Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 15(2): 132–161.
DOI: 10.3934/cam.2023008
Received: 23 March 2023
Revised: 24 April 2023
Accepted: 27 April 2023
Published: 05 May 2023

Research article

Existence and blow-up of solutions for finitely degenerate semilinear parabolic equations with singular potentials

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Abstract: In this article, we investigate the initial-boundary value problem for a class of finitely degenerate semilinear parabolic equations with singular potential term. By applying the Galerkin method and Banach fixed theorem we establish the local existence and uniqueness of the weak solution. On the other hand, by constructing a family of potential wells, we prove the global existence, the decay estimate and the finite time blow-up of solutions with subcritical or critical initial energy.

Keywords: Finitely degenerate parabolic equation; singular potentials; local existence; potential well; global existence; blow-up; decay estimate
Mathematics Subject Classification: 35K58, 35K65

1. Introduction

In this article, we study the initial-boundary value problem for the class of finitely degenerate semilinear parabolic equations with singular potential term as follows

$$\begin{cases} u_t - \Delta_X u - \mu V(x)\, u = g(x)\, |u|^{p-1} u, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $X = (X_1, \cdots, X_m)$ is a system of real smooth vector fields defined on open set U in $\mathbb{R}^n$ for $n \geq 3$, $\Omega \subset\subset U$ is a bounded open domain, $X_j = \sum_{k=1}^{n} a_{jk}(x)\, \partial_{x_k}$, $a_{jk} \in C^\infty(U)$, $j = 1, \cdots, m$, and $(x_1, \ldots, x_n)$ is the coordinate system of U. In general, X_j is different from the position vector field $x = \sum_{k=1}^{n} x_k \partial_{x_k}$ of U in $\mathbb{R}^n$. In the whole paper, we always suppose that the system of vector fields X is finitely degenerate, i.e., it satisfies the following Hörmander's condition [13] with $Q > 1$.

(H) $X_1, X_2, \ldots, X_m$ together with their commutators of length at most Q can span the tangent space $T_x(U)$ at each point $x \in U$, where Q is the Hörmander index of U with respect to X.

The sum of square operator $\Delta_X := \sum_{j=1}^{m} X_j^2$, also called the Hörmander type operator, is finitely degenerate elliptic operator if $Q > 1$, while it is the usual elliptic operator and $m \geq n$ if $Q = 1$. Here, we further pose the following hypotheses:

($H_{\partial\Omega}$) $\partial\Omega$ is smooth and non-characteristic for the system of vector fields X, i.e., for any $x \in \partial\Omega$, there exists at least one vector field X_j such that $X_j(x) \notin T_x(\partial\Omega)$.

$\left(H_p\right)$ $1 < p \leq \frac{\tilde{\nu}}{\tilde{\nu}-2}$, where $\tilde{\nu} \geq 3$ is the generalized Métivier index (cf. Definition 2.2).

(H_V) $\mu \in \left(0, 1/C_*^2\right)$ is constant, and the positive singular potential function $V(x) \in L^\infty(\Omega) \cap C(\Omega)$ satisfies the Hardy's inequality

$$\int_\Omega V(x)|u|^2\,dx \leq C\int_\Omega |Xu|^2\,dx \tag{1.2}$$

for any u of the Hilbert space $H_{X,0}^1(\Omega)$ (cf. Section 2), where

$$C_* := \sup_{u\in H_{X,0}^1(\Omega)\setminus\{0\}} \frac{\|\sqrt{V(x)}u\|}{\|Xu\|}. \tag{1.3}$$

$\left(H_g\right)$ $g(x) \in L^\infty(\Omega) \cap C(\Omega)$ is a non-negative weighted function.

Finitely degenerate elliptic operators originate from physical applications and mathematical problems, e.g., Lewy's example [19], the stochastic differential equations [30], $\bar{\partial}$-Neumann problem in complex geometry [17], Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$ in quantum mechanics [4]. Hörmander [13] proved the hypoellipticity and the subelliptic estimates of Δ_X, and thus Δ_X is still called the subelliptic operator. Bony [3] obtained the maximum principle and the Harnack inequality of Δ_X, and Rothschild and Stein [29] gave the regularity estimates of Δ_X. By Hörmander condition one can define a Carnot-Carathéodory metric induced by X, which is paid attentions by scholars in sub-Riemannian geometry [27]. Moreover, the Poincaré inequality [14], the Sobolev embedding theorem [6, 33, 39], heat kernel and Green kernel estimates [15] were well investigated.

