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Research article

Eigenvalues of the bi-Xin-Laplacian on complete Riemannian manifolds

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Abstract: The clamped plate problem describes the vibration of a clamped plate in the classical elastic mechanics, and the Xin-Laplacian is an important elliptic operator for understanding the geometric structure of translators of mean curvature flow(MCF for short). In this article, we investigate the clamped plate problem of the bi-Xin-Laplacian on Riemannian manifolds isometrically immersed in the Euclidean space. On one hand, we obtain some eigenvalue inequalities of the bi-Xin-Laplacian on some important Riemannian manifolds admitting some special functions. Let us emphasize that, this class of manifolds contains some interesting examples: Cartan-Hadamard manifolds, some types of warp product manifolds and homogenous spaces. On the other hand, we also consider the eigenvalue problem of the bi-Xin-Laplacian on the cylinders and obtain an eigenvalue inequality. In particular, we can give an estimate for the lower order eigenvalues on the cylinders.

Keywords: bi-Xin-Laplacian; eigenvalue; Riemannian manifold; Cartan-Hadamard manifold; homogenous manifold; cylinder
Mathematics Subject Classification: 35P15, 53C40

1. Introduction

In elastic mechanics, a fundamental theme is to describe vibrations of a clamped plate. To this end, we usually consider a clamped plate problem of bi-Laplacian Δ^2 as follows:

$$\begin{cases} \Delta^2 u = \Lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \boldsymbol{n}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ, $\Omega$and $\boldsymbol{n}$ denote the Laplacian, the bounded domain on the Euclidean space $\mathbb{R}^n$, and normal vector field to the boundary $\partial\Omega$, respectively. In 1956, Payne, Pólya and Weinberger [1] considered eigenvalue problem (1.1) of biharmonic operator Δ^2 and established an interesting universal inequality as follows:

$$\Lambda_{k+1} - \Lambda_k \leq \frac{8(n+2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\Lambda_i. \tag{1.2}$$

In 1984, Hile and Yeh [2] improved (1.2) to the following:

$$\sum_{i=1}^{k}\frac{\Lambda_i^{1/2}}{\Lambda_{k+1}-\Lambda_i} \geq \frac{n^2k^{3/2}}{8(n+2)}\left(\sum_{i=1}^{k}\Lambda_i\right)^{-1/2}, \tag{1.3}$$

by virtue of an improved techniques due to Hile and Protter [3]. In 1990, Hook [4], Chen and Qian [5] also studied eigenvalue problem (1.1), and they independently established

$$\frac{n^2k^2}{8(n+2)} \leq \left[\sum_{i=1}\frac{\Lambda_i^{1/2}}{\Lambda_{k+1}-\Lambda_i}\right]\sum_{i=1}^{k}\Lambda_i^{1/2}. \tag{1.4}$$

In 2006, Cheng and Yang [6] made an affirmative answer to Ashbaugh's problem proposed in a survey paper [7], where he asked whether one can establish eigenvalue inequalities for the clamped plate problem which are analogous inequalities of Yang in the case of the Dirichlet eigenvalue problem of the Laplace operator. More precisely, they proved

$$\Lambda_{k+1} - \frac{1}{k}\sum_{i=1}^{k}\Lambda_i \leq \left[\frac{8(n+2)}{n^2}\right]^{1/2}\frac{1}{k}\sum_{i=1}^{k}\left[\Lambda_i\left(\Lambda_{k+1}-\Lambda_i\right)\right]^{1/2}, \tag{1.5}$$

which improved a universe bound established by Payne, Pólya and Weinberger in [1]. In 2007, Xia and Wang [8] made an important attribution to the universal inequality of Yang type. More concretely, they proved

$$\sum_{i=1}^{k}\left(\Lambda_{k+1}-\Lambda_i\right)^2 \leqslant \frac{8}{n}\left(\sum_{i=1}^{k}\left(\Lambda_{k+1}-\Lambda_i\right)\Lambda_i^{3/2}\right)^{1/2}\left(\sum_{i=1}^{k}\left(\Lambda_{k+1}-\Lambda_i\right)\Lambda_i^{1/2}\right)^{1/2}.$$

Next, we suppose that $X : \mathcal{M}^n \to \mathbb{R}^{n+p}$ is an n-dimensional, isometrically immersed submanifold with mean curvature H. In 2011, Wang and Xia [9] proved

$$\begin{aligned}\sum_{i=1}^{k}\left(\Lambda_{k+1}-\Lambda_i\right)^2 \leq &\frac{4}{n}\left\{\sum_{i=1}^{k}\left(\Lambda_{k+1}-\Lambda_i\right)^2\left[\left(\frac{n}{2}+1\right)\Lambda_i^{1/2}+C_0\right]\right\}^{1/2}\\ &\times\left\{\sum_{i=1}^{k}\left(\Lambda_{k+1}-\Lambda_i\right)\left(\Lambda_i^{1/2}+C_0\right)\right\}^{1/2},\end{aligned} \tag{1.6}$$

where $C_0 = \frac{1}{4}\inf_{\sigma\in\Pi}\max_{\overline{\Omega}}\left(n^2H^2\right)$, and Π represents a set of all isometric immersions from $\mathcal{M}^n$ into $\mathbb{R}^{n+p}$. In 2013, Wang and Xia [10] considered the fourth order Steklov eigenvalue problems on the compact Riemannian manifolds and obtained some interesting lower bounds of the first non-zero eigenvalue.

