Theorem 2.4 and taking the limit as $h \to \infty$ we can derive

$$
\begin{aligned}
|\nabla_{\mathcal{L}^\alpha} 1_{E_t}|(\Omega_1) &\le \limsup_{h\to\infty} \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} v_h(x)| d\mu_\alpha(x) \\
&= h \limsup_{h\to\infty} \int_{\{x\in\Omega_1 : t<f(x)\le t+1/h\}} \Big| \sum_{i=1}^{d} \sqrt{x_i}\frac{\partial}{\partial x_i} f(x) \Big| d\mu_\alpha(x) \\
&= m'(t).
\end{aligned} \tag{2.11}
$$

Integrating (2.11) reaches

$$
\begin{aligned}
\int_{-\infty}^{+\infty} P_{\mathcal{L}^\alpha}(E_t,\Omega_1) dt &\le \int_{-\infty}^{+\infty} m'(t) dt \\
&= \int_{\Omega_1} \Big| \sum_{i=1}^{d} \sqrt{x_i}\frac{\partial}{\partial x_i} f(x) \Big| d\mu_\alpha(x) \\
&\lesssim \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} f(x)| d\mu_\alpha(x).
\end{aligned}
$$

Ultimately, through approximation and using the lower semicontinuity of the $\mathcal{L}^\alpha$-perimeter, we can deduce that (2.10) is valid for every $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)$.

We can eventually establish the Sobolev inequality and the isoperimetric inequality for $\mathcal{L}^\alpha$-BV functions. Since the domain Ω_1 is a reasonable substitute of Ω, we can obtain the isoperimetric inequality and the Sobolev inequality for $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)$, where Ω_1 is given in (1.3).

Theorem 2.12.

(i) *(Sobolev inequality). Let Ω_1 be an open set defined in (1.3). Then for all $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)$, we have*

$$
\|f\|_{L^{\frac{d}{d-1}}(\Omega_1, d\mu_\alpha)} \lesssim |\nabla_{\mathcal{L}^\alpha} f|(\Omega_1). \tag{2.12}
$$

(ii) *(Isoperimetric inequality). Suppose that E is a bounded set having finite $\mathcal{L}^\alpha$-perimeter in Ω_1. Then*

$$
\mu_\alpha(E)^{\frac{d-1}{d}} \lesssim P_{\mathcal{L}^\alpha}(E,\Omega_1). \tag{2.13}
$$

(iii) *The two statements mentioned above are equivalent.*

Proof. (i) Let

$$f_k \in C_c^\infty(\Omega_1) \cap \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1),\ k = 1, 2, \ldots,$$

such that

$$\begin{cases} f_k \to f \text{ in } L^1(\Omega_1, d\mu_\alpha), \\ \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} f_k(x)| d\mu_\alpha(x) \to \| \nabla_{\mathcal{L}^\alpha} f \| (\Omega_1). \end{cases}$$

Since $\Omega_1 = \Omega \setminus \{x \in \mathbb{R}_+^d : \exists i \in 1, \cdots, d \text{ such that } \sqrt{x_i} < 1\}$, then for any $i = 1, \cdots, d$, we obtain $\sqrt{x_i} \geq 1$. It is easy to see that

$$|\nabla f(x)| \leq |\nabla_{\mathcal{L}^\alpha} f(x)| = \left(\sum_{i=1}^d \left(\sqrt{x_i} \frac{\partial}{\partial x_i} f(x) \right)^2 \right)^{\frac{1}{2}}. \tag{2.14}$$

After applying Fatou's lemma and the weighted Gagliardo-Nirenberg-Sobolev inequality, we get

$$\begin{aligned} \|f\|_{L^{\frac{d}{d-1}}(\Omega_1, d\mu_\alpha)} &\leq \liminf_{k\to\infty} \|f_k\|_{L^{\frac{d}{d-1}}(\Omega_1, d\mu_\alpha)} \\ &\lesssim \lim_{k\to\infty} \|\nabla f\|_{L^1(\Omega_1, d\mu_\alpha)} \\ &\lesssim \lim_{k\to\infty} \|\nabla_{\mathcal{L}^\alpha} f\|_{L^1(\Omega_1, d\mu_\alpha)} = |\nabla_{\mathcal{L}^\alpha} f|(\Omega_1), \end{aligned}$$

where the relation between the gradient ∇ and the Laguerre gradient $\nabla_{\mathcal{L}^\alpha}$ has been applied in (2.14).