Furthermore, under the Métivier's condition Métivier [26] studied the eigenvalues problems of Δ_X, and defined the Métivier index ν, also namely the Hausdorff dimension of Ω related to X. For example, let $X = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ on the Heisenberg group $\mathbb{H}^n \subset \mathbb{R}^{2n+1}$, where $X_j = \partial_{x_j} + 2y_j\partial_t$, $Y_j = \partial_{y_j} + 2x_j\partial_t$, $j = 1, \ldots, n$. Then X satisfies the Hörmander's condition for $Q = 2$, the Métivier's condition for $\nu = 2n + 2$, and the Kohn Laplacian $\Delta_X = \sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right)$ is a finitely degenerate elliptic operator. Unfortunately, in the finitely degenerate case, if no Métivier's condition there are no Rellich-Kondrachov compact embedding results, while such compact embedding results play an important role when one discusses the existence of solutions for the Dirichlet problem of semilinear subelliptic equations. To deal with this case, Chen and Luo [8] defined the generalized Métivier index $\tilde{\nu}$, also named non-isotropic dimension of Ω associated with X [39], which is exactly the Métivier index under the Métivier's condition. Note that X always has the generalized Métivier index $\tilde{\nu}$ on Ω even without the Métivier's condition. For example, the Grushin type vector fields $X = \left(\partial_{x_1}, \ldots, \partial_{x_{n-1}}, x_1^i\partial_{x_n}\right)$, $n \geq 2$, $i \in \mathbb{Z}^+$, defined on a domain $\Omega \subset \mathbb{R}^n$. If $\Omega \cap \{x_1 = 0\} \neq \emptyset$, then X satisfies the Hörmander's condition with $Q = i + 1$, the Grushin type operator $\Delta_X = \sum_{j=1}^{m} X_j^2$ is finitely degenerate, and $\tilde{\nu} = n + i$.

For the vector fields $X = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$, Δ_X is exactly the usual Laplacian operator Δ, and the equation in (1.1) is the heat equation with singular potentials, which has attracted attentions since the work of Baras and Goldstein [2] in 1984. In fact, they studied the initial-boundary value problem for the linear heat equation

$$u_t - \Delta u - V(x) u = f(x, t) \tag{1.4}$$

with singular potential $V(x) = c/|x|^2$, and proved that under the initial data $u_0 > 0$, if $c \leq \frac{(n-2)^2}{4}$ it has a global weak solution, otherwise it has no solution [2] and even no local solution [5]. Particularly, if $V(x) = 0$, the equation in (1.1) becomes

$$u_t - \triangle u = f(u), \tag{1.5}$$

which has been popularly studied. For the initial-boundary value problem (1.5), Liu in [23] improved the potential wells method of Payne and Sattinger [28], and obtained the global existence and blow-up of solutions with subcritical initial energy in [24]. Then, Xu [34] studied this problem with critical initial energy, and Gazzola and Weth [12] further discussed the high initial energy. Since the family of potential wells was proposed in [23], it has been used to study various important and interesting nonlinear evolution equations, including hyperbolic [20, 22, 32, 38], the system of coupled parabolic equation [35], and the pseudo-parabolic equation [21, 36, 37].

On singular manifolds, Alimohammady and Kalleji [1] studied the initial-boundary value problem of the semilinear evolution equation as follows

$$\partial_t^k u - \Delta_{\mathbb{B}} u - \varrho V(x) u = g(x) |u|^{p-1} u, \ k \geq 1 \tag{1.6}$$

with $\varrho = 1$, obtained the global existence and the finite time blow-up of weak solutions on cone type Sobolev spaces. However, for the case $k \geq 2$ the results in [1] are invalid, hence later Luo, Xu and Yang [25] considered the case $k = 2$ and ϱ in some value range, proved the local existence and uniqueness of the solution by using the contraction mapping principle, and obtained the existence of global solutions and finite time blow-up of solutions on the cone-type Sobolev spaces. On the other hand, the edge-degenerate parabolic equation with singular potentials was studied by Chen and Liu [7].

In this article, under the assumptions (H), $(\mathrm{H}_{\partial\Omega})$, (H_V), $\left(\mathrm{H}_g\right)$ and $\left(\mathrm{H}_p\right)$, by the known properties of Δ_X we establish the local and global existence, decay and finite time blow-up of the solutions for problem (1.1). This article is organized as follows. After introducing some notions and results on the finite degenerate vector fields in Section 2, by applying the Galerkin method and Banach fixed theorem we establish the local existence and uniqueness of the weak solution of problem (1.1) in Section 3. In Section 4, by constructing a family of potential wells, we prove some auxiliary results for it. In Section 5, by potential well method we obtain the global existence, the decay estimate and the finite time blow-up of solutions with subcritical or critical initial energy.

2. Preliminaries

In this section, we recall some notions and properties of the finite degenerate vector fields X.

First, by X we define the Sobolev space (cf. [33])

$$H_X^1(U) = \left\{u \in L^2(U) \mid X_i u \in L^2(U), \ i = 1, \cdots, m\right\}.$$

This is a Hilbert space equipped with the norm

$$\|u\|^2_{H^1_X(U)} = \|u\|^2_{L^2(U)} + \|Xu\|^2_{L^2(U)},$$

where $\|Xu\|^2_{L^2(U)} = \sum_{i=1}^m \|X_i u\|^2_{L^2(U)}$. We denote by $H^1_{X,0}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1_X(U)$, which is still a Hilbert space. For simplicity, from now on, we write $\|\cdot\|_{H^1_{X,0}} = \|\cdot\|_{H^1_{X,0}(\Omega)}$, $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for $1 \le p \le \infty$, and also let $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Moreover, we denote by (u, v) the inner product in $L^2(\Omega)$, and follow the convention that C is an arbitrary positive constant, which may be different from line to line.