In what follows, we assume that $\nu \in \mathbb{R}^{n+p}$ is a vector field defined on $\mathcal{M}^n$ with $|\nu|_{g_0}$ = constant, where $|\cdot|_{g_0}^2$ is a Euclidean norm with respect to the standard inner product $\langle\cdot,\cdot\rangle_{g_0}$ and g_0 is a Euclidean metric on $\mathbb{R}^{n+p}$. Also, we use the following notations: $\langle\cdot,\cdot\rangle_g$, $|\cdot|_g^2$, ∇, Δ, div and $\nu^\top$ to denote the Riemannian inner product associated with induced metric g, norm with respect to the inner product $\langle\cdot,\cdot\rangle_g$, gradient, Laplacian, divergence on $\mathcal{M}^n$ and the projection of vector field ν on the tangent bundle of $\mathcal{M}^n$, respectively. Recently, Xin introduced [11] an elliptic differential operator defined by

$$\mathfrak{L}_\nu(\cdot) = \Delta(\cdot) + \langle \nu, \nabla(\cdot)\rangle_{g_0} = e^{-\langle \nu, X\rangle_{g_0}}\mathrm{div}(e^{\langle \nu, X\rangle_{g_0}}\nabla(\cdot)), \tag{1.7}$$

which is called the Xin-Laplacian. We refer the reader to the excellent survey [12] for detailed introduction to this operator, where Xin reviewed briefly some important progress on singularities of MCF. We note that the Xin-Laplacian is similar to the Witten Laplacian that appeared in [13–18] and $\mathfrak{L}$ operator introduced by Colding and Minicozzi in [19] (or see [20]), and all of those operators play a critical role in understanding the singularities of geometric flows. In particular, Xin-Laplacian is a very important elliptic differential operator for understanding the geometric structure of translator of MCF. See [11, 21, 22] and the references therein. Let us emphasize that, from a more analytic perspective, just like the Witten Laplacian and $\mathcal{L}$ operator, it is of great importance to prove some analytic properties of the Xin-Laplacian. For example, we can prove some mean value inequalities and Liouville properties by maximum principle in terms of the Xin-Laplacian. Of course, one can also consider Gauss maps, heat kernel associated with the Xin-Laplacian and so on. It is the main task of this paper to study the following eigenvalue problem of the bi-Xin-Laplacian on the complete Riemannian manifold $\mathcal{M}^n$:

$$\begin{cases} \mathfrak{L}_\nu^2 u = \Gamma u, & \text{in} \quad \Omega, \\ u = \frac{\partial u}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.8}$$

where $\boldsymbol{n}$ denotes the outward unit normal to the boundary $\partial\Omega$. Let Γ_k be the k^{th} eigenvalue according to the eigenfunction u_k. Moreover, we always assume that the boundary $\partial\Omega$ of bounded domain Ω is piecewise smooth to avoid some possible technical difficulties. Clearly, eigenvalue problem (1.8) has discrete and real spectrum satisfying the following connections:

$$0 \le \Gamma_1 \le \Gamma_2 \le \cdots \nearrow +\infty,$$

where each Γ_i has finite multiplicity which is repeated according to its multiplicity. Recently, in the separate papers [23, 24], the second author investigated eigenvalue problem (1.8) of the bi-Xin-Laplacian on the complete Riemannian submanifolds isometrically embedded into $\mathbb{R}^{n+p}$ with arbitrary codimension. Specially, the author obtained some universal bounds in the case of translating solitons. Motivated by the works done in [8, 9, 17], the present paper continues to contribute on the spectra of bi-Xin-Laplacian on the Riemannian manifolds. Essentially, comparing the cases of Laplacian or its weighted version, some of our results are intrinsic without considering the extra term.

The remainder of the paper is structured as follows. In Section 2, we recall some known results and prove some key technical lemmas. In Section 3, we investigate the eigenvalues of bi-Xin-Laplacian on the manifolds admitting some special functions. In fact, many important examples satisfy those conditions in Theorem 3.1 and Theorem 3.5. As another important and interesting manifold, we discuss the the eigenvalues on cylinders in Section 4.

2. Preliminaries

In this section, we would like to prove several key auxiliary lemmas.

Let $x_1, x_2, \cdots, x_{n+1}$ be $(n+1)$ coordinate functions defined on the Euclidean space $\mathbb{R}^{n+1}$. Then, for any point $x \in \Omega$ (cf. [25, 26]),

$$\sum_{p=1}^{n+1} \langle \nabla x^p, \nabla u_i \rangle_g^2 = |\nabla u_i|_g^2. \tag{2.1}$$

A direct calculation shows that (cf. [23]),

$$\sum_{\alpha=1}^{n+1} \langle \nabla x_\alpha, \nu \rangle_{g_0}^2 = |\nu^\top|_{g_0}^2. \tag{2.2}$$

By making use of Cauchy-Schwarz inequality and (2.2), we have

$$\sum_{\alpha=1}^{n+1} \langle \nabla x_\alpha, \nabla u_i \rangle_g \langle \nabla x_\alpha, \nu \rangle_{g_0} \leq |\nabla u_i|_g \, |\nu^\top|_{g_0}. \tag{2.3}$$

Lemma 2.1. *Let w be a smooth function defined on $\mathcal{M}^n$, then*

$$\langle \nu, \nabla w \rangle_{g_0} \leq |\nu^\top|_{g_0} |\nabla w|_g. \tag{2.4}$$

Proof. We choose a new coordinate system $\bar{x} = \left(\bar{x}^1, \cdots, \bar{x}^{n+p}\right)$ of $\mathbb{R}^{n+p}$ given by $x - x(P) = \bar{x}A$, such that, at the point P, $\left(\frac{\partial}{\partial \bar{x}^1}\right)_P, \cdots, \left(\frac{\partial}{\partial \bar{x}^n}\right)_P$ span a tangent space $T_P\mathcal{M}^n$ and $\langle \frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial \bar{x}^j} \rangle_g = \delta_{ij}$, where $A = \left(a_\beta^\alpha\right) \in O(n+p)$ is an orthogonal matrix of $(n+p) \times (n+p)$ type. Let $\nu = \sum_{\theta=1}^{n+p} \nu_\theta \frac{\partial}{\partial \bar{x}^\theta} \in \mathbb{R}^{n+p}$, and $g_{0\alpha\beta} = \langle \frac{\partial}{\partial \bar{x}^\alpha}, \frac{\partial}{\partial \bar{x}^\beta} \rangle_{g_0}$. Let $w \in C^\infty(\mathcal{M}^n)$, and $\bar{x} = \left(\bar{x}^1, \cdots, \bar{x}^n\right)$ be a local coordinate system. On one hand, under this coordinate system, a straightforward computation shows that