(ii) By setting $f = 1_E$ in (2.12), it can be demonstrated that (2.13) is true.

(iii) Apparently, the implication from (i) to (ii) has been proved. The statement below demonstrates that (ii) implies (i). Let $0 \leq f \in C_c^\infty(\Omega_1)$. Applying the coarea formula from Theorem 2.11 and (ii), we obtain

$$\int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} f(x)| d\mu_\alpha(x) = \int_0^{+\infty} |\nabla_{\mathcal{L}^\alpha} 1_{E_t}|(\Omega_1)\, dt \gtrsim \int_0^{+\infty} |E_t|^{\frac{d-1}{d}} dt,$$

where $E_t = \{x \in \Omega_1 : f(x) > t\}$. Let

$$f_t = \min\{t, f\} \ \&\ \chi(t) = \left(\int_{\Omega_1} f_t^{\frac{d}{d-1}}(x) d\mu_\alpha(x) \right)^{\frac{d-1}{d}},\ \forall\, t \in \mathbb{R}.$$

One can easily observe that

$$\lim_{t\to\infty} \chi(t) = \left(\int_{\Omega_1} |f(x)|^{\frac{d}{d-1}} d\mu_\alpha(x) \right)^{\frac{d-1}{d}}.$$

Moreover, we can verify that $\chi(t)$ increases monotonically on $(0, \infty)$ and for any positive h,

$$0 \leq \chi(t+h) - \chi(t) \leq \left(\int_{\Omega_1} |f_{t+h}(x) - f_t(x)|^{\frac{d}{d-1}} d\mu_\alpha(x) \right)^{\frac{d-1}{d}} \leq h|E_t|^{\frac{d-1}{d}}.$$

Then $\chi(t)$ can be considered a Lipschitz function locally and $\chi'(t) \leq |E_t|^{\frac{d-1}{d}}$ for a.e. $t \in (0, \infty)$. Thus,

$$\begin{aligned} \left(\int_{\Omega_1} |f(x)|^{\frac{d}{d-1}} d\mu_\alpha(x) \right)^{\frac{d-1}{d}} = \int_0^\infty \chi'(t) dt &\leq \int_0^\infty |E_t|^{\frac{d-1}{d}} dt \\ &\lesssim \int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} f(x)| d\mu_\alpha(x). \end{aligned}$$

Finally, Theorem 2.4 establishes the validity of (2.12) for all $f \in \mathcal{BV}_{\mathcal{L}^\alpha}(\Omega_1)$.

As a direct result of the proof of (i) in Theorem 2.12, we can get the following corollary.

Corollary 2.13. *Let $1 < p < d$ and let Ω_1 be an open set defined in (1.3). For any $f \in W^{1,1}_{\mathcal{L}^\alpha}(\Omega_1)$ one has*

$$\|f\|_{L^{\frac{dp}{d-p}}(\Omega_1, d\mu_\alpha)} \lesssim \|\nabla_{\mathcal{L}^\alpha} f\|_{L^p(\Omega_1, d\mu_\alpha)}. \tag{2.15}$$

Proof. For some $\gamma > 1$ to be fixed later, via the Lemma 2.1 (i) and Hölder inequality we obtain

$$\begin{aligned}
&\left(\int_{\Omega_1} |f(x)|^{\frac{\gamma d}{d-1}} d\mu_\alpha(x)\right)^{\frac{d-1}{d}} \\
&\quad \lesssim \int_{\Omega_1} |f(x)|^{\gamma-1} |\nabla_{\mathcal{L}^\alpha} f(x)| d\mu_\alpha(x) \\
&\quad \lesssim \left(\int_{\Omega_1} |f(x)|^{\frac{p(\gamma-1)}{p-1}} d\mu_\alpha(x)\right)^{1-\frac{1}{p}} \left(\int_{\Omega_1} |\nabla_{\mathcal{L}^\alpha} f(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}.
\end{aligned}$$

Choosing

$$\gamma = \frac{p(d-1)}{d-p}$$

and noting

$$\gamma - 1 = \frac{d(p-1)}{d-p},$$

then we conclude that (2.15) holds true.