Next, we introduce the Métivier's condition [26] and the generalized Métivier index [8] as follows.

Definiton 2.1 (Métivier's condition)**.** *Under the Hörmander's condition* (H) *for the vector fields X, let $V_i(x)$ be the subspace of the tangent space at $x \in \bar{\Omega}$ spanned by all commutators of $X_1, \cdots, X_m$ with length at most i for $1 \le i \le Q$. If each $\nu_i = \dim V_i(x)$ is constant on some neighborhood of every $x \in \bar{\Omega}$, we call X satisfying the Métivier's condition on Ω, and define the Métivier index by*

$$\nu := \sum_{i=1}^{Q} i(\nu_i - \nu_{i-1}), \qquad \nu_0 := 0,$$

namely also the Hausdorff dimension of Ω related to the subelliptic metric induced by X.

Definiton 2.2 (Generalized Métivier index)**.** *Under the Hörmander's condition* (H)*, by the notations in Definition 2.1, let $\nu_i(x)$ be the dimension of vector space $V_i(x)$ at point $x \in \bar{\Omega}$, we define the pointwise homogeneous dimension at x by*

$$\nu(x) := \sum_{i=1}^{Q} i(\nu_i(x) - \nu_{i-1}(x)), \qquad \nu_0(x) := 0. \tag{2.1}$$

Then, the generalized Métivier index of Ω is defined by

$$\tilde{\nu} := \max_{x \in \bar{\Omega}} \nu(x), \tag{2.2}$$

which is also named the non-isotropic dimension of Ω (cf. [39])*.*

For $Q > 1$ we see from (2.1) that $3 \le n + Q - 1 \le \tilde{\nu} < nQ$, and $\tilde{\nu}$ is exactly ν under the Métivier's condition.

Now, we recall the weighted Poincaré inequality and weighted Sobolev embedding theorem related to X as follows.

Proposition 2.1 (Weighted Poincaré inequality [16])**.** *Under the assumptions* (H) *and* ($H_{\partial\Omega}$)*, the first eigenvalue λ_1 of $-\Delta_X$ is strictly positive, and*

$$\lambda_1 \|u\|^2 \le \|Xu\|^2, \ \forall\, u \in H^1_{X,0}(\Omega). \tag{2.3}$$

Proposition 2.2 (Weighted Sobolev embedding theorem [39])**.** *Under the assumptions* (H) *and* ($H_{\partial\Omega}$)*, for arbitrary $u \in C^\infty(\bar{\Omega})$ we have*

$$\|u\|_{p^*} \le C(\|Xu\|_p + \|u\|_p),$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\tilde{\nu}}$, $p \in [1, \tilde{\nu})$ related to the generalized Métivier index $\tilde{\nu}$, and $C = C(\Omega, X)$ is a positive constant.

Remark 2.1. *As $\tilde{\nu} \geq 3$, by Proposition 2.2 for $p = 2$ we see that $H^1_{X,0}(\Omega) \hookrightarrow L^q(\Omega)$ is a bounded embedding for any $1 \leq q \leq 2^*_{\tilde{\nu}} := \frac{2\tilde{\nu}}{\tilde{\nu}-2}$.*

Proposition 2.3 (compact embedding theorem, cf. [9])**.** *Under the assumptions* (H) *and* $(\mathrm{H}_{\partial\Omega})$*, for $1 \leq q < 2^*_{\tilde{\nu}}$, the embedding*

$$H^1_{X,0}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact.

Note from (H_p) and Remark 2.1 that $2 < p+1 < 2^*_{\tilde{\nu}}$. Together with the Poincaré inequality (2.3), Proposition 2.3 and (H_g), we can deduce the following inequality.

Lemma 2.1. *Under the assumptions* (H)*,* $(\mathrm{H}_{\partial\Omega})$*,* (H_p) *and* (H_g)*, for arbitrary $u \in H^1_{X,0}(\Omega)$, we have*

$$\|g(x)^{\frac{1}{p+1}} u\|_{p+1} \leq C\|Xu\|.$$

Thanks to Lemmas 2.1, for $1 < p \leq \frac{\tilde{\nu}}{\tilde{\nu}-2}$ we can define a positive constant

$$C_X := \sup_{u \in H^1_{X,0}(\Omega)\setminus\{0\}} \frac{\|g(x)^{\frac{1}{p+1}} u\|_{p+1}}{\|Xu\|}. \tag{2.4}$$

Proposition 2.4 (cf. [9])**.** *Under the assumptions* (H) *and* $(\mathrm{H}_{\partial\Omega})$*, the subelliptic Dirichlet problem*

$$\begin{cases} -\Delta_X u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2.5}$$

is well-defined, i.e., $-\Delta_X$ possesses a sequence of discrete eigenvalues $\{\lambda_k\}_{k\geq 1}$ such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$, and $\lambda_k \to +\infty$ as $k \to +\infty$. Denote the corresponding eigenfunctions by $\{\phi_k\}_{k\geq 1}$, which forms an orthonormal basis of $L^2(\Omega)$ and also an orthogonal basis of the Hilbert space $H^1_{X,0}(\Omega)$.