$$\nu^\top = \sum_{\theta=1}^{n} \nu_\theta \frac{\partial}{\partial \bar{x}^\theta}, \tag{2.5}$$

and

$$\begin{aligned}
\langle \nu, \nabla w \rangle_{g_0}^2 &= \left(\sum_{i,j=1}^{n} \nu_i \frac{\partial w}{\partial \bar{x}^j} \langle \frac{\partial}{\partial \bar{x}^i}, \frac{\partial}{\partial \bar{x}^j} \rangle_{g_0} \right)^2 \\
&= \left(\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{n+p} \nu_i \frac{\partial w}{\partial \bar{x}^j} \langle \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial}{\partial x^\alpha}, \frac{\partial x^\beta}{\partial \bar{x}^j} \frac{\partial}{\partial x^\beta} \rangle_{g_0} \right)^2 \\
&= \left(\sum_{i,j=1}^{n} \sum_{\alpha=1}^{n+p} \nu_i \frac{\partial w}{\partial \bar{x}^j} \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\alpha}{\partial \bar{x}^j} \right)^2 \\
&= \left(\sum_{i,j=1}^{n} \sum_{\alpha,\beta,\gamma=1}^{n+p} a_\beta^\alpha a_\gamma^\alpha \nu_i \frac{\partial w}{\partial \bar{x}^j} \frac{\partial \bar{x}^\beta}{\partial \bar{x}^i} \frac{\partial \bar{x}^\gamma}{\partial \bar{x}^j} \right)^2.
\end{aligned} \tag{2.6}$$

On the other hand, we have

$$
\begin{aligned}
\left(\sum_{i,j=1}^{n}\sum_{\alpha,\beta,\gamma=1}^{n+p} a_{\beta}^{\alpha}a_{\gamma}^{\alpha}\nu_i\frac{\partial w}{\partial\bar{x}^j}\frac{\partial\bar{x}^{\beta}}{\partial\bar{x}^i}\frac{\partial\bar{x}^{\gamma}}{\partial\bar{x}^j}\right)^2 &= \left(\sum_{i,j=1}^{n}\sum_{\alpha,\beta,\gamma=1}^{n+p} a_{\beta}^{\alpha}a_{\gamma}^{\alpha}\nu_i\frac{\partial w}{\partial\bar{x}^j}\delta_{\beta i}\delta_{\gamma j}\right)^2 \\
&= \left(\sum_{i,j=1}^{n}\sum_{\alpha=1}^{n+p} a_{i}^{\alpha}a_{j}^{\alpha}\nu_i\frac{\partial w}{\partial\bar{x}^j}\right)^2 \\
&= \left[\sum_{i,j=1}^{n}\nu_i\frac{\partial w}{\partial\bar{x}^j}\left(\sum_{\alpha=1}^{n+p} a_{i}^{\alpha}a_{j}^{\alpha}\right)\right]^2 \\
&= \left(\sum_{i=1}^{n}\nu_i\frac{\partial w}{\partial\bar{x}^i}\right)^2.
\end{aligned}
\tag{2.7}
$$

From (2.6) and (2.7), it holds that

$$
\langle\nu,\nabla w\rangle_{g_0}^2 = \left(\sum_{i=1}^{n}\nu_i\frac{\partial w}{\partial\bar{x}^i}\right)^2. \tag{2.8}
$$

Furthermore, Cauchy-Schwarz inequality implies that

$$
\left(\sum_{\theta=1}^{n}\nu_{\theta}\frac{\partial w}{\partial\bar{x}^{\theta}}\right)^2 \le \left(\sum_{\theta=1}^{n}\nu_{\theta}^2\right)\cdot\sum_{\theta=1}^{n}\left(\frac{\partial w}{\partial\bar{x}^{\theta}}\right)^2. \tag{2.9}
$$

Combining (2.8), (2.5) and (2.9), we get (2.4). This ends the proof.

In addition, we need the following lemma, which was proved in [23].

Lemma 2.2. (General Formula) *Let $\mathcal{M}^n$ be an n-dimensional, complete, Riemannian manifold equipped with smooth metric g, and Ω a bounded domain on $\mathcal{M}^n$. Assume that h is a function defined on $\bar{\Omega}$, i.e., $\Omega\cup\partial\Omega$), with $h\in C^4(\Omega)\cap C^3(\partial\Omega)$, and*

$$
\begin{cases}
\mathfrak{L}_{\nu}^2u_i = \Gamma_iu_i, & \text{in } \Omega,\\
u_i = \frac{\partial u_i}{\partial\boldsymbol{n}} = 0, & \text{on } \partial\Omega,\\
\int_{\Omega}u_iu_je^{\langle\nu,X\rangle_{g_0}}dv = \delta_{ij}, & \forall i,j=1,2,\ldots,
\end{cases}
$$

where $\boldsymbol{n}$ denotes the outward normal vector field to the boundary $\partial\Omega$. For any $k\in\mathbb{Z}^+$ and any $\delta>0$, it holds that

$$
\begin{aligned}
\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_{\Omega}u_i^2|\nabla h|_g^2e^{\langle\nu,X\rangle_{g_0}}dv \le{}& \sum_{i=1}^{k}\delta(\Gamma_{k+1}-\Gamma_i)^2\int_{\Omega}\Psi_i(h)e^{\langle\nu,X\rangle_{g_0}}dv \\
&+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_{\Omega}\Theta_i(h)e^{\langle\nu,X\rangle_{g_0}}dv,
\end{aligned}
\tag{2.10}
$$

where

$$
\Psi_i(h) = -2|\nabla h|_g^2u_i\mathfrak{L}_{\nu}u_i + 4u_i\mathfrak{L}_{\nu}h\langle\nabla h,\nabla u_i\rangle_g + 4\langle\nabla h,\nabla u_i\rangle_g^2 + u_i^2(\mathfrak{L}_{\nu}h)^2, \tag{2.11}
$$

and

$$\Theta_i(h) = \left(\langle \nabla h, \nabla u_i \rangle_g + \frac{u_i \mathfrak{L}_\nu h}{2} \right)^2. \tag{2.12}$$

Lemma 2.3. *Under the same assumption of Lemma* 2.2*, we have*

$$\int_\Omega |\nabla u_i|_g^2 \, e^{\langle \nu, X \rangle_{g_0}} dv \le \Gamma_i^{1/2}, \tag{2.13}$$

and

$$-\int_\Omega u_i \langle \nabla u_i, \nu \rangle_{g_0} \, e^{\langle \nu, X \rangle_{g_0}} dv \le C_1 \Gamma_i^{1/4}, \tag{2.14}$$

where $C_1 = \max_{\overline{\Omega}} |\nu^\top|_{g_0}$.