3. Laguerre mean curvature

The main concern of this section is to determine if the mean curvature of every set with finite $\mathcal{L}^\alpha$-perimeter in $\Omega_1 \subseteq \mathbb{R}^d_+$ belongs to $L^1(\Omega_1, d\mu_\alpha)$. To obtain comprehensive information on the classical case, kindly consult [13]. In order to prove Theorem 3.1, it is necessary to use the important result for the Laguerre perimeter in Corollary 2.8. Therefore, we assume that the dimension $d \geq 3$ via Remark 2.5.

For a given $u \in L^1(\Omega_1, d\mu_\alpha)$, the functional corresponding to the $\mathcal{L}^\alpha$-perimeter, known as Massari type, is given by

$$\mathscr{F}_{u,\mathcal{L}^\alpha}(E) := P_{\mathcal{L}^\alpha}(E, \Omega_1) + \int_E u(x) d\mu_\alpha(x),$$

where an arbitrary set of finite $\mathcal{L}^\alpha$-perimeter in $\mathbb{R}^d_+$ is denoted by E.

Theorem 3.1. *For every set $E \subset \mathbb{R}^d_+$ that has finite $\mathcal{L}^\alpha$-perimeter, a function u belonging to $L^1(\mathbb{R}^d_+, d\mu_\alpha)$ exists such that*

$$\mathscr{F}_{u,\mathcal{L}^\alpha}(E) \leq \mathscr{F}_{u,\mathcal{L}^\alpha}(F)$$

is satisfied for every set $F \subset \Omega_1$ with finite $\mathcal{L}^\alpha$-perimeter.

Proof. Initially, we must identify a function $u \in L^1(\Omega_1, d\mu_\alpha)$ for a specified set E such that

$$\mathscr{F}_{u,\mathcal{L}^\alpha}(E) \leq \mathscr{F}_{u,\mathcal{L}^\alpha}(F) \tag{3.1}$$

is true for every F with either $F \subset E$ or $E \subset F$, then Theorem 3.1 is demonstrated, indicating that (3.1) applies to every $F \subset \Omega_1$. By including the inequality (3.1) that pertains to the test sets $E \cap F$ and $E \cup F$, we have

$$\begin{cases} P_{\mathcal{L}^\alpha}(E, \Omega_1) + \int_E u(x)d\mu_\alpha(x) \leq P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + \int_{E\cap F} u(x)d\mu_\alpha(x), \\ P_{\mathcal{L}^\alpha}(E, \Omega_1) + \int_E u(x)d\mu_\alpha(x) \leq P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) + \int_{E\cup F} u(x)d\mu_\alpha(x). \end{cases}$$

After taking that

$$P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) \leq P_{\mathcal{L}^\alpha}(E, \Omega_1) + P_{\mathcal{L}^\alpha}(F, \Omega_1), \tag{3.2}$$

we can get

$$\begin{aligned} &2P_{\mathcal{L}^\alpha}(E, \Omega_1) + 2\int_E u(x)d\mu_\alpha(x) \\ &\leq P_{\mathcal{L}^\alpha}(E \cap F, \Omega_1) + P_{\mathcal{L}^\alpha}(E \cup F, \Omega_1) + \int_{E\cap F} u(x)d\mu_\alpha(x) + \int_{E\cup F} u(x)d\mu_\alpha(x) \\ &\leq P_{\mathcal{L}^\alpha}(E, \Omega_1) + P_{\mathcal{L}^\alpha}(F, \Omega_1) + \int_E u(x)d\mu_\alpha(x) + \int_F u(x)d\mu_\alpha(x), \end{aligned}$$

that is, (3.1) holds for arbitrary F. Moreover, if (3.1) is vaild for a set $F \subset E$, then it is also applicable to the set F such that $E \subset F$, i.e. $\Omega_1 \backslash F \subset \Omega_1 \backslash E$,