Lemma 2.2. *For $n \geq 3$, $C_0^\infty(\Omega\setminus\{0\})$ is dense in $H^1_{X,0}(\Omega)$.*

Proof. As $C_0^\infty(\Omega)$ is dense in $H^1_{X,0}(\Omega)$, we just need to prove that

$$C_0^\infty(\Omega) \subset \overline{C_0^\infty(\Omega\setminus\{0\})}^{\|\cdot\|_{H^1_{X,0}}}.$$

Denote by φ a smooth function such that

$$\varphi(x) = \begin{cases} 0, & 0 < x \leq 1, \\ 1, & x \geq 2. \end{cases}$$

Now, taking a sufficiently small $\epsilon > 0$ and defining $u_\epsilon(x) = \varphi\left(\frac{|x|}{\epsilon}\right)u(x)$ for $u \in C_0^\infty(\Omega)$, we have $u_\epsilon(x) \in C_0^\infty(\Omega\setminus\{0\})$ and

$$\|u_\epsilon - u\|^2_{H^1_{X,0}} = \|u_\epsilon - u\|^2 + \|X(u_\epsilon - u)\|^2.$$

It follows from the dominated convergence theorem that

$$\|u_\epsilon - u\|^2 \xrightarrow{\epsilon\to 0} 0, \quad \int_\Omega \left|\varphi\left(\frac{|x|}{\epsilon}\right) - 1\right|^2 |Xu(x)|^2\, dx \xrightarrow{\epsilon\to 0} 0.$$

Moreover,

$$\begin{aligned}\int_{\Omega}\left|X\left(\frac{|x|}{\epsilon}\right)\right|^{2}\left|\nabla\varphi\left(\frac{|x|}{\epsilon}\right)\right|^{2}|u(x)|^{2}\,dx \leq & \frac{C}{\epsilon^{2}}\int_{\Omega}\left|\nabla\varphi\left(\frac{|x|}{\epsilon}\right)\right|^{2}|u(x)|^{2}\,dx\\ \leq & \frac{C}{\epsilon^{2}}\|u\|_{\infty}^{2}\|\nabla\varphi\|_{\infty}^{2}\int_{\{\epsilon\leq|x|\leq 2\epsilon\}}dx\\ \leq & C\epsilon^{n-2}\xrightarrow{\epsilon\to 0}0.\end{aligned}$$

Lemma 2.2 has been proved. □

By the Hardy inequality on $C_0^1(\Omega\backslash\{0\})$ related to degenerate elliptic differential operators [10] and Lemma 2.2, we immediately see that there exists a positive singular potential function $V(x)\in L^{\infty}(\Omega)\cap C(\Omega)$ such that Hardy inequality (1.2) holds for any $u\in H_{X,0}^1(\Omega)$. Therefore, the assumption (H_V) is reasonable. From (H_V) and the Poincaré inequality (2.3) we see that the operator $-\Delta_X-\mu V(x)$ is a positive operator on $H_{X,0}^1(\Omega)$. Moreover, we have the following result.

Proposition 2.5. *Under the assumptions* (H), ($\mathrm{H}_{\partial\Omega}$) *and* ($\mathrm{H}_V$)*, the Dirichlet eigenvalue problem*

$$\begin{cases}-\Delta_X u-\mu V(x)\,u=\eta u, & x\in\Omega,\\ u=0, & x\in\partial\Omega\end{cases}\tag{2.6}$$

is well-defined, i.e., $-\Delta_X-\mu V(x)$ possesses a sequence of discrete Dirichlet eigenvalues $\{\eta_k\}_{k\geq 1}$ such that $0<\eta_1\leq\eta_2\leq\eta_3\leq\cdots\leq\eta_k\leq\cdots$, and $\eta_k\to+\infty$ as $k\to+\infty$. Denote the corresponding eigenfunctions by $\{\varphi_k\}_{k\geq 1}$, which is an orthonormal basis of $L^2(\Omega)$ and also an orthogonal basis of the Hilbert space $H_{X,0}^1(\Omega)$.

