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Research article

Generalized Ricci solitons and Einstein metrics on weak K-contact manifolds

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Abstract: We study so-called "weak" metric structures on a smooth manifold, which generalize the metric contact and K-contact structures and allow a new look at the classical theory. We characterize weak K-contact manifolds among all weak contact metric manifolds using the property well known for K-contact manifolds, as well as find when a Riemannian manifold endowed with a unit Killing vector field is a weak K-contact manifold. We also find sufficient conditions for a weak K-contact manifold with a parallel Ricci tensor or with a generalized Ricci soliton structure to be an Einstein manifold.

Keywords: weak K-contact manifold; unit Killing vector field; generalized Ricci soliton; Einstein metric; curvature
Mathematics Subject Classification: 53C15, 53C25, 53D15

1. Introduction

The growing interest in contact geometry is associated with its important role in mechanics in explaining physical phenomena. In addition, many recent articles have been motivated by the question of to what extent can self-similar solutions of the Ricci flow equation, i.e., Ricci solitons, be applicable for contact metric geometry. Some of them have established the conditions for when a contact manifold equipped with a Ricci-type soliton structure carries a canonical (e.g., Einstein or constant curvature) metric, e.g.. [1–4].

K-contact manifolds (i.e., contact metric manifolds, whose Reeb vector field is Killing) have been studied by several geometers, e.g., [5, 6], and it can be seen that the K-contact structure is intermediate between the contact and Sasakian structures. The Reeb vector field ξ of the K-contact structure is a unit Killing vector field, and the influence of constant-length Killing vector fields on the Riemannian geometry has been studied by several authors from different points of view, e.g., [7–9]. An interesting result related to the above question is that a K-contact manifold equipped with a generalized Ricci soliton structure has an Einstein metric, e.g., [2].

In [10], we introduced the "weakened" metric structures on a smooth manifold (replacing the complex structure on the characteristic distribution with a nonsingular skew-symmetric tensor). This previous paper generalized the metric contact, K-contact, Sasakian and cosymplectic structures and allowed for a new look at the classical theory. In [10], we build a retraction of weak structures with positive partial Ricci curvature onto a set of classical structures. In [11] (where the definition of weak structures is a bit more general than in [10]), we proved that the weak Sasakian structure is weak K-contact, and that a weak almost contact metric manifold is weak Sasakian if and only if it is a Sasakian manifold. In this article, we study weak K-contact manifolds using their sectional and Ricci curvature in the ξ-direction. Our goal is to show that the weak K-contact structure can be a useful tool for studying unit Killing vector fields on Riemannian manifolds, and that some results for K-contact manifolds can be extended to the case of weak K-contact manifolds. For example, we answer the question of when a weak K-contact manifold carries a generalized Ricci soliton structure, or just an Einstein metric.

The article is organized as follows. In Section 2, following the introductory Section 1, we recall the basics of weak contact metric manifolds. Section 3 contains the main results. In Subsection 3.1, we characterize (in Theorem 2) weak K-contact manifolds among all weak contact metric manifolds by using the property $\phi = -\nabla\xi$ (well known for K-contact manifolds), and we find (in Theorem 3) when a Riemannian manifold endowed with a unit Killing vector field forms a weak K-contact structure. In Subsection 3.2, for a weak K-contact manifold, we calculate (in Proposition 2) the Ricci curvature in the ξ-direction, and then find (in Theorem 4) the sufficient condition for such a manifold with a parallel Ricci tensor to be an Einstein manifold. In Subsection 3.3, we find (in Theorem 5) sufficient conditions for a weak K-contact manifold admitting a generalized Ricci soliton structure to be an Einstein manifold.

2. Preliminaries

Here, we recall the basics of some metric structures that generalize the almost contact metric structure (see [10], or, equivalently, [11] with $\nu = 1$). A *weak almost contact structure* on a smooth manifold M^{2n+1} is a set (ϕ, Q, ξ, η), where ϕ is a rank-(1,1) tensor, Q is a nonsingular $(1,1)$-tensor, ξ is the Reeb vector field and η is a dual 1-form, i.e., $\eta(\xi) = 1$, satisfying

$$\phi^2 = -Q + \eta\otimes\xi, \qquad Q\,\xi = \xi. \tag{2.1}$$

The form η determines a smooth $2n$-dimensional distribution $\mathcal{D} := \ker\eta$, which is the collection of subspaces $\mathcal{D}_m = \{U \in T_m M : \eta(U) = 0\}$ for $m \in M$. We assume that $\mathcal{D}$ is ϕ-invariant, i.e.,

$$\phi U \in \mathcal{D}, \quad U \in \mathcal{D}, \tag{2.2}$$

as in the theory of almost contact structure [5,6], where $Q = \mathrm{id}_{TM}$. By (2.1) and (2.2), our distribution $\mathcal{D}$ is invariant for Q: $Q(\mathcal{D}) = \mathcal{D}$. If there is a Riemannian metric g on M such that

$$g(\phi U, \phi V) = g(U, Q\,V) - \eta(U)\,\eta(V), \quad U, V \in \mathfrak{X}_M, \tag{2.3}$$

then (ϕ, Q, ξ, η, g) is called a *weak almost contact metric structure* on M, and g is called a *compatible* metric. A weak almost contact manifold $M(\phi, Q, \xi, \eta)$ endowed with a compatible Riemannian metric g is called a *weak almost contact metric manifold* and is denoted by $M(\phi, Q, \xi, \eta, g)$. Some sufficient conditions for the existence of a compatible metric on a weak almost contact manifold are given in [10].

Putting $V = \xi$ in (2.3) and using $Q\xi = \xi$, we get, as in the classical theory, $\eta(U) = g(\xi, U)$. In particular, ξ is g-orthogonal to $\mathcal{D}$ for any compatible metric g.