$$\begin{aligned} &P_{\mathcal{L}^\alpha}(E, \Omega_1) + \int_E u(x)d\mu_\alpha(x) \\ &= P_{\mathcal{L}^\alpha}(\Omega_1 \backslash E, \Omega_1) + \int_{\Omega_1\backslash E} u(x)d\mu_\alpha(x) - \int_{\Omega_1\backslash E} u(x)d\mu_\alpha(x) + \int_E u(x)d\mu_\alpha(x) \\ &\leq P_{\mathcal{L}^\alpha}(\Omega_1 \backslash F, \Omega_1) + \int_{\Omega_1\backslash F} u(x)d\mu_\alpha(x) - \int_{\Omega_1\backslash E} u(x)d\mu_\alpha(x) + \int_E u(x)d\mu_\alpha(x) \\ &= P_{\mathcal{L}^\alpha}(F, \Omega_1) + \int_{\Omega_1\backslash F} u(x)d\mu_\alpha(x) - \int_{\Omega_1\backslash E} u(x)d\mu_\alpha(x) + \int_E u(x)d\mu_\alpha(x) \\ &= \mathscr{F}_{u,\mathcal{L}^\alpha}(F) - \int_F u(x)d\mu_\alpha(x) + \int_{\Omega_1\backslash F} u(x)d\mu_\alpha(x) - \int_{\Omega_1\backslash E} u(x)d\mu_\alpha(x) \\ &\quad + \int_E u(x)d\mu_\alpha(x) \\ &= \mathscr{F}_{u,\mathcal{L}^\alpha}(F) + \int_{F\backslash E} u(x)d\mu_\alpha(x) - \int_{(\Omega_1\backslash E)/(\Omega_1\backslash F)} u(x)d\mu_\alpha(x) \\ &= \mathscr{F}_{u,\mathcal{L}^\alpha}(F), \end{aligned}$$

where we have utilized lemma 2.10 along with the property that u equals zero outside of the set E. Therefore, it is sufficient to prove that the integrability of u on E is established and that (3.1) is valid for every $F \subset E$.

Step I. Let $h(\cdot)$ be a measurable function on E such that $h > 0$ and $\int_E h(x)d\mu_\alpha(x) < \infty$, and let Λ be a measure that is both positive and totally finite:

$$\Lambda(F) = \int_F h(x)d\mu_\alpha(x),\ F \subset E.$$

Since $\Lambda(F) = 0$ if and only if $\mu_\alpha(F) = 0$ is clearly true. For $\lambda > 0$ and $F \subset E$, we will examine the following functional

$$\mathscr{F}_\lambda(F) := P_{\mathcal{L}^\alpha}(F, \Omega_1) + \lambda\Lambda(E \setminus F).$$

A commonly recognized fact is that any minimizing sequence is compact within $L^1_{loc}(\Omega_1, d\mu_\alpha)$, and this functional is lower semi-continuous in regards to the same convergence. Thus, we can deduce that, for any positive value of λ, there is a solution E_λ to the problem:

$$\mathscr{F}_\lambda(F) \to \min,\ F \subset E.$$

Select a strictly increasing sequence of positive numbers $\{\lambda_i\}$ that tend to ∞ and use $E_i \equiv E_{\lambda_i}$ to refer to the associated solutions, so that $\forall i \geq 1$,

$$\mathscr{F}_{\lambda_i}(E_i) \leq \mathscr{F}_{\lambda_i}(F),\ \forall\ F \subset E. \tag{3.3}$$

Given $i < j$. Let $F = E_i \cap E_j$. It follows from (3.3) that

$$\mathscr{F}_{\lambda_i}(E_i) \leq \mathscr{F}_{\lambda_i}(E_i \cap E_j),$$

that is,

$$P_{\mathcal{L}^\alpha}(E_i, \Omega_1) + \lambda_i\Lambda(E \setminus E_i) \leq P_{\mathcal{L}^\alpha}(E_i \cap E_j, \Omega_1) + \lambda_i\Lambda(E \setminus (E_i \cap E_j)),$$

this suggests

$$P_{\mathcal{L}^\alpha}(E_i, \Omega_1) + \lambda_i \int_{E\setminus E_i} h(x)d\mu_\alpha(x) \leq P_{\mathcal{L}^\alpha}(E_i \cap E_j, \Omega_1) + \lambda_i \int_{E\setminus(E_i\cap E_j)} h(x)d\mu_\alpha(x).$$

A direct computation gives

$$P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \leq \lambda_i \int_{E_i\setminus E_j} h(x)d\mu_\alpha(x) + P_{\mathcal{L}^\alpha}(E_i \cap E_j, \Omega_1).$$