Proof. Define the bilinear form

$$a[u,v]=\left(L_\mu u,v\right):H_{X,0}^1(\Omega)\times H_{X,0}^1(\Omega)\to\mathbb{R},$$

where $L_\mu:=-\Delta_X-\mu V(x)$ is an operator defined on the Hilbert space $H_{X,0}^1(\Omega)$. By combining with the Hölder inequality, (1.3) and the Poincaré inequality (2.3) we have

$$\begin{aligned}|a[u,v]|=&|(-\Delta_X u-\mu V(x)\,u,v)|\\ \leq&\left|\int_{\Omega}XuXvdx\right|+\left|\mu\int_{\Omega}V(x)\,uvdx\right|\\ \leq&\|Xu\|\|Xv\|+\mu\|\sqrt{V(x)}u\|\|\sqrt{V(x)}v\|\\ \leq&\left(1+\mu C_*^2\right)\|Xu\|\|Xv\|\\ \leq&\left(1+\mu C_*^2\right)\|u\|_{H_{X,0}^1}\|v\|_{H_{X,0}^1},\ \forall\,u,v\in H_{X,0}^1(\Omega),\end{aligned}$$

and

$$\begin{aligned}a[u,u]=&(-\Delta_X u-\mu V(x)\,u,u)=\|Xu\|^2-\mu\int_{\Omega}V(x)\,|u|^2\,dx\\ \geq&\left(1-\mu C_*^2\right)\|Xu\|^2\\ \geq&\left(1-\mu C_*^2\right)\frac{\lambda_1}{1+\lambda_1}\|u\|_{H_{X,0}^1}^2,\ \forall\,u\in H_{X,0}^1(\Omega).\end{aligned}$$

It follows from the Lax-Milgram theorem that for any $g \in H_X^{-1}(\Omega)$, the Dirichlet problem

$$\begin{cases} L_\mu u = -\Delta_X u - \mu V(x)\, u = g, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

has a unique solution $u \in H_{X,0}^1(\Omega)$, where $H_X^{-1}(\Omega)$ is the dual space of $H_{X,0}^1(\Omega)$ with the norm

$$\|g\|_{H_X^{-1}(\Omega)} = \sup_{\varphi \in H_{X,0}^1(\Omega), \varphi \neq 0} \frac{|\langle g, \varphi \rangle|}{\|\varphi\|_{H_{X,0}^1}},$$

and $L_\mu : H_{X,0}^1(\Omega) \to H_X^{-1}(\Omega)$ is continuous. Therefore, the inverse operator $L_\mu^{-1} = (-\Delta_X - \mu V(x))^{-1}$ of L_μ is well-defined and is a continuous map from $H_X^{-1}(\Omega)$ into $H_{X,0}^1(\Omega)$.

Since that the embedding $i : H_{X,0}^1(\Omega) \to L^2(\Omega)$ is compact and the embedding $i^* : L^2(\Omega) \to H_X^{-1}(\Omega)$ is continuous, we deduce that

$$K_\mu := L_\mu^{-1} \circ i^* \circ i : H_{X,0}^1(\Omega) \to H_{X,0}^1(\Omega)$$

is a compact and self-adjoint operator. Therefore, K_μ possesses a sequence of discrete eigenvalues $\{\mu_k\}_{k\geq 1}$ such that $\mu_k > 0$, decreasing on k and $\mu_k \to 0$ as $k \to +\infty$. Denote the corresponding eigenfunctions by $\{\varphi_k\}_{k\geq 1}$, then

$$K_\mu \varphi_k = \mu_k \varphi_k, \ \forall\, k \geq 1$$

and $\{\varphi_k\}_{k\geq 1}$ form an orthonormal basis of $H_{X,0}^1(\Omega)$. Proposition 2.5 has been proved. □

Finally, we give the definition of weak solutions.

Definiton 2.3 (Weak solution)**.** *A function $u = u(x,t)$ is called a weak solution of problem* (1.1) *on $\Omega \times [0,T)$, if $u \in L^\infty(0,T; H_{X,0}^1(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ satisfies $u(0,x) = u_0(x) \in H_{X,0}^1(\Omega)$ and*

$$(u_t, w) + (Xu, Xw) - (\mu V(x)\, u, w) = \left(g(x)\, |u|^{p-1} u, w\right) \tag{2.7}$$

for any $w \in H_{X,0}^1(\Omega)$, $0 < t < T$, where T is the maximum existence time of the solution.

3. Local existence of the solution

In this section, we will prove the existence and uniqueness of the local solution for the problem (1.1). First, we consider the linear problem of (1.1)

$$\begin{cases} v_t - \Delta_X v - \mu V(x)\, v = g(x)\, |u|^{p-1} u, & x \in \Omega, t > 0, \\ v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.1}$$

For a given $T > 0$ and any $\mu \in \left(0, \frac{1}{C_*^2}\right)$, define the Banach space

$$\mathcal{H} := \left\{ u \mid u \in C\left([0,T]; H_{X,0}^1(\Omega)\right),\ u_t \in L^2\left([0,T]; L^2(\Omega)\right) \right\}$$

equipped with the norm

$$\|u\|_{\mathcal{H}}^2 := \sup_{t \in [0,T]} \left(1 - \mu C_*^2\right) \|Xu\|^2. \tag{3.2}$$

By the Galerkin method we establish the local existence result of the problem (3.1) as follows.