By (2.3), we get $g(U, Q\,U) = g(\phi U, \phi U) > 0$ for any nonzero vector $U \in \mathcal{D}$; thus, Q is positive definite. For a weak almost contact structure on a smooth manifold M, the tensor ϕ has rank $2n$ and

$$\phi\,\xi = 0, \quad \eta\circ\phi = 0, \quad \eta\circ Q = \eta, \quad [Q,\,\phi] = 0;$$

moreover, for a weak almost contact metric structure, ϕ is skew-symmetric and Q is self-adjoint.

A *weak contact metric structure* is defined as a weak almost contact metric structure satisfying $\Phi = d\eta$ (thus, $d\Phi = 0$), where $\Phi(U, V) = g(U, \phi V)$ ($U, V \in \mathfrak{X}_M$) is called the fundamental 2-form, and

$$d\eta(U, V) = \frac{1}{2}\,\{U(\eta(V)) - V(\eta(U)) - \eta([U, V])\}, \quad U, V \in \mathfrak{X}_M, \tag{2.4}$$

$$\begin{aligned} d\Phi(U, V, Z) = \frac{1}{3}\,\{&U\,\Phi(V, Z) + V\,\Phi(Z, U) + Z\,\Phi(U, V) \\ &- \Phi([U, V], Z) - \Phi([Z, U], V) - \Phi([V, Z], U)\}, \quad U, V, Z \in \mathfrak{X}_M. \end{aligned} \tag{2.5}$$

Remark 1. A differential *k-form* on a manifold M is a skew-symmetric tensor field ω of type $(0, k)$. According to the conventions of, e.g., [12], the formula

$$\begin{aligned} d\omega(U_1, \dots, U_{k+1}) = \tfrac{1}{k+1} \textstyle\sum_{i=1}^{k+1}(-1)^{i+1} U_i(\omega(U_1, \dots, \hat{U}_i \dots, U_{k+1})) \\ + \textstyle\sum_{i<j}(-1)^{i+j}\omega([U_i, U_j], U_1, \dots, \hat{U}_i, \dots, \hat{U}_j, \dots, U_{k+1}), \end{aligned} \tag{2.6}$$

where $U_1, \dots, U_{k+1} \in \mathfrak{X}_M$ and $\hat{}$ denotes the operator of omission, defines a $(k+1)$-form $d\omega$ called the *exterior differential* of ω. Note that (2.4) and (2.5), as given in [5], correspond to (2.6) with $k = 1$ and $k = 2$.

For a weak contact metric structure, the distribution $\mathcal{D}$ is non-integrable (i.e., has no integral hypersurfaces), since $g([U, \phi U], \xi) = 2\,d\eta(\phi U, U) = g(\phi U, \phi U) > 0$ for any nonzero $U \in \mathcal{D}$.

The Nijenhuis torsion $[\phi, \phi]$ of ϕ is given by

$$[\phi, \phi](U, V) = \phi^2[U, V] + [\phi U, \phi V] - \phi[\phi U, V] - \phi[U, \phi V], \quad U, V \in \mathfrak{X}_M.$$

A weak almost contact structure (ϕ, Q, ξ, η) is called *normal* if the following tensor is zero:

$$N^{(1)}(U, V) = [\phi, \phi](U, V) + 2\,d\eta(U, V)\,\xi, \quad U, V \in \mathfrak{X}_M. \tag{2.7}$$

The following tensors $N^{(2)}, N^{(3)}$ and $N^{(4)}$ are well known in the classical theory; see [5, 6]:

$$\begin{aligned} N^{(2)}(U, V) &= (\pounds_{\phi U}\,\eta)(V) - (\pounds_{\phi V}\,\eta)(U) \overset{(2.4)}{=} 2\,d\eta(\phi U, V) - 2\,d\eta(\phi V, U), \\ N^{(3)}(U) &= (\pounds_{\xi}\,\phi)U = [\xi, \phi U] - \phi[\xi, U], \\ N^{(4)}(U) &= (\pounds_{\xi}\,\eta)(U) = \xi(\eta(U)) - \eta([\xi, U]) \overset{(2.4)}{=} 2\,d\eta(\xi, U), \end{aligned}$$

where $\pounds_\xi$ is the Lie derivative in the ξ-direction.

Remark 2. Let $M(\phi, Q, \xi, \eta)$ be a weak almost contact manifold. Consider the product manifold $\bar M = M\times\mathbb{R}$, where $\mathbb{R}$ has the Euclidean basis ∂_t, and define tensor fields $\bar\phi$ and $\bar Q$ on $\bar M$ by putting

$$\bar\phi(U,\, a\,\partial_t) = (\phi U - a\,\xi,\ \eta(U)\,\partial_t),$$
$$\bar Q(U,\, a\,\partial_t) = (QU,\, a\,\partial_t),$$

where $a \in C^\infty(M)$. Thus, $\bar\phi(U,0) = (\phi U,0)$, $\bar Q(U,0) = (QU,0)$ for $U \perp \ker\phi$, $\bar\phi(\xi,0) = (0,\partial_t)$, $\bar Q(\xi,0) = (\xi,0)$ and $\bar\phi(0,\partial_t) = (-\xi,0)$, $\bar Q(0,\partial_t) = (0,\partial_t)$. By the above, $\bar\phi^{\,2} = -\bar Q$. The tensors $N^{(i)}$ $(i = 1,2,3,4)$ appear when we derive the integrability condition $[\bar\phi,\bar\phi] = 0$ (i.e., vanishing of the Nijenhuis torsion of $\bar\phi$) and express the normality condition $N^{(1)} = 0$ of (ϕ, Q, ξ, η) on M.