Proof. Utilizing Cauchy-Schwarz inequality, the divergence theorem and the condition $\frac{\partial u_i}{\partial \mathbf{n}} = 0$ on $\partial\Omega$, since $\mathcal{L}_\nu$ is self-adjoint with respect to the weighted measure $e^{\langle \nu, X \rangle_{g_0}} dv$, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u_i|_g^2 \, e^{\langle \nu, X \rangle_{g_0}} dv &= -\int_\Omega u_i \mathcal{L}_\nu u_i e^{\langle \nu, X \rangle_{g_0}} dv \\
&\le \left\{ \int_\Omega u_i^2 e^{\langle \nu, X \rangle_{g_0}} dv \right\}^{1/2} \left\{ \int_\Omega (\mathcal{L}_\nu u_i)^2 \, e^{\langle \nu, X \rangle_{g_0}} dv \right\}^{1/2} = \Gamma_i^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
-\int_\Omega u_i \langle \nabla u_i, \nu \rangle_{g_0} \, e^{\langle \nu, X \rangle_{g_0}} dv &\le \int_\Omega |u_i|_a \, |\nabla u_i|_g \, |\nu^\top|_{g_0} e^{\langle \nu, X \rangle_{g_0}} dv \\
&\le C_1 \left(\int_\Omega u_i^2 e^{\langle \nu, X \rangle_{g_0}} dv \right)^{1/2} \left(\int_\Omega |\nabla u_i|_g^2 \, e^{\langle \nu, X \rangle_{g_0}} dv \right)^{1/2} \\
&\le C_1 \Gamma_i^{1/4}.
\end{aligned}$$

Thus, we finish the proof of this lemma.

Next, we assume that $\mathcal{M}^n$ is an n-dimensional unit round cylinder $\mathbb{R}^{n-m} \times \mathbb{S}^m(1)$ and denote the position vector of $\mathbb{R}^{n-m} \times \mathbb{S}^m(1)$ in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ by

$$\mathbf{x} = (\mathbf{v}, \ \mathbf{w}) = (x^1, x^2, \ldots, x^{n-m}, x^{n-m+1}, x^{n-m+2} \cdots, x^n, x^{n+1}),$$

where $\mathbf{v} = (x^1, x^2, \ldots, x^{n-m})$, $\mathbf{w} = (x^{n-m+1}, x^{n-m+2} \cdots, x^n, x^{n+1})$. A simple calculation shows that

$$\sum_{\alpha=n-m+1}^{n+1} (x^\alpha)^2 = 1, \ \sum_{\alpha=1}^{n+1} |\nabla x^\alpha|_g^2 = n. \tag{2.15}$$

By (2.15), it is easy to verify four expressions as follows:

$$\sum_{\beta=n-m+1}^{n+1} |\nabla x^\beta|_g^2 = -\sum_{\alpha=1}^{n+1} x^\alpha \Delta x^\alpha = m. \tag{2.16}$$

Noticing that equation (2.16) implies that $n^2 H^2 = m^2$, the following lemmas are some immediately consequences of Lemma 3.2 and Lemma 3.3 in [23].

Lemma 2.4. *Let $x_1, x_2, \ldots, x_{n+1}$ be $(n+1)$ coordinate functions of $\mathbb{R}^{n+1}$. For any $i = 1, 2, \cdots k$ and $\alpha = 1, 2, \cdots, n+1$, where k is an arbitrary positive integer, let*

$$\widehat{\Psi}_{i,\alpha} := \int_\Omega \Psi_i(x_\alpha) e^{\langle \nu, X\rangle_{g_0}} dv,$$

where function Ψ_i is given by (2.11). *Then,*

$$\begin{aligned}\sum_{\alpha=1}^{n+1} \widehat{\Psi}_{i,\alpha} \leq & \int_\Omega \left[-2nu_i \mathfrak{L}_\nu u_i + 4\,|\nabla u_i|_g^2 + u_i^2 \left(m^2 + |\nu^\top|_{g_0}^2\right)\right] e^{\langle \nu, X\rangle_{g_0}} dv \\ & + 4\Gamma_i^{1/4} \left(\int_\Omega u_i^2 |\nu^\top|_{g_0}^2 e^{\langle \nu, X\rangle_{g_0}} dv\right)^{1/2}.\end{aligned} \tag{2.17}$$

Lemma 2.5. *Let $x_1, x_2, \ldots, x_{n+1}$ be $(n+1)$the standard coordinate functions of $\mathbb{R}^{n+1}$. For any $i = 1, 2, \cdots k$ and $\alpha = 1, 2, \cdots, n+1$, where k is an arbitrary positive integer, let*

$$\widehat{\Theta}_{i,\alpha} := \int_\Omega \Theta_i(x_\alpha) e^{\langle \nu, X\rangle_{g_0}} dv,$$

where function Θ_i is given by (2.12). *Then,*

$$\sum_{\alpha=1}^{n+1} \widehat{\Theta}_{i,\alpha} \leq \int_\Omega \left[|\nabla u_i|_g^2 + \frac{1}{4} u_i^2 \left(m^2 + |\nu^\top|_{g_0}^2\right)\right] e^{\langle \nu, X\rangle_{g_0}} dv + \Gamma_i^{1/4} \left[\int_\Omega (u_i |\nu^\top|_{g_0})^2 e^{\langle \nu, X\rangle_{g_0}} dv\right]^{1/2}. \tag{2.18}$$

3. Eigenvalues on manifolds admitting special functions

In this section, we consider the eigenvalue problem on some manifolds admitting certain special function. Next, let us establish the first theorem.