Lemma 3.1. *Under the assumptions* (H)*,* $(H_{\partial\Omega})$*,* (H_V)*,* $\left(H_g\right)$ *and* $\left(H_p\right)$*, for every* $u_0 \in H^1_{X,0}(\Omega)$ *and* $u \in \mathcal{H}$*, the problem* (3.1) *has a unique local solution* $v \in \mathcal{H}$*.*

Proof. By Proposition 2.5, we see that $\{\eta_i\}_{i\geq 1}$ are the eigenvalues of the positive operator $L_\mu = -\Delta_X - \mu V(x)$ of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_X \varphi_i - \mu V(x)\varphi_i = \eta_i \varphi_i, & x \in \Omega, \\ \varphi_i = 0, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

where $\|\varphi_i\| = 1$ for all i, and the eigenfunctions $\{\varphi_i\}_{i\geq 1}$ are the orthogonal basis of both $H^1_{X,0}(\Omega)$ and $L^2(\Omega)$. Let $W_m = \text{Span}\{\varphi_1, \cdots, \varphi_m\}$, $m \in \mathbb{N}^+$. For each $m \in \mathbb{N}^+$, we can construct the approximate solutions of problem (3.1) as follows

$$v_m(t) = \sum_{i=1}^{m} h_{im}\varphi_i, \tag{3.4}$$

which satisfies the following Cauchy problem in W_m

$$\begin{cases} (v_{mt} - \Delta_X v_m - \mu V(x) v_m, \varphi_i) = \left(g(x)|u|^{p-1}u, \varphi_i\right), \\ v_m(x,0) = u_{m0} = \sum_{i=1}^{m}(u_0,\varphi_i)\varphi_i \xrightarrow{m\to\infty} u_0 \text{ in } H^1_{X,0}(\Omega). \end{cases} \tag{3.5}$$

By taking (3.4) into (3.5), we get the Cauchy problem of the ordinary differential equation with respect to $h_{im}(t)$ as follows

$$\begin{cases} h'_{im}(t) + \eta_i h_{im}(t) = \left(g(x)|u|^{p-1}u, \varphi_i\right), \ i = 1, 2, \cdots, m, \\ h_{im}(0) = (u_0, \varphi_i). \end{cases} \tag{3.6}$$

Thanks to the theory of ordinary differential equations, the problem (3.6) has a solution $h_{im} \in C^1[0,T]$ for each i. Multiplying both sides of the equation in (3.5) by $h'_{im}(t)$, summing for i and integrating over $[0,t]$, one has

$$\begin{aligned} &2\int_0^t \|v_{m\tau}\|^2 d\tau + \|Xv_m\|^2 - \int_\Omega \mu V(x)|v_m|^2 dx \\ =&\|Xu_{m0}\|^2 - \int_\Omega \mu V(x)|u_{m0}|^2 dx + 2\int_0^t\int_\Omega g(x)|u|^{p-1}uv_{m\tau}dx. \end{aligned} \tag{3.7}$$

Next, according to the Hölder inequality, $\left(H_g\right)$, the Sobolev embedding $H^1_{X,0}(\Omega) \hookrightarrow L^{2p}(\Omega)$, the Poincaré inequality (2.3) and the Cauchy inequality with ϵ, we can estimate the last term of (3.7) as follows

$$\begin{aligned} &2\int_0^t\int_\Omega g(x)|u|^{p-1}uv_{m\tau}dxd\tau \\ \leq&2\|g\|_\infty \int_0^t \|u\|^p_{2p}\|v_{m\tau}\|d\tau \\ \leq&2C\|g\|_\infty \int_0^t \|u\|^p_{H^1_{X,0}}\|v_{m\tau}\|d\tau \\ \leq&\frac{C}{2\epsilon}\left(1+\frac{1}{\lambda_1}\right)^p\|g\|_\infty\int_0^t\|Xu\|^{2p}d\tau + 2C\epsilon\|g\|_\infty\int_0^t\|v_{m\tau}\|^2d\tau \\ \leq&CT + 2C\epsilon\|g\|_\infty\int_0^t\|v_{m\tau}\|^2d\tau, \end{aligned} \tag{3.8}$$

where the positive constant C may be different from line to line. By choosing $\epsilon > 0$ such that $2C\epsilon\|g\|_\infty = 1$, we see from (1.3), (3.7) and (3.8) that

$$\begin{aligned}&\int_0^t \|v_{m\tau}\|^2 d\tau + \left(1 - \mu C_*^2\right)\|Xv_m\|^2 \\ \le &\int_0^t \|v_{m\tau}\|^2 d\tau + \|Xv_m\|^2 - \int_\Omega \mu V(x)|v_m|^2 dx \\ &= \|Xu_{m0}\|^2 - \int_\Omega \mu V(x)|u_{m0}|^2 dx + CT \\ \le &CT.\end{aligned} \tag{3.9}$$