Conversely, by choosing $F = E_i \cup E_j \subset E$ from (3.3), we can obtain $\mathscr{F}_{\lambda_j}(E_j) \leq \mathscr{F}_{\lambda_j}(E_i \cup E_j)$. Hence,

$$P_{\mathcal{L}^\alpha}(E_j, \Omega_1) + \lambda_j \int_{E\setminus E_j} h(x)d\mu_\alpha(x) \leq P_{\mathcal{L}^\alpha}(E_i \cup E_j, \Omega_1) + \lambda_j \int_{E\setminus(E_i\cup E_j)} h(x)d\mu_\alpha(x),$$

equivalently,

$$P_{\mathcal{L}^\alpha}(E_j, \Omega_1) + \lambda_j \int_{E_i\setminus E_j} h(x)d\mu_\alpha(x) \leq P_{\mathcal{L}^\alpha}(E_i \cup E_j, \Omega_1)$$

which implies that

$$P_{\mathcal{L}^\alpha}(E_i, \Omega_1) + P_{\mathcal{L}^\alpha}(E_j, \Omega_1) + \lambda_j \int_{E_i\setminus E_j} h(x)d\mu_\alpha(x)$$

$$\leq P_{\mathcal{L}^\alpha}(E_i \cup E_j, \Omega_1) + \lambda_i \int_{E_i \setminus E_j} h(x) d\mu_\alpha(x) + P_{\mathcal{L}^\alpha}(E_i \cap E_j, \Omega_1).$$

Remember that h is a positive number. The previous estimate, along with (3.2) and the condition $\lambda_i < \lambda_j$, suggests that

$$(\lambda_j - \lambda_i)\Lambda(E_i \setminus E_j) = (\lambda_j - \lambda_i) \int_{E_i \setminus E_j} h(x) d\mu_\alpha(x) = 0,$$

i.e., $E_i \subset E_j$ and the sequence of minimizers $\{E_i\}$ is monotonically increasing. Conversely, by letting $F = E$, we get

$$P_{\mathcal{L}^\alpha}(E_i, \Omega_1) + \lambda_i \Lambda(E \setminus E_i) \leq P_{\mathcal{L}^\alpha}(E, \Omega_1) + \lambda_i \Lambda(E \setminus E) = P_{\mathcal{L}^\alpha}(E, \Omega_1) \ \ \forall\, i \geq 1,$$

which infers that E_i converges to E in a monotonic manner and within $L^1_{loc}(\mathbb{R}^d_+, d\mu_\alpha)$. Using Lemma 2.1 (ii), we have

$$\begin{cases} P_{\mathcal{L}^\alpha}(E, \Omega_1) \leq \liminf\limits_{i\to\infty} P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \leq P_{\mathcal{L}^\alpha}(E, \Omega_1), \\ P_{\mathcal{L}^\alpha}(E, \Omega_1) \leq \liminf\limits_{i\to\infty} P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \leq \limsup\limits_{i\to\infty} P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \leq P_{\mathcal{L}^\alpha}(E), \end{cases}$$

which means

$$P_{\mathcal{L}^\alpha}(E, \Omega_1) = \lim_{i\to\infty} P_{\mathcal{L}^\alpha}(E_i, \Omega_1). \tag{3.4}$$

Step II. Define $\lambda_0 = 0$ and $E_0 = \emptyset$, and let

$$u(x) = \begin{cases} -\lambda_i h(x),\ x \in E_i \setminus E_{i-1},\ i \geq 1, \\ 0,\ \text{otherwise}. \end{cases}$$

It is evident that u is negative almost everywhere on E, and

$$\begin{aligned} \int_{\mathbb{R}^d_+} |u(x)| d\mu_\alpha(x) &= \int_{\cup_{i=0}^\infty E_{i+1} \setminus E_i} |u(x)| d\mu_\alpha(x) \\ &= \sum_{i=0}^\infty \int_{E_{i+1} \setminus E_i} \lambda_{i+1} h(x) d\mu_\alpha(x) \\ &= \sum_{i=0}^\infty \lambda_{i+1} \Lambda(E_{i+1} \setminus E_i). \end{aligned}$$