Theorem 1 (see [11] with $\nu = 1$)**.** (a) *For a weak almost contact metric structure* (ϕ, Q, ξ, η, g)*, the vanishing of* $N^{(1)}$ *implies that* $N^{(3)}$ *and* $N^{(4)}$ *vanish and* $N^{(2)}(U,V) = \eta([\tilde QU,\, \phi V])$.

(b) *For a weak contact metric manifold, the tensors* $N^{(2)}$ *and* $N^{(4)}$ *vanish and the trajectories of* ξ *are geodesics, i.e.,* $\nabla_\xi\,\xi = 0$*; moreover,* $N^{(3)} \equiv 0$ *if and only if* ξ *is a Killing vector field.*

Definition 1 (see [11] with $\nu = 1$)**.** Two weak almost contact structures (ϕ, Q, ξ, η) and (ϕ', Q', ξ, η) on M are said to be homothetic if the following is valid for some real $\lambda > 0$:

$$\phi = \sqrt{\lambda}\,\phi', \tag{2.8a}$$
$$Q\,|_{\mathcal D} = \lambda\, Q'|_{\mathcal D}. \tag{2.8b}$$

Two weak contact metric structures (ϕ, Q, ξ, η, g) and $(\phi', Q', \xi, \eta, g')$ on M are said to be homothetic if they satisfy conditions (2.8a,b) and

$$g|_{\mathcal D} = \lambda^{-\frac12}\, g'|_{\mathcal D}, \quad g(\xi,\,\cdot) = g'(\xi,\,\cdot). \tag{2.8c}$$

Lemma 1 (see [11] with $\nu = 1$)**.** *Let* (ϕ, Q, ξ, η) *be a weak almost contact structure such that*

$$Q\,|_{\mathcal D} = \lambda\,\mathrm{id}_{\mathcal D}$$

for some real $\lambda > 0$*. Then, the following is true:*
- (ϕ', ξ, η) *is an almost contact structure, where* ϕ' *is given by* (2.8a).
- *If* (ϕ, Q, ξ, η, g) *is a weak contact metric structure and* ϕ', g' *satisfy* (2.8a,c)*, then* (ϕ', ξ, η, g') *is a contact metric structure.*

A *weak K-contact manifold* is defined as a weak contact metric manifold, whose Reeb vector field ξ is Killing (or, infinitesimal isometry, such as in [12]), i.e.,

$$(\pounds_\xi\, g)(U,V) := \xi(g(U,V)) - g([\xi,U],V) - g(U,[\xi,V]) = g(\nabla_U\,\xi, V) + g(\nabla_V\,\xi, U) = 0. \tag{2.10}$$

Here, ∇ is the Levi-Civita connection. A normal weak contact metric manifold is called a *weak Sasakian manifold*. Recall that a weak Sasakian structure is weak K-contact; see [11, Proposition 4.1], furthermore, a weak almost contact metric structure is weak Sasakian if and only if it is homothetic to a Sasakian structure; see [11, Theorem 4.1].

The relationships between the different classes of weak structures (considered in this article) can be summarized in the diagram (well known in the case of classical structures):

$$\begin{pmatrix} weak \\ almost\ contact \end{pmatrix} \overset{metric}{\longrightarrow} \begin{pmatrix} weak\ almost \\ contact\ metric \end{pmatrix} \overset{\Phi=d\eta}{\longrightarrow} \begin{pmatrix} weak \\ contact\ metric \end{pmatrix} \overset{\xi-Killing}{\longrightarrow} \begin{pmatrix} weak \\ K-contact \end{pmatrix}.$$

A "small" (1,1)-tensor $\tilde{Q} = Q - \mathrm{id}$ is a measure of the difference between a weakly contact structure and a contact one, and $\tilde{Q} = 0$ denotes the classical contact geometry. Note that $[\tilde{Q}, \phi] = 0$ and $\tilde{Q}\,\xi = 0$.

Lemma 2 ([11] with $\nu = 1$)**.** *For a weak contact metric manifold, we get*

$$g((\nabla_U \phi)V, Z) = \frac{1}{2}\, g(N^{(1)}(V, Z), \phi U) + g(\phi U, \phi V)\, \eta(Z) - g(\phi U, \phi Z)\, \eta(V) + \frac{1}{2}\, N^{(5)}(U, V, Z), \quad (2.11)$$

where the tensor $N^{(5)}(U, V, Z)$ is skew-symmetric with respect to V and Z, given for a weak contact metric manifold by the formula

$$\begin{aligned} N^{(5)}(U, V, Z) &= (\phi Z)\,(g(U, \tilde{Q}V)) - (\phi V)\,(g(U, \tilde{Q}Z)) \\ &\quad + g([U, \phi Z], \tilde{Q}V) - g([U, \phi V], \tilde{Q}Z) + g([V, \phi Z] - [Z, \phi V] - \phi[V, Z],\ \tilde{Q}U). \end{aligned}$$

Remark 3. For a contact metric structure, (2.11) (with $N^{(5)} = 0$) gives the result in [5, Corollary 6.1]. Note that only one new tensor $N^{(5)}$, which supplements the sequence of well-known tensors $N^{(i)},\ i = 1, 2, 3, 4$, is needed for further study of a weak contact metric structure. In particular, by (2.11), we get $g((\nabla_\xi \phi)V, Z) = \frac{1}{2}\, N^{(5)}(\xi, V, Z)$ and

$$\begin{aligned} &N^{(5)}(\cdot\,, \xi, Z) = \tilde{Q}N^{(3)}(Z), \\ &N^{(5)}(\xi, V, Z) = g([\xi, \phi Z], \tilde{Q}V) - g([\xi, \phi V], \tilde{Q}Z), \\ &N^{(5)}(\xi, \xi, Z) = N^{(5)}(\xi, V, \xi) = 0. \end{aligned} \quad (2.12)$$

3. Results

The results are given in the following three subsections.