Let $\overset{w^*}{\longrightarrow}$ be the weakly star convergence. By (3.9) we have a subsequence, also denoted by $\{v_m\}$, satisfying as $m \to \infty$,

$$v_m \overset{w^*}{\longrightarrow} v \text{ in } L^\infty\left([0,T]; H^1_{X,0}(\Omega)\right), \tag{3.10}$$

$$v_{mt} \overset{w^*}{\longrightarrow} v_t \text{ in } L^2\left([0,T]; L^2(\Omega)\right). \tag{3.11}$$

These imply that

$$v \in H^1\left([0,T]; L^2(\Omega)\right).$$

Then one has from Evans Theorem ([11], 5.9.2. Theorem 2, p. 304) that

$$v \in C\left([0,T]; L^2(\Omega)\right). \tag{3.12}$$

By Proposition 2.3 and Remark 2.1, the injection $H^1_{X,0} \hookrightarrow L^2(\Omega)$ is continuous and compact, which together with (3.12) and Temam lemma ([31], Section II, Lemma 3.3) shows that

$$v \in C\left([0,T]; H^1_{X,0}(\Omega)\right). \tag{3.13}$$

It follows from (3.5) and (3.10) that

$$v_{mt} \overset{w^*}{\longrightarrow} v_t \text{ in } L^\infty([0,T]; H^{-1}_X(\Omega)). \tag{3.14}$$

For fixed i, letting $m \to \infty$, taking the limit in (3.5), by (3.10)-(3.11) we get

$$(v_t, \varphi_i) + (Xv, X\varphi_i) - (\mu V(x)v, \varphi_i) = \left(g(x)|u|^{p-1}u, \varphi_i\right), \quad \forall\, i \ge 1.$$

Since $\{\varphi_i\}_{i\ge1}$ is a base of $H^1_{X,0}(\Omega)$, we deduce that $v \in \mathcal{H}$ satisfies the equation in (3.1).

Finally, we prove the uniqueness of solutions. Otherwise, assume that w_1 and w_2 are two solutions of problem (3.1). Let $\tilde{w} = w_1 - w_2$, there holds

$$\begin{cases}\tilde{w}_t - \Delta_X\tilde{w} - \mu V(x)\tilde{w} = 0, & x \in \Omega, t > 0,\\ \tilde{w}(x,t) = 0, & x \in \partial\Omega, t > 0,\\ \tilde{w}(x,0) = 0, & x \in \Omega.\end{cases}$$

Multiplying both sides of $\tilde{w}_t - \Delta_X \tilde{w} - \mu V(x)\tilde{w} = 0$ by $\tilde{w}_t$, and integrating it over $\Omega \times (0,t)$, we have

$$\begin{aligned}&2\int_0^t \|\tilde{w}_\tau\|^2 d\tau + \|X\tilde{w}\|^2 - \int_\Omega \mu V(x)|\tilde{w}|^2 dx\\ =&\|X\tilde{w}(x,0)\|^2 - \int_\Omega \mu V(x)|\tilde{w}(x,0)|^2 dx = 0.\end{aligned}$$

It follows from (H_V) that

$$\begin{aligned}0 \le\ & 2\int_0^t \|\tilde{w}_\tau\|^2 d\tau + \left(1-\mu C_*^2\right)\|X\tilde{w}\|^2\\ \le\ & \|X\tilde{w}(x,0)\|^2 - \int_\Omega \mu V(x)|\tilde{w}(x,0)|^2 dx \equiv 0,\end{aligned}$$

and thus $\tilde{w} = 0$ a.e. in Ω, i.e., $w_1 \equiv w_2$. The conclusion follows. □

Theorem 3.1 (Local existence)**.** *Under the assumptions* (H), ($H_{\partial\Omega}$), (H_V), $\left(H_g\right)$ *and* $\left(H_p\right)$*, if* $u_0 \in H^1_{X,0}(\Omega)$*, there exists* $T > 0$ *such that the problem* (1.1) *has a unique weak solution*

$$u \in C\left([0,T]; H^1_{X,0}(\Omega)\right), \quad u_t \in L^2\left([0,T]; L^2(\Omega)\right). \tag{3.15}$$

Proof. For any $T > 0$, we define the set

$$\mathcal{M}_T := \{u \in \mathcal{H} \mid u(0) = u_0,\ \|u\|_{\mathcal{H}} \le \rho\}, \tag{3.16}$$

where

$$\rho^2 = 2\left(\|Xu_0\|^2 - \mu\|\sqrt{V(x)}u_0\|^2\right).$$

By Lemma 3.1 we can define the mapping Ψ on $\mathcal{M}_T$, such that $\Psi(u)$ is the unique solution of the problem (3.1), i.e., $\Psi(u) = v$. We will prove that $\Psi : \mathcal{M}_T \to \mathcal{M}_T$ is a contractive mapping for small enough T.