Theorem 3.1. *Assume that $\mathcal{M}^n$ is an n-dimensional, isometrically immersed, complete submanifold of the Euclidean space $\mathbb{R}^{n+p}$ and g is a induced metric from the immersed map $X : \mathcal{M}^n \to \mathbb{R}^{n+p}$. Let Ω be a bounded domain on $\mathcal{M}^n$ with piecewise smooth boundary $\partial\Omega$. Provided that there exist a function $\varphi : \Omega \to \mathbb{R}$ and a positive constant D_1 satisfy*

$$|\nabla \varphi|_g = 1, \text{ and } |\Delta \varphi|_a \leq D_1, \tag{3.1}$$

where $|w|_a$ denotes the absolute value of w. Then, the eigenvalues Γ_k of the eigenvalue problem (1.8)*, where $k = 1, 2, \cdots$, satisfy*

$$\begin{aligned}\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq & \left\{\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \left[6\Gamma_i^{1/2} + 4\,(C_1 + D_1)\,\Gamma_i^{1/4} + (C_1 + D_1)^2\right]\right\}^{1/2} \\ & \times \left\{\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \left[4\Gamma_i^{1/2} + 4\,(C_1 + D_1)\,\Gamma_i^{1/4} + (C_1 + D_1)^2\right]\right\}^{\frac{1}{2}},\end{aligned} \tag{3.2}$$

where $C_1 = \max_{\overline{\Omega}} |\nu^\top|_{g_0}$.

Remark 3.2. *We further suppose that Ricci curvature of $\mathcal{M}^n$ is bounded from below by a uniform nonnegative constant $-(n-1)\kappa^2(\kappa \geq 0)$, i.e.,* $\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)\kappa^2, \kappa \geq 0$. *If there exists a function $\varphi \in C^\infty(\mathcal{M}^n)$ such that $|\nabla\varphi|_g = 1$, then, by Remark 3.6 in* [27], $|\Delta\varphi|_a \leq (n-1)\kappa^2$. *Let $\xi : [0, +\infty) \to M$ be a normal geodesic ray, namely a unit speed geodesic with $d(\xi(s), \xi(t)) = t - s$ for any $t > s > 0$. Then, Busemann function b_ξ w.r.t. geodesic ray ξ is defined as $b_\xi(q) := \lim_{t\to+\infty}(d(q, \xi(t)) - t)$. Under the assumption that $\mathcal{M}^n$ is an Hadamard manifold, b_ξ is a convex function of class C^2 with $|\nabla b_\xi|_g \equiv 1$ and these conditions characterize Busemann functions. Here, we refer the reader to* [28, 29] *for more detailed information. Obviously, Busemann functions defined on Cartan-Hadamard manifolds, whose Ricci curvature is bounded from below, satisfy those conditions in Theorem* 3.1.

Remark 3.3. *We assume that $\mathcal{N}^{n-1}$ is complete Riemannian manifold with Ricci curvature bounded below and $\mathcal{M}^n = \mathcal{N}^{n-1} \times \mathbb{R}$ is the product of $\mathcal{N}$ and $\mathbb{R}$ with the product metric, and then the function $f : \mathcal{M}^n \to \mathbb{R}$ given by $f(p, t) = t$ satisfies the conditions of Theorem* 3.1.

Remark 3.4. *Let $\mathcal{M}^n = \mathbb{R} \times \mathcal{N}^{n-1}$ be an n-dimensional complete manifold with warped product metric $ds^2_{\mathcal{M}} = dt^2 + \exp(2t)ds^2_{\mathcal{N}}$, where $\mathcal{N}^{n-1}$ is an $(n-1)$-dimensional complete Riemannian manifold with* $\mathrm{Ric}_{\mathcal{N}^{n-1}} \geq 0$. *Then, it is easy to verify that* $\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)$. *We refer the readers to* [30] *for details. Therefore, the function $\varphi : \mathcal{M}^n \to \mathbb{R}$ given by $\varphi(p, t) = t$ satisfies conditions $|\nabla\varphi|_g = 1$ and $|\Delta\varphi|_a \leq n - 1$.*

Proof of Theorem 3.1. Substituting $h = \varphi$ into (2.10), we get

$$\begin{aligned}
&\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \int_\Omega u_i^2 |\nabla\varphi|_g^2 e^{\langle \nu, X\rangle_{g_0}} dv \\
&\leq \sum_{i=1}^{k} \delta (\Gamma_{k+1} - \Gamma_i)^2 \int_\Omega \Big(-2|\nabla\varphi|_g^2 u_i \mathfrak{L}_\nu u_i + 4u_i \mathfrak{L}_\nu \varphi \langle \nabla\varphi, \nabla u_i\rangle_g \\
&\quad +4 \langle \nabla\varphi, \nabla u_i\rangle_g^2 + u_i^2 (\mathfrak{L}_\nu\varphi)^2 \Big) e^{\langle \nu, X\rangle_{g_0}} dv \\
&\quad + \sum_{i=1}^{k} \frac{(\Gamma_{k+1} - \Gamma_i)}{\delta} \int_\Omega \left(\langle \nabla\varphi, \nabla u_i\rangle_g + \frac{u_i \mathfrak{L}_\nu\varphi}{2} \right)^2 e^{\langle \nu, X\rangle_{g_0}} dv,
\end{aligned} \tag{3.3}$$

where δ is any positive constant. According to (2.4), (3.1) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\mathfrak{L}_\nu\varphi \langle \nabla\varphi, \nabla u_i\rangle_g &= (\Delta\varphi + \langle \nu, \nabla\varphi\rangle_{g_0}) \langle \nabla\varphi, \nabla u_i\rangle_g \\
&\leq |\Delta\varphi|_a |\nabla\varphi|_g |\nabla u_i|_g + |\nabla\varphi|_g^2 |\nu^\top|_{g_0} |\nabla u_i|_g \\
&\leq (C_1 + D_1) |\nabla u_i|_g ,
\end{aligned} \tag{3.4}$$