3.1. Unit Killing vector fields

Proposition 1. *On a weak K-contact manifold, we get $N^{(1)}(\xi, \cdot) = 0$ and*

$$\begin{aligned} N^{(5)}(\xi, \cdot\,, \cdot) &= N^{(5)}(\cdot\,, \xi, \cdot) = 0, && (3.1) \\ \pounds_\xi Q &= \nabla_\xi Q = 0, && (3.2) \\ \nabla_\xi \phi &= 0. && (3.3) \end{aligned}$$

Proof. By (2.7) and $d\eta(\xi, \cdot) = \Phi(\xi, \cdot) = 0$, we get

$$N^{(1)}(\xi, U) = [\phi, \phi](U, \xi) = \phi^2[U, \xi] - \phi[\phi U, \xi] = \phi N^{(3)}(U) = 0.$$

By [11, Lemma 3.1] with $\nu = 1$ and $h = \frac{1}{2}\, N^{(3)} = 0$, we get $N^{(5)}(\xi, \cdot\,, \cdot) = 0$, $\pounds_\xi Q = 0$ and

$$g(Q\, \nabla_U\, \xi, Z) = g(\phi Z, QU) - \frac{1}{2}\, N^{(5)}(U, \xi, \phi Z).$$

By $(2.12)_1$ with $N^{(3)} = 0$, we get $N^{(5)}(\cdot\,, \xi, \cdot) = 0$. We use $[\phi, Q] = 0$ to obtain $\nabla_\xi Q = 0$:

$$(\pounds_\xi\, Q)U = [\xi, QU] - Q[\xi, U] = (\nabla_\xi Q)U + [\phi, Q]U = (\nabla_\xi\, Q)U.$$

This completes the proof of (3.1) and (3.2). Next, from (2.11) with $U = \xi$, we get (3.3).

In the next theorem, we characterize weak K-contact manifolds among all weak contact metric manifolds by using the following well-known property of K-contact manifolds; see [5]:

$$\nabla\xi = -\phi. \tag{3.4}$$

Theorem 2. *A weak contact metric manifold is weak K-contact* (*that is, ξ is a Killing vector field*) *if and only if* (3.4) *is valid.*

Proof. Let a weak contact metric manifold satisfy (3.4). By the skew-symmetry of ϕ, we get that $(\pounds_\xi\, g)(U,V) = g(\nabla_U\,\xi, V) + g(\nabla_V\,\xi, U) = -g(\phi U, V) - g(\phi V, U) = 0$; thus, ξ is a Killing vector field.

Conversely, let our manifold be weak K-contact. By (2.11) with $V=\xi$, using $N^{(1)}(\xi,\,\cdot) = 0$ and $N^{(5)}(U,\xi,Z) = 0$ (see (3.1)) we get

$$g((\nabla_U\,\phi)\,\xi, Z) = \frac{1}{2}\,g(N^{(1)}(\xi,Z),\phi U) - g(\phi U,\phi Z) + \frac{1}{2}\,N^{(5)}(U,\xi,Z) = g(\phi^2 U, Z).$$

Hence, $(\nabla_U\,\phi)\,\xi = \phi^2 U$. From this and $0 = \nabla_U\,(\phi\,\xi) = (\nabla_U\,\phi)\,\xi + \phi\nabla_U\,\xi$, we obtain $\phi(\nabla_U\,\xi + \phi U) = 0$. Since $\nabla_U\,\xi + \phi U \in \mathcal{D}$ and ϕ is invertible when restricted on $\mathcal{D}$, we get that $\nabla_U\,\xi = -\phi U$.

If a plane contains ξ, then its sectional curvature is called *ξ-sectional curvature*. It is well known that the ξ-sectional curvature of a K-contact manifold is a constant equal to 1. Recall that a Riemannian manifold with a unit Killing vector field and the property $R_{U,\xi}\,\xi = U$ $(U\perp\xi)$ is a K-contact manifold, e.g., [6, Theorem 3.1] or [5, Proposition 7.4]. We generalize this result below.

Theorem 3. *A Riemannian manifold (M^{2n+1}, g) admitting a unit Killing vector field ξ such that the $(1,1)$-tensor $\nabla\xi$ has constant rank $2n$ and ξ-sectional curvature is positive is a weak K-contact manifold $M(\phi, Q, \xi, \eta, g)$ with $\eta = g(\cdot\,,\xi)$ and the following structural tensors: $\phi = -\nabla\,\xi$ (see* (3.4)*) and $QU = R_{U,\xi}\,\xi$ for $U\in\ker\eta$.*

Proof. Let $\eta = g(\,\cdot\,,\xi)$ and $\mathcal{D} = \ker\eta$. Put $\phi U = -\nabla_U\,\xi$ and $QU = R_{U,\xi}\,\xi$ for $U\in\mathcal{D}$. Since ξ is a Killing vector field, we obtain the property $d\eta = \Phi$ for $\Phi(U,V) = g(U,\phi V)$:

$$d\eta(U,V) = \frac{1}{2}\,(g(\nabla_U\,\xi, V) - g(\nabla_V\,\xi, U)) = -g(\nabla_V\,\xi, U) = g(U,\phi V).$$

Since ξ is a unit Killing vector field, we get that $\nabla_\xi\,\xi = 0$ and $\nabla_U\nabla_V\,\xi - \nabla_{\nabla_U V}\,\xi = R_{\xi,U}\,V$. Thus, $\phi\,\xi = 0$, ϕ has constant rank $2n$ and

$$\phi^2 U = \nabla_{\nabla_U\,\xi}\,\xi = R_{\xi,U}\,\xi = -QU \quad (U\in\mathcal{D}).$$

Put $Q\,\xi = \xi$. Therefore, (2.1) is valid and Q is positive definite; that completes the proof.