First, for sufficiently small T we show that Ψ is a mapping from $\mathcal{M}_T$ to itself. For any $u \in \mathcal{M}_T$, similar to (3.7) and (3.8) the unique solution $v = \Psi(u)$ satisfies

$$\begin{aligned}&2\int_0^t \|v_\tau\|^2 d\tau + \|Xv\|^2 - \int_\Omega \mu V(x)|v|^2 dx\\ =&\|Xu_0\|^2 - \int_\Omega \mu V(x)|u_0|^2 dx + 2\int_0^t\int_\Omega g(x)|u|^{p-1}uv_\tau dx\\ \le&\frac{1}{2}\rho^2 + C^2\left(1+\frac{1}{\lambda_1}\right)^p \|g\|_\infty^2 \int_0^t \|Xu\|^{2p} d\tau + \int_0^t \|v_\tau\|^2 d\tau\\ \le&\frac{1}{2}\rho^2 + C^2\left(1+\frac{1}{\lambda_1}\right)^p \|g\|_\infty^2 \frac{\rho^{2p}}{(1-\mu C_*^2)^p}T + \int_0^t \|v_\tau\|^2 d\tau.\end{aligned} \tag{3.17}$$

It follows from (1.3) that

$$\begin{aligned}&\left(1-\mu C_*^2\right)\|Xu\|^2\\ \le&\int_0^t \|v_\tau\|^2 d\tau + \|Xv\|^2 - \int_\Omega \mu V(x)|v|^2 dx\\ \le&\rho^2\left(\frac{1}{2} + C^2\left(1+\frac{1}{\lambda_1}\right)^p \|g\|_\infty^2 \frac{\rho^{2(p-1)}}{(1-\mu C_*^2)^p}T\right).\end{aligned} \tag{3.18}$$

Then by (3.2) we obtain

$$\|u\|_{\mathcal{H}}^2 \le \rho^2 \left(\frac{1}{2} + C^2 \left(1 + \frac{1}{\lambda_1}\right)^p \|g\|_\infty^2 \frac{\rho^{2(p-1)}}{(1-\mu C_*^2)^p} T\right).$$

Therefore, for T small enough $\|u\|_{\mathcal{H}}^2 \le \rho^2$, i.e., $\Psi(\mathcal{M}_T) \subseteq \mathcal{M}_T$.

Now, we will show that Ψ is a contraction mapping. Let $u_1, u_2 \in \mathcal{M}_T$ and $v_1 = \Psi(u_1)$, $v_2 = \Psi(u_2)$. By taking $\tilde{v} := v_1 - v_2$, we see that $\tilde{v}$ satisfies the following problem

$$\begin{cases} \tilde{v}_t - \Delta_X \tilde{v} - \mu V(x)\tilde{v} = g(x)\left(|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2\right), & x \in \Omega, t > 0, \\ \tilde{v}(x,t) = 0, & x \in \partial\Omega, t > 0, \\ \tilde{v}(x,0) = 0, & x \in \Omega. \end{cases} \tag{3.19}$$

Multiplying the equation above by $\tilde{v}_t$, and integrating it over $\Omega \times (0,t)$, we deduce

$$\begin{aligned} & 2\int_0^t \|\tilde{v}_\tau\|^2 d\tau + \|X\tilde{v}\|^2 - \int_\Omega \mu V(x)|\tilde{v}|^2 dx \\ =& \|X\tilde{v}_0\|^2 - \int_\Omega \mu V(x)|\tilde{v}_0|^2 dx + 2\int_0^t \int_\Omega g(x)\left(|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2\right)\tilde{v}_\tau dx \\ =& 2\int_0^t \int_\Omega g(x)\left(|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2\right)\tilde{v}_\tau dx. \end{aligned} \tag{3.20}$$

Note from Lemma 4 of [32] that $|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \le p(|u_1| + |u_2|)^{p-1}|u_1 - u_2|$. Together with the Minkowski inequality, similar to (3.8) we have

$$\begin{aligned} & 2\int_0^t \int_\Omega g(x)\left(|u_1|^{p-1} u_1 - |u_2|^{p-1} u_2\right)\tilde{v}_\tau dx \\ \le& 2p\|g\|_\infty \int_0^t \|(|u_1| + |u_2|)^{p-1}\|_{\frac{2p}{p-1}} \|u_1 - u_2\|_{2p} \|\tilde{v}_\tau\| d\tau \\ \le& 2p\|g\|_\infty \int_0^t \left(\|u_1\|_{2p} + \|u_2\|_{2p}\right)^{p-1} \|u_1 - u_2\|_{2p} \|\tilde{v}_\tau\| d\tau \\ \le& 2C\|g\|_\infty \int_0^t \left(\|u_1\|_{H_{X,0}^1} + \|u_2\|_{H_{X,0}^1}\right)^{p-1} \|u_1 - u_2\|_{H_{X,0}^1} \|\tilde{v}_\tau\| d\tau \\ \le& \frac{C}{2\epsilon}\left(\frac{1+\lambda_1}{\lambda_1(1-\mu C_*^2)}\right)^p \|g\|_\infty \int_0^t (\|u_1\|_{\mathcal{H}} + \|u_2\|_{\mathcal{H}})^{2(p-1)} \|u_1 - u_2