$$(\mathfrak{L}_\nu\varphi)^2 = (\Delta\varphi + \langle \nu, \nabla\varphi\rangle_{g_0})^2 \leq (|\Delta\varphi|_a + |\nu^\top|_{g_0} |\nabla\varphi|_g)^2 \leq (C_1 + D_1)^2 , \tag{3.5}$$

and

$$\begin{aligned}
\left(\langle \nabla\varphi, \nabla u_i\rangle_g + \frac{u_i \mathfrak{L}_\nu\varphi}{2} \right)^2 &= \langle \nabla\varphi, \nabla u_i\rangle_g^2 + \langle \nabla\varphi, \nabla u_i\rangle_g u_i \mathfrak{L}_\nu\varphi + \frac{1}{4} u_i^2 (\mathfrak{L}_\nu\varphi)^2 \\
&\leq |\nabla u_i|_g^2 + (C_1 + D_1) \big| \nabla u_i|_g |u_i \big|_a + \frac{1}{4} (C_1 + D_1)^2 u_i^2.
\end{aligned} \tag{3.6}$$

Substituting (3.4)-(3.6) into (3.3), we infer that,

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2$$
$$\leq \sum_{i=1}^{k}\delta(\Gamma_{k+1}-\Gamma_i)^2\int_{\Omega}\Big[-2u_i\mathfrak{L}_\nu u_i+4(C_1+D_1)|\nabla u_i|_g|u_i|_a$$
$$+4|\nabla u_i|_g^2+u_i^2(C_1+D_1)^2\Big]e^{\langle\nu,X\rangle_{g_0}}dv$$
$$+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_{\Omega}\left[|\nabla u_i|_g^2+(C_1+D_1)|\nabla u_i|_g|u_i|_a+\frac{1}{4}(C_1+D_1)^2u_i^2\right]e^{\langle\nu,X\rangle_{g_0}}dv.$$

Furthermore, inserting (2.13) and (2.14) into the above inequality, we derive

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\leq{}&\sum_{i=1}^{k}\delta(\Gamma_{k+1}-\Gamma_i)^2\left[6\Gamma_i^{1/2}+4(C_1+D_1)\Gamma_i^{1/4}+(C_1+D_1)^2\right]\\&+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\left[\Gamma_i^{1/2}+(C_1+D_1)\Gamma_i^{1/4}+\frac{1}{4}(C_1+D_1)^2\right].\end{aligned}$$

Therefore, to get (3.2), the undetermined positive constant δ could be taken by

$$\delta=\frac{\left\{\sum_{i=1}^{n}(\Gamma_{k+1}-\Gamma_i)\left[\Gamma_i^{1/2}+(C_1+D_1)\Gamma_i^{1/4}+\frac{1}{4}(C_1+D_1)^2\right]\right\}^{1/2}}{\left\{\sum_{i=1}^{n}(\Gamma_{k+1}-\Gamma_i)^2\left[6\Gamma_i^{1/2}+4(C_1+D_1)\Gamma_i^{1/4}+(C_1+D_1)^2\right]\right\}^{1/2}}>0,$$

since the eigenvalues are monotonically increasing and the first eigenvalue is simple. This completes the proof of Theorem 3.1.

The second part of this section is to establish the following theorem.

Theorem 3.5. *Assume that Ω is a bounded domain with piecewise smooth boundary in an n-dimensional complete Riemannian manifold $\mathcal{M}^n$ isometrically immersed into the Euclidean space $\mathbb{R}^{n+p}$ via a map $X:\mathcal{M}^n\to\mathbb{R}^{n+p}$. Let Γ_i be the i-th eigenvalue of the problem* (1.8). *If the bounded domain Ω admits an eigenmap $f=(f_1,f_2,\ldots,f_{m+1})$ from Ω to the unit sphere $\mathbb{S}^m(1)$ corresponding to an eigenvalue η, that is,*

$$\Delta f_\alpha=-\eta f_\alpha,\ \ \textit{where}\ \ \alpha=1,\ldots,m+1,\tag{3.7}$$

and

$$\sum_{\alpha=1}^{m+1}f_\alpha^2=1.\tag{3.8}$$

Then,

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\leq{}&\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(6\Gamma_i^{1/2}+4C_1\Gamma_i^{1/4}+\left(C_1^2+\eta\right)\right)\right\}^{1/2}\\&\times\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(4\Gamma_i^{1/2}+4C_1\Gamma_i^{1/4}+\left(C_1^2+\eta\right)\right)\right\}^{1/2},\end{aligned}\tag{3.9}$$

where $\mathbb{S}^m(1)$ is a unit sphere with dimension m and $C_1=\max_{\overline{\Omega}}|\nu^\top|_{g_0}$.

Remark 3.6. *Assume that Riemannian manifold $\mathcal{M}^n$ is compact and homogeneous, and then it admits eigenmaps to some unit spheres for the first positive eigenvalue of the Laplacian (cf.* [31, Corollary 4]*), which means that all conditions presented in Theorem* 3.5 *are satisfied for any compact homogeneous Riemannian manifold.*

Proof of Theorem 3.5. It follows by taking $h = f_\alpha$ in (2.10) and summing over α that

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{\alpha=1}^{m+1}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega u_i^2\,|\nabla f_\alpha|_g^2\,e^{\langle\nu,X\rangle_{g_0}}dv\\
&\leq\sum_{i=1}^{k}\sum_{\alpha=1}^{m+1}\delta\,(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega\Big[-2\,|\nabla f_\alpha|_g^2\,u_i\mathcal{L}_\nu u_i\\
&\quad+4u_i\mathcal{L}_\nu f_\alpha\,\langle\nabla f_\alpha,\nabla u_i\rangle_g+4\,\langle\nabla f_\alpha,\nabla u_i\rangle_g^2+u_i^2\,(\mathcal{L}_\nu f_\alpha)^2\Big]e^{\langle\nu,X\rangle_{g_0}}dv\\
&\quad+\sum_{i=1}^{k}\sum_{\alpha=1}^{m+1}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega\left(\langle\nabla f_\alpha,\nabla u_i\rangle_g+\frac{u_i\mathcal{L}_\nu f_\alpha}{2}\right)^2e^{\langle\nu,X\rangle_{g_0}}dv.
\end{aligned}\tag{3.10}$$