In (3.3), taking $F = E_{i+1}$, we have

$$P_{\mathcal{L}^\alpha}(E_i, \Omega_1) + \lambda_i \Lambda(E \setminus E_i) \leq P_{\mathcal{L}^\alpha}(E_{i+1}, \Omega_1) + \lambda_i \Lambda(E \setminus E_{i+1}),$$

that is, for every $i \geq 0$,

$$\lambda_i \Lambda(E_{i+1} \setminus E_i) \leq P_{\mathcal{L}^\alpha}(E_{i+1}, \Omega_1) - P_{\mathcal{L}^\alpha}(E_i, \Omega_1).$$

For values of N that are large enough, we have

$$\sum_{i=0}^N \lambda_i \Lambda(E_{i+1} \setminus E_i) \leq \sum_{i=0}^N \left[P_{\mathcal{L}^\alpha}(E_{i+1}, \Omega_1) - P_{\mathcal{L}^\alpha}(E_i, \Omega_1) \right]$$

$$= P_{\mathcal{L}^\alpha}(E_N,\Omega_1) - P_{\mathcal{L}^\alpha}(E_0,\Omega_1)$$
$$= P_{\mathcal{L}^\alpha}(E_N,\Omega_1).$$

Letting $N \to \infty$, (3.4) indicates that

$$\sum_{i=0}^{\infty} \lambda_i \Lambda(E_{i+1} \backslash E_i) \leq P_{\mathcal{L}^\alpha}(E,\Omega_1).$$

Let's assume an additional condition that $0 < \lambda_{i+1} - \lambda_i \leq c$, $i \geq 0$, where c is a constant that doesn't depend on i, we can say that for any $N > 0$,

$$\begin{aligned}\sum_{i=0}^{N} (\lambda_{i+1} - \lambda_i)\Lambda(E_{i+1} \backslash E_i) &\leq c \sum_{i=0}^{N} \Lambda(E_{i+1} \backslash E_i) \\ &= c \sum_{i=0}^{N} \int_{E_{i+1} \backslash E_i} h(x) d\mu_\alpha(x) \\ &= c \int_{\cup_{i=0}^{N}(E_{i+1} \backslash E_i)} h(x) d\mu_\alpha(x),\end{aligned}$$

which gives

$$\sum_{i=0}^{\infty} (\lambda_{i+1} - \lambda_i)\Lambda(E_{i+1} \backslash E_i) \leq c\Lambda(E).$$

Then

$$\begin{aligned}\int_{\mathbb{R}^d_+} |u(x)| d\mu_\alpha(x) &= \sum_{i=0}^{\infty} \lambda_{i+1} \Lambda(E_{i+1} \backslash E_i) \\ &= \sum_{i=0}^{\infty} (\lambda_{i+1} - \lambda_i)\Lambda(E_{i+1} \backslash E_i) + \sum_{i=0}^{\infty} \lambda_i \Lambda(E_{i+1} \backslash E_i) \\ &\leq c\Lambda(E) + P_{\mathcal{L}^\alpha}(E,\Omega_1) < \infty.\end{aligned}$$

In conclusion, $u \in L^1(\mathbb{R}^d_+, d\mu_\alpha)$.

Step III. We contend that the inequality

$$P_{\mathcal{L}^\alpha}(E_i,\Omega_1) \leq P_{\mathcal{L}^\alpha}(F,\Omega_1) + \sum_{j=1}^{i} \lambda_j \Lambda((E_j \backslash E_{j-1}) \backslash F) \tag{3.5}$$

is vaild for all $F \subset E$ and every $i \geq 1$.

If $i = 1$, then $E_{i-1} = E_0 = \emptyset$. Substituting this into (3.5) yields

$$P_{\mathcal{L}^\alpha}(E_1,\Omega_1) \leq P_{\mathcal{L}^\alpha}(F,\Omega_1) + \lambda_1 \Lambda(E_1 \backslash F),$$

which coincides with (3.3) for $i = 1$.