Example 1. By applying Theorem 3, we can search for examples of weak K-contact (not K-contact) manifolds among Riemannian manifolds with positive sectional curvature that admit unit Killing vector fields. Indeed, let M be a convex hypersurface (ellipsoid) with induced metric g of the Euclidean space $\mathbb{R}^{2n+2}$, and

$$M = \Big\{(u_1,\ldots,u_{2n+2})\in\mathbb{R}^{2n+2} : \sum\nolimits_{i=1}^{n+1} u_i^2 + a\sum\nolimits_{i=n+2}^{2n+1} u_i^2 = 1\Big\},$$

where $0 < a = const \neq 1$ and $n \geq 1$ is odd. The sectional curvature of (M, g) is positive. It follows that

$$\xi = (-u_2, u_1, \ldots, -u_{n+1}, u_n, -\sqrt{a}\, u_{n+3},\ \sqrt{a}\, u_{n+2}, \ldots, -\sqrt{a}\, u_{2n+2},\ \sqrt{a}\, u_{2n+1})$$

is a Killing vector field on $\mathbb{R}^{2n+2}$, whose restriction to M has unit length. Since ξ is tangent to M (i.e., M is invariant under the flow of ξ), ξ is a unit Killing vector field on (M, g); see [8, p. 5]. For $a \to 1$, such vector field ξ converges to a unit Killing vector field ξ_1 on a unit sphere, and $\mathrm{rank}(\nabla\xi_1) = 2n$. Thus, $\mathrm{rank}(\nabla\xi) = 2n$ for $a \approx 1$. For $n = 1$, we get a weak K-contact manifold $M^3 = \{u_1^2 + u_2^2 + au_3^2 + au_4^2 = 1\} \subset \mathbb{R}^4$ with $\xi = (-u_2, u_1, -\sqrt{a}\, u_4, \sqrt{a}\, u_3)$.

Other examples of weak K-contact (not K-contact) manifolds can be obtained from [7, Theorem 12]: "On every sphere S^{2n-1}, $n \geq 2$, for any $\varepsilon > 0$, there exists a (real analytic) Riemannian metric g of cohomogeneity 1 and a (real analytic) Killing vector field ξ of unit length on (S^{2n-1}, g) such that

1) all sectional curvatures of (S^{2n-1}, g) differ from 1 by at most ε;
2) the vector field ξ has both closed and non-closed integral trajectories".

Corollary 1. *A weak K-contact structure (ϕ, Q, ξ, η, g) with constant positive ξ-sectional curvature, $K(\xi, U) = \lambda > 0$, for some $\lambda = const \in \mathbb{R}$ and all $U \in \mathcal{D}$, is homothetic to a K-contact structure (ϕ', ξ, η, g') after the transformation described by* (2.8a–c).

Proof. Note that $K(\xi, U) = \lambda$ $(U \in \mathcal{D})$ if and only if $R_{U,\xi}\,\xi = \lambda\, U$ $(U \in \mathcal{D})$. By $QU = R_{U,\xi}\,\xi$ $(U \in \mathcal{D})$ (see Theorem 3) we get that $QU = \lambda U$ $(U \in \mathcal{D})$. By Lemma 1, (ϕ', ξ, η, g') is a contact metric structure. Using (2.10), we get that $(\pounds_\xi\, g')(U, V) = \lambda(\pounds_\xi\, g)(U, V)$ $(U, V \in \mathcal{D})$ and $(\pounds_\xi\, g')(\xi, \cdot) = 0$. By applying $\pounds_\xi\, g = 0$, we get that $\pounds_\xi\, g' = 0$; thus, (ϕ', ξ, η, g') is a K-contact structure.

3.2. The Ricci curvature in the characteristic direction

Denote by $R_{U,V}\, Z$ the curvature tensor and by $\mathrm{Ric}^\sharp$ the Ricci operator of g associated with the Ricci tensor Ric and given by $\mathrm{Ric}(U, V) = g(\mathrm{Ric}^\sharp\, U, V)$ for all $U, V \in \mathfrak{X}_M$. The Ricci curvature in the ξ-direction is given by $\mathrm{Ric}(\xi, \xi) = \sum_{i=1}^{2n} g(R_{e_i,\xi}\,\xi, e_i)$, where (e_i) is any local orthonormal basis of $\mathcal{D}$.

In the next proposition, we generalize three particular properties of K-contact manifolds to weak K-contact manifolds.

Proposition 2. *For a weak K-contact manifold, the following equalities are true:*

$$R_{\xi,\, U} = \nabla_U\, \phi, \tag{3.5}$$

$$R_{U,\xi}\,\xi = -\phi^2 U, \tag{3.6}$$

$$\mathrm{Ric}(\xi, \xi) = \mathrm{trace}\ Q = 2n + \mathrm{trace}\ \tilde{Q}. \tag{3.7}$$

Proof. Using (3.4), we derive

$$\begin{aligned} R_{Z,\,U}\,\xi &= \nabla_Z(\nabla_U\,\xi) - \nabla_U(\nabla_Z\,\xi) - \nabla_{[Z,U]}\,\xi) \\ &= \nabla_U(\phi Z) - \nabla_Z(\phi U) + \phi([Z, U]) = (\nabla_U\,\phi)Z - (\nabla_Z\,\phi)U. \end{aligned} \tag{3.8}$$

Note that $(\nabla_U\,\Phi)(V, Z) = g((\nabla_U\,\phi)Z, V) = -g((\nabla_U\,\phi)V, Z)$. Using the condition that $d\Phi = d^2\eta = 0$, we get

$$(\nabla_U\,\Phi)(V, Z) + (\nabla_V\,\Phi)(Z, U) + (\nabla_Z\,\Phi)(U, V) = 0. \tag{3.9}$$