Taking the Laplacian of the equation (3.8) and noticing that $\Delta f_\alpha = -\eta f_\alpha, \alpha = 1,\ldots,m+1$, a straightforward calculation shows that

$$\sum_{\alpha=1}^{m+1}|\nabla f_\alpha|_g^2=\eta.\tag{3.11}$$

Computing the gradient of two sides of equation (3.8), we assert that

$$\sum_{\alpha=1}^{m+1}f_\alpha\nabla f_\alpha=\mathbf{0}.\tag{3.12}$$

Synthesizing (3.11), (3.12), Lemma 2.1 and the Cauchy-Schwarz inequality, we derive

$$\begin{aligned}
\sum_{\alpha=1}^{m+1}\mathfrak{L}_\nu f_\alpha\,\langle\nabla f_\alpha,\nabla u_i\rangle_g&=\sum_{\alpha=1}^{m+1}\left(\Delta f_\alpha+\langle\nu,\nabla f_\alpha\rangle_{g_0}\right)\langle\nabla f_\alpha,\nabla u_i\rangle_g\\
&\leq\sum_{\alpha=1}^{m+1}\left(-\eta f_\alpha\,\langle\nabla f_\alpha,\nabla u_i\rangle_g+|\nabla f_\alpha|_g^2\,|\nu^\top|_{g_0}\,|\nabla u_i|_g\right)\\
&\leq C_1\eta\,|\nabla u_i|_g\,,
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
\sum_{\alpha=1}^{m+1}(\mathfrak{L}_\nu f_\alpha)^2&=\sum_{\alpha=1}^{m+1}\left(\Delta f_\alpha+\langle\nu,\nabla f_\alpha\rangle_{g_0}\right)^2\\
&=\sum_{\alpha=1}^{m+1}\left((\Delta f_\alpha)^2+2\Delta f_\alpha\,\langle\nu,\nabla f_\alpha\rangle_{g_0}+\langle\nu,\nabla f_\alpha\rangle_{g_0}^2\right)\\
&\leq\eta C_1^2+\eta^2,
\end{aligned}\tag{3.14}$$

and

$$
\begin{aligned}
&\sum_{\alpha=1}^{m+1}\left(\langle\nabla f_\alpha,\nabla u_i\rangle_g+\frac{u_i\mathfrak{L}_\nu f_\alpha}{2}\right)^2\\
&=\sum_{\alpha=1}^{m+1}\left[\langle\nabla f_\alpha,\nabla u_i\rangle_g^2+\langle\nabla f_\alpha,\nabla u_i\rangle_g\,u_i\mathfrak{L}_\nu f_\alpha+\frac{1}{4}u_i^2\left(\mathfrak{L}_\nu f_\alpha\right)^2\right]\\
&\leq\eta\,|\nabla u_i|_g^2+C_1\eta\left|\nabla u_i|_g|u_i\right|_a+\frac{1}{4}\left(\eta C_1^2+\eta^2\right)u_i^2.
\end{aligned}\tag{3.15}
$$

Furthermore, substituting (3.7), (3.11)-(3.15) into (3.10), with the aid of (2.13) and (2.14), we arrive at

$$
\begin{aligned}
\eta\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)^2\leq&\sum_{i=1}^{k}\delta\left(\Gamma_{k+1}-\Gamma_i\right)^2\int_\Omega^2\Big[-2\eta u_i\mathfrak{L}_\nu u_i+4C_1\eta u_i\,|\nabla u_i|_g\\
&+4\eta\,|\nabla u_i|_g^2+u_i^2\left(\eta C_1^2+\eta^2\right)\Big]e^{\langle\nu,X\rangle_{g_0}}dv+\sum_{i=1}^{k}\frac{\left(\Gamma_{k+1}-\Gamma_i\right)}{\delta}\\
&\times\int_\Omega\left[\eta\,|\nabla u_i|_g^2+C_1\eta\left|\nabla u_i|_g|u_i\right|_a+\frac{1}{4}\left(\eta C_1^2+\eta^2\right)u_i^2\right]e^{\langle\nu,X\rangle_{g_0}}dv\\
\leq&\sum_{i=1}^{k}\delta\left(\Gamma_{k+1}-\Gamma_i\right)^2\left[6\eta\Gamma_i^{1/2}+4C_1\eta\Gamma_i^{1/4}+\left(\eta C_1^2+\eta^2\right)\right]\\
&+\sum_{i=1}^{k}\frac{\left(\Gamma_{k+1}-\Gamma_i\right)}{\delta}\left[\eta\Gamma_i^{1/2}+C_1\eta\Gamma_i^{1/4}+\frac{1}{4}\left(\eta C_1^2+\eta^2\right)\right].
\end{aligned}\tag{3.16}
$$

Finally, we put

$$
\delta=\frac{\left\{\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)\left[\eta\Gamma_i^{1/2}+C_1\eta\Gamma_i^{1/4}+\frac{1}{4}\left(\eta C_1^2+\eta^2\right)\right]\right\}^{1/2}}{\left\{\sum_{i=1}^{k}\left(\Gamma_{k+1}-\Gamma_i\right)^2\left[6\eta\Gamma_i^{1/2}+4C_1\eta\Gamma_i^{1/4}+\left(\eta C_1^2+\eta^2\right)\right]\right\}^{1/2}}>0,
$$

and insert it into (3.16) to obtain desired inequality (3.9).

4. Eigenvalues on cylinders

In this section, we investigate eigenvalue problem (1.8) on an n-dimensional cylinder $\mathbb{R}^{n-m}\times\mathbb{S}^m$.