Now we assume that (3.5) holds for a fixed $i \geq 1$ and every $F \subset E$. Take $F \cap E_i$ as a test set. Observe that $\{E_j\}$ is increasing. It is evidently clear to show that

$$(E_j \backslash E_{j-1}) \backslash (F \cap E_i) = (E_j \backslash E_{j-1}) \backslash F.$$

Then

$$P_{\mathcal{L}^\alpha}(E_i,\Omega_1) \le P_{\mathcal{L}^\alpha}(F\cap E_i,\Omega_1) + \sum_{j=1}^{i}\lambda_j\Lambda((E_j\backslash E_{j-1})\backslash(F\cap E_i))$$
$$= P_{\mathcal{L}^\alpha}(F\cap E_i,\Omega_1) + \sum_{j=1}^{i}\lambda_j\Lambda((E_j\backslash E_{j-1})\backslash F).$$

Conversely, $\mathscr{F}_{\lambda_{i+1}}$ is minimized by E_{i+1}. Hence,

$$\mathscr{F}_{\lambda_{i+1}}(E_{i+1}) \le \mathscr{F}_{\lambda_{i+1}}(F\cup E_i),$$

and noticing that

$$E\backslash E_i = (E\backslash E_{i+1})\cup(E_{i+1}\backslash E_i),$$

it is possible for us to obtain

$$E\backslash(F\cup E_i) = ((E\backslash E_{i+1})\backslash F)\cup((E_{i+1}\backslash E_i)\backslash F).$$

This deduces

$$\begin{aligned} P_{\mathcal{L}^\alpha}(E_{i+1},\Omega_1) + \lambda_{i+1}\Lambda(E\backslash E_{i+1}) &\le P_{\mathcal{L}^\alpha}(F\cup E_i,\Omega_1) + \lambda_{i+1}\Lambda(E\backslash(F\cup E_i)) \\ &\le P_{\mathcal{L}^\alpha}(F\cup E_i,\Omega_1) + \lambda_{i+1}\Lambda((E\backslash E_{i+1})\backslash F) \\ &\quad + \lambda_{i+1}\Lambda((E_{i+1}\backslash E_i)\backslash F). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} &P_{\mathcal{L}^\alpha}(E_i,\Omega_1) + P_{\mathcal{L}^\alpha}(E_{i+1},\Omega_1) + \lambda_{i+1}\Lambda(E\backslash E_{i+1}) \\ &\le P_{\mathcal{L}^\alpha}(F\cap E_i,\Omega_1) + \sum_{j=1}^{i}\lambda_j\Lambda((E_j\backslash E_{j-1})\backslash F) \\ &\quad + P_{\mathcal{L}^\alpha}(F\cup E_i,\Omega_1) + \lambda_{i+1}\Lambda((E\backslash E_{i+1})\backslash F) + \lambda_{i+1}\Lambda((E_{i+1}\backslash E_i)\backslash F) \\ &\le P_{\mathcal{L}^\alpha}(E_i,\Omega_1) + P_{\mathcal{L}^\alpha}(F,\Omega_1) + \sum_{j=1}^{i+1}\lambda_j\Lambda((E_j\backslash E_{j-1})\backslash F) + \lambda_{i+1}\Lambda((E\backslash E_{i+1})\backslash F) \\ &\le P_{\mathcal{L}^\alpha}(E_i,\Omega_1) + P_{\mathcal{L}^\alpha}(F,\Omega_1) + \sum_{j=1}^{i+1}\lambda_j\Lambda((E_j\backslash E_{j-1})\backslash F) + \lambda_{i+1}\Lambda(E\backslash E_{i+1}), \end{aligned}$$

i.e., (3.5) is true for $i+1$. Last but not least,

$$\begin{aligned} P_{\mathcal{L}^\alpha}(E,\Omega_1) &= \lim_{i\to\infty} P_{\mathcal{L}^\alpha}(E_i,\Omega_1) \\ &\le P_{\mathcal{L}^\alpha}(F,\Omega_1) + \lim_{i\to\infty}\sum_{j=1}^{i}\lambda_j\Lambda((E_j\backslash E_{j-1})\backslash F) \\ &= P_{\mathcal{L}^\alpha}(F,\Omega_1) - \int_{\cup_{j=0}^{\infty}(E_j\backslash E_{j-1})\backslash F} u(x)d\mu_\alpha(x) \\ &= P_{\mathcal{L}^\alpha}(F,\Omega_1) - \int_{E\backslash F} u(x)d\mu_\alpha(x), \end{aligned}$$

which gives (3.3).

Acknowledgments

This work was supported by National Natural Science Foundation of China (No. 11671031, No. 12271042) and Beijing Natural Science Foundation of China (No. 1232023).

Conflict of interest

The authors declare there is no conflict of interest.

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