From (3.8), using (3.9) and the skew-symmetry of Φ, we get (3.5):

$$g(R_{\xi,U}V,Z) = g(R_{V,Z}\,\xi,U) \overset{(3.8)}{=} (\nabla_Z\,\Phi)(U,V) + (\nabla_V\,\Phi)(Z,U) \overset{(3.9)}{=} -(\nabla_U\,\Phi)(V,Z) = g((\nabla_U\,\phi)V,Z).$$

By applying (3.5) with $V=\xi$, $\phi\,\xi=0$ and (3.4), we find that

$$R_{\xi,U}\,\xi = (\nabla_U\,\phi)\,\xi = -\phi\nabla_U\,\xi = \phi^2 U.$$

This and $(2.1)_1$ yield (3.6). By this, for any local orthonormal basis (e_i) of $\mathcal{D}$, we get

$$\mathrm{Ric}(\xi,\xi) \overset{(3.6)}{=} -\sum\nolimits_{i=1}^{2n} g(\phi^2 e_i, e_i) \overset{(2.1)}{=} \sum\nolimits_{i=1}^{2n} g(Qe_i, e_i).$$

By the above and the equality trace $Q = 2n +$ trace $\tilde{Q}$, (3.7) is true.

Note that if a Riemannian manifold admits a unit Killing vector field ξ, then $K(\xi,U)\ge 0$ ($U\perp\xi$, $U\ne 0$); thus, $\mathrm{Ric}(\xi,\xi)\ge 0$; moreover, $\mathrm{Ric}(\xi,\xi)\equiv 0$ if and only if ξ is parallel: $\nabla\xi\equiv 0$, see for example, [9]. In the case of K-contact manifolds, $K(\xi,U)=1$; see [5, Theorem 7.2].

Corollary 2. *For a weak K-contact manifold, the ξ-sectional curvature is positive:*

$$K(\xi,U) = g(QU,U) > 0 \quad (U\in\mathcal{D},\ \|U\|=1), \tag{3.10}$$

therefore, for the Ricci curvature, we get that $\mathrm{Ric}(\xi,\xi)>0$.

Proof. For any unit vector $U\in\mathcal{D}$, by (3.6), we get

$$0 < g(\phi U,\phi U) = -g(\phi^2 U,U) = g(QU,U);$$

thus, $K(\xi,U)>0$ and trace $Q>0$. Therefore, from (3.7), we obtain $\mathrm{Ric}(\xi,\xi)>0$.

By Theorem 3 and Corollary 2, and using [10, Corollary 3], we conclude the following.

Corollary 3. *A weak K-contact manifold $M(\phi,Q,\xi,\eta,g_0)$ admits a smooth family of metrics g_t ($t\in\mathbb{R}$) such that $M(\phi_t,Q_t,\xi,\eta,g_t)$ are weak K-contact manifolds with certainly defined ϕ_t and Q_t; moreover, g_t converges exponentially fast, as $t\to-\infty$, to a limit metric $\hat{g}$ that gives a K-contact structure.*

The following theorem generalizes a well-known result, see, for example, [6, Proposition 5.1].

Theorem 4. *A weak K-contact manifold with the conditions that* $(\nabla\,\mathrm{Ric})(\xi,\cdot)=0$ (*in particular, the Ricci tensor is parallel*) *and* trace $Q = $ *const is an Einstein manifold of scalar curvature* $(2n+1)\,$trace Q.

Proof. Differentiating (3.7) and using (3.4) and the above-mentioned conditions, we get

$$0 = \nabla_V\,(\mathrm{Ric}(\xi,\xi)) = (\nabla_V\,\mathrm{Ric})(\xi,\xi) + 2\,\mathrm{Ric}(\nabla_V\,\xi,\xi)) = -2\,\mathrm{Ric}(\phi V,\xi));$$

hence, $\mathrm{Ric}(V,\xi) = \eta(V)\,\mathrm{Ric}(\xi,\xi) = \eta(V)\,$trace Q. Differentiating this, and then using

$$U(\eta(V)) = g(\nabla_U\xi,V) = -g(\phi U,V) + g(\nabla_U V,\xi)$$

and assuming $\nabla_U V = 0$ at $x \in M$, gives

$$(\text{trace } Q)\, g(\phi V, U) = \nabla_U (\operatorname{Ric}(V,\xi)) = (\nabla_U \operatorname{Ric})(V,\xi) + 2\operatorname{Ric}(V, \nabla_U \xi) = -2\operatorname{Ric}(V, \phi U);$$

hence, $\operatorname{Ric}(V, \phi U) = (\text{trace } Q)\, g(V, \phi U)$. Therefore, we obtain

$$\operatorname{Ric}(U, V) = (\text{trace } Q)\, g(U, V)$$

for any vector fields U and V on M, which means that (M, g) is an Einstein manifold. Using the definition of scalar curvature, $\tau = \text{trace Ric}$, we find that $\tau = (2n + 1)\,\text{trace } Q$.

Remark 4. For a weak K-contact manifold, by (3.5) and $\nabla_\xi \phi = \frac{1}{2} N^{(5)}(\xi, V, Z) = 0$ (see Remark 3 and (3.1)), we get the following equality (well known for K-contact manifolds, such as in [5]):

$$\operatorname{Ric}^\sharp(\xi) = \sum\nolimits_{i=1}^{2n} (\nabla_{e_i} \phi)\, e_i,$$

where (e_i) is any local orthonormal basis of $\mathcal{D}$, and for contact manifolds we have $\sum_{i=1}^{2n} (\nabla_{e_i} \varphi)\, e_i = 2n\,\xi$. For K-contact manifolds, this gives $\operatorname{Ric}^\sharp(\xi) = 2n\,\xi$ (see [5, Proposition 7.2]) and $\operatorname{Ric}(\xi, \xi) = 2n$; moreover, the last condition characterizes K-contact manifolds among all contact metric manifolds.