Theorem 4.1. *Let $\mathcal{M}^n$ be an n-dimensional cylinder $\mathbb{R}^{n-m}\times\mathbb{S}^m(1)$ equipped with smooth metric $g=\langle,\rangle_{\mathbb{R}^{n-m}}+\langle,\rangle_{\mathbb{S}^m(1)}$ and Ω a bounded domain on this product manifold. Let Γ_i be the i-th eigenvalue of* *the problem* (1.8). *Then,*

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\\
&\leq \frac{4}{n}\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left[\left(\frac{n}{2}+1\right)\Gamma_i^{1/2}+4C_3\Gamma_i^{1/4}+4C_3^2+\frac{m^2}{4}\right]\right\}^{1/2}\\
&\times\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/2}+4C_3\Gamma_i^{\frac{1}{4}}+4C_3^2+\frac{m^2}{4}\right)\right\}^{1/2},
\end{aligned}\tag{4.1}$$

where $C_1=\max_{\overline{\Omega}}|\nu^{\top}|_{g_0}$.

Remark 4.2. *In fact, inequality* (4.1) *can be regard as a bound of Yang type, and also be compared with the following eigenvalue inequality for the version of drifting Laplacian:*

$$\begin{aligned}
\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq &\frac{4}{n}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left[\left(\frac{n}{2}+1\right)\Lambda_i^{1/2}+C_0\right]\right\}^{1/2}\\
&\times\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(\Lambda_i^{1/2}+C_0\right)\right\}^{1/2},
\end{aligned}$$

established by Wang and Xia in [8].

Proof of theorem 4.1. For each $\alpha\in\{1,2,\cdots,n+1\}$, applying $h=x_\alpha$ to Lemma 2.2, we assert that

$$\begin{aligned}
\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_{\Omega}u_i^2|\nabla x_\alpha|^2e^{\langle\nu,X\rangle_{g_0}}dv \leq &\sum_{i=1}^{k}\delta\,(\Gamma_{k+1}-\Gamma_i)^2\,\widehat{\Psi}_{i,\alpha}\\
&+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\widehat{\Theta}_{i,\alpha}.
\end{aligned}\tag{4.2}$$

Utilizing (2.15), we arrive at

$$\int_{\Omega}u_i^2\sum_{\alpha=1}^{n+1}|\nabla x_\alpha|_g^2\,e^{\langle\nu,X\rangle_{g_0}}dv=n.$$

Hence, summing over α from 1 to $n+1$ for (4.2), one has

$$\begin{aligned}
n\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 &\leq \sum_{i=1}^{k}\sum_{\alpha=1}^{n+1}\delta\,(\Gamma_{k+1}-\Gamma_i)^2\,\widehat{\Psi}_{i,\alpha}+\sum_{i=1}^{k}\sum_{\alpha=1}^{n+1}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\widehat{\Theta}_{i,\alpha}\\
&=\sum_{i=1}^{k}\delta\,(\Gamma_{k+1}-\Gamma_i)^2\sum_{\alpha=1}^{n+1}\widehat{\Psi}_{i,\alpha}+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\sum_{\alpha=1}^{n+1}\widehat{\Theta}_{i,\alpha}.
\end{aligned}\tag{4.3}$$

Next, let us estimate the upper bounds for $\widehat{\Psi}_{i,\alpha}$ and $\widehat{\Theta}_{i,\alpha}$. Letting $C_1=\max_{\overline{\Omega}}|\nu^{\top}|_{g_0}$, from (2.17) and (2.18), using (2.13) and proceeding as in the proof of Theorem 3.1, we conclude that

$$\sum_{\alpha=1}^{n+1} \widehat{\Psi}_{i,\alpha} \leq (2n+4)\Gamma_i^{1/2} + \left(16C_1\Gamma_i^{\frac{1}{4}} + 16C_1^2 + m^2\right), \tag{4.4}$$

and

$$\sum_{\alpha=1}^{n+1} \widehat{\Theta}_{i,\alpha} \leq \Gamma_i^{1/2} + \frac{1}{4}\left(16C_1\Gamma_i^{1/4} + 16C_1^2 + m^2\right). \tag{4.5}$$

Thus, substituting (4.4) and (4.5) into (4.3) yields

$$n\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq \sum_{i=1}^{k} \delta\,(\Gamma_{k+1} - \Gamma_i)^2\left[(2n+4)\Gamma_i^{1/2} + 4\overline{C}_3\right] + \sum_{i=1}^{k} \frac{\Gamma_{k+1} - \Gamma_i}{\delta}\left(\Gamma_i^{1/2} + \overline{C}_3\right), \tag{4.6}$$

where $\overline{C}_3 = \frac{1}{4}\left(16C_1\Gamma_i^{\frac{1}{4}} + 16C_1^2 + m^2\right)$. The remainder step is to take

$$\delta = \frac{\left[\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)\left(\Gamma_i^{1/2} + \overline{C}_3\right)\right]^{1/2}}{\left[\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2\left((2n+4)\Gamma_i^{1/2} + 4\overline{C}_3\right)\right]^{1/2}} > 0,$$

and insert it into (4.6), which gets desired inequality (4.1).

□

Remark 4.3. *Recall that the second author proved another general formula. See Lemma 2.2 in* [24]. *According to this formula and slightly modifying the proof of Theorem* 4.1*, we can give the following estimate for the eigenvalues with lower order of $\mathfrak{L}_\nu^2$ operator on the round cylinder $\mathbb{R}^{n-m} \times \mathbb{S}^m(1)$:*

$$\sum_{i=1}^{n} (\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} \leq 4\left\{\left[\left(\frac{n}{2} + 1\right)\Gamma_1^{1/2} + 4C_1\Gamma_1^{1/4} + 4C_1^2 + \frac{m^2}{4}\right]\left(\Gamma_1^{1/2} + 4C_1\Gamma_1^{\frac{1}{4}} + 4C_1^2 + \frac{m^2}{4}\right)\right\}^{1/2},$$

where $C_1 = \max_{\overline{\Omega}} |\nu^\top|_{g_0}$.

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Conflict of interest

The authors declare there is no conflict of interest.

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