3.3. Generalized Ricci solitons on weak K-contact manifolds

The *generalized Ricci soliton* equation in a Riemannian manifold (M, g) is defined by [4], i.e.,

$$\frac{1}{2}\, £_U\, g = -c_1 U^\flat \otimes U^\flat + c_2 \operatorname{Ric} + \lambda\, g, \tag{3.11}$$

for some smooth vector field U and real c_1, c_2 and λ. If $U = \nabla f$ in (3.11) for some $f \in C^\infty(M)$; then by the definition $\operatorname{Hess}_f(U, V) = \frac{1}{2}\,(£_{\nabla f}\, g)(U,V)$, we get the *generalized gradient Ricci soliton* equation

$$\operatorname{Hess}_f = -c_1 df \otimes df + c_2 \operatorname{Ric} + \lambda\, g. \tag{3.12}$$

Each equation above is a generalization of the Einstein metric, $\operatorname{Ric} + \lambda\, g = 0$. For different values of c_1, c_2 and λ, (3.11) is a generalization of the Killing equation ($c_1 = c_2 = \lambda = 0$), equation for homotheties ($c_1 = c_2 = 0$), Ricci soliton equation ($c_1 = 0,\ c_2 = -1$) and vacuum near-horizon geometry equation ($c_1 = 1,\ c_2 = 1/2$), such as in [2].

First, we formulate some lemmas.

Lemma 3. *For a weak K-contact manifold, we get*

$$(£_\xi(£_U\, g))(V, \xi) = g(U, V) + g(\nabla_\xi \nabla_\xi U, V) + V g(\nabla_\xi U, \xi)$$

for all smooth vector fields U, V with V orthogonal to ξ.

Proof. This uses the equalities $\nabla_\xi \xi = 0$ and (3.6), and is the same as for [2, Lemma 3.1].

Lemma 4 (see, for example, [2])**.** *Let $(M; g)$ be a Riemannian manifold and $f \in C^\infty(M)$. Then, the following holds for each vector field ξ, V on M:*

$$£_\xi(df \otimes df)(V, \xi) = V(\xi(f))\,\xi(f) + V(f)\,\xi(\xi(f)).$$

Recall that the Ricci curvature of any K-contact manifold satisfies the following condition:

$$\mathrm{Ric}(\xi, U) = 0 \quad (U \in \mathcal{D}). \tag{3.13}$$

Lemma 5. *Let a weak K-contact manifold satisfy* (3.13) *and admit the generalized gradient Ricci soliton structure given by* (3.12). *Then,*

$$\nabla_\xi \nabla f = (\lambda + c_2 \,\mathrm{trace}\ Q)\, \xi - c_1 \xi(f)\, \nabla f.$$

Proof. This uses (3.7) and (3.12) and is analogous to the proof of [2, Lemma 3.3]. By (3.7) and (3.13), we get

$$\lambda\, \eta(V) + c_2\, \mathrm{Ric}(\xi, V) = (\lambda + c_2 \,\mathrm{trace}\ Q)\, \eta(V). \tag{3.14}$$

Using (3.12) and (3.14), we get

$$\mathrm{Hess}_f(\xi, V) = -c_1 \xi(f)\, g(\nabla f, V) + (\lambda + c_2 \,\mathrm{trace}\ Q)\, \eta(V). \tag{3.15}$$

Thus, (3.15) and the condition (3.12) for the Hessian complete the proof.

The next theorem generalizes [2, Theorem 3.1].

Theorem 5. *Let a weak K-contact manifold with* trace Q = *const satisfy the generalized gradient Ricci soliton equation* (3.12) *with* $c_1(\lambda + c_2 \,\mathrm{trace}\ Q) \neq -1$. *Suppose that the condition* (3.13) *is true. Then,* $f = const$. *Furthermore, if* $c_2 \neq 0$, *then the manifold is an Einstein one.*

Proof. Let $V \in \mathcal{D}$. Then, by Lemma 3 with $U = \nabla f$, we obtain

$$2\, (\pounds_\xi(\mathrm{Hess}_f))(V, \xi) = V(f) + g(\nabla_\xi \nabla_\xi \nabla f, V) + V g(\nabla_\xi \nabla f, \xi). \tag{3.16}$$

Using Lemma 5 in (3.16) and the properties $\nabla_\xi \xi = 0$ and $g(\xi, \xi) = 1$ yields

$$\begin{aligned} 2\, (\pounds_\xi(\mathrm{Hess}_f))(V, \xi) &= V(f) + (\lambda + c_2 \,\mathrm{trace}\ Q)\, g(\nabla_\xi \xi, V) \\ &\quad - c_1 g(\nabla_\xi(\xi(f) \nabla f), V) + (\lambda + c_2 \,\mathrm{trace}\ Q)\, V(g(\xi, \xi)) - c_1 V(\xi(f)^2) \\ &= V(f) - c_1 g(\nabla_\xi(\xi(f) \nabla f), V) - c_1 V(\xi(f)^2). \end{aligned} \tag{3.17}$$

Using Lemma 5 with $V \in \mathcal{D}$, from (3.17), it follows that

$$2\, (\pounds_\xi(\mathrm{Hess}_f))(V, \xi) = V(f) - c_1 \xi(\xi(f))\, V(f) + c_1^2 \xi(f)^2 V(f) - c_1 V(\xi(f)^2). \tag{3.18}$$

Since ξ is a Killing vector field, $\pounds_\xi\, g = 0$, which implies that $\pounds_\xi \mathrm{Ric} = 0$. Using the above fact and applying the Lie derivative to (3.12), gives

$$2\, (\pounds_\xi(\mathrm{Hess}_f))(V, \xi) = -2\, c_1 (\pounds_\xi(df \otimes df))(V, \xi). \tag{3.19}$$

Using (3.18), (3.19) and Lemma 4, we obtain

$$V(f)(1 + c_1 \xi(\xi(f)) + c_1^2\, \xi(f)^2) = 0. \tag{3.20}$$

By Lemma 5, we get

$$c_1 \xi(\xi(f)) = c_1\, \xi(g(\xi, \nabla f)) = c_1 g(\xi, \nabla_\xi \nabla f) = c_1(\lambda + c_2 \,\mathrm{trace}\ Q) - c_1^2\, \xi(f)^2. \tag{3.21}$$

Using (3.20) in (3.21), we get that $V(f)(c_1(\lambda + c_2 \operatorname{trace} Q) + 1) = 0$. This implies that $V(f) = 0$, provided that $c_1(\lambda + c_2 \operatorname{trace} Q) + 1 \neq 0$. Hence, ∇f is parallel to ξ. Taking the covariant derivative of $\nabla f = \xi(f)\,\xi$ and using (3.4), we obtain

$$g(\nabla_Z \nabla f, V) = Z(\xi(f))\,\eta(V) - \xi(f)\,g(\phi Z, V), \quad Z, V \in \mathfrak{X}_M.$$

From this, by symmetry of Hess_f, i.e., $g(\nabla_Z \nabla f, V) = g(\nabla_V \nabla f, Z)$, we get that $\xi(f)\,g(\phi Z, V) = 0$. For $V = \phi Z$ for some $Z \neq 0$, since $g(\phi Z, \phi Z) > 0$, we get $\xi(f) = 0$; so, $\nabla f = 0$, i.e., $f = const$. Thus, from (3.12) and $c_2 \neq 0$, we conclude that the manifold is an Einstein manifold.

Remark 5. The following generalization of the gradient Ricci soliton equation was given in [3]:

$$\operatorname{Hess}_{f_1} = -c_1 df_2 \otimes df_2 + c_2 \operatorname{Ric} + \lambda\, g \tag{3.22}$$

for some functions $f_1, f_2 \in C^\infty(M)$ and real c_1, c_2 and λ. For $f_1 = f_2$, (3.22) reduces to (3.12).

Let a weak K-contact manifold with trace $Q = const$ satisfy (3.13) and admit the generalized Ricci soliton structure given by (3.22) with $c_1(\lambda + c_2 \operatorname{trace} Q) \neq -1$. Then, similar to Lemma 5, we get

$$\nabla_\xi \nabla f_1 = (\lambda + c_2 \operatorname{trace} Q)\,\xi - c_1 \xi(f_2)\,\nabla f_2. \tag{3.23}$$

Using (3.23) and Lemmas 3 and 4, and, slightly modifying the proof of Theorem 5, we find that the vector field ∇f is parallel to ξ, where $f = f_1 + c_1(\lambda + c_2 \operatorname{trace} Q)\, f_2$. Thus, $df = 0$, i.e.,

$$df_1 = -c_1(\lambda + c_2 \operatorname{trace} Q)\, df_2.$$

Using this in (3.22) and denoting $a := \lambda + c_2 \operatorname{trace} Q$, we get

$$-c_1 a \operatorname{Hess}_{f_2} = -c_1\, df_2 \otimes df_2 + c_2 \operatorname{Ric} + \lambda\, g. \tag{3.24}$$

Then we obtain the following assertions (with three cases) that generalizes Theorem 5.

1. If $c_1 a \neq 0$, then (3.24) reduces to $\operatorname{Hess}_{f_2} = \frac{1}{a}\, df_2 \otimes df_2 - \frac{c_2}{c_1 a} \operatorname{Ric} - \frac{\lambda}{c_1 a}\, g$. By Theorem 5, if $c_1 a \neq -1$, then $f_2 = const$; moreover, if $c_2 \neq 0$, then (M, g) is an Einstein manifold.

2. If $a = 0$ and $c_1 \neq 0$, then (3.24) reduces to

$$0 = c_2 \operatorname{Ric} - c_1\, df_2 \otimes df_2 + \lambda\, g.$$

If $c_2 \neq 0$ and $f_2 \neq const$, then we get a gradient quasi-Einstein manifold. The concept of a quasi-Einstein manifold was introduced in [13] based on the condition that $\operatorname{Ric}(U, V) = a\, g(U, V) + b\,\mu(U)\,\mu(V)$ for all vector fields U, V, where a and $b \neq 0$ are real scalars and μ is a 1-form of unit norm.

3. If $c_1 = 0$, then (3.24) reduces to $0 = c_2 \operatorname{Ric} + \lambda\, g$, and for $c_2 \neq 0$, we get an Einstein manifold.

4. Conclusion

It is shown that the weak K-contact structure is a useful tool for studying unit Killing vector fields on Riemannian manifolds, and that some results for K-contact manifolds can be extended to the case of weak K-contact manifolds. Inspired by [5, Theorems 7.1 and 7.2], the following question can be posed: is the condition (3.10), or the weaker condition (3.7), sufficient for a weak contact metric manifold to be weak K-contact? To answer the question, some well-known results for contact metric manifolds, such as [5, Proposition 7.1], must be generalized for weak contact metric manifolds. In conclusion, we pose the following questions (about "weak" analogues of results mentioned in [2, Remark 3.2]): is a compact weak K-contact Einstein manifold a Sasakian manifold? Thus, is a compact weak K-contact manifold admitting a generalized Ricci soliton structure a Sasakian manifold?

Conflict of interest

The author declares that there is no conflict of interest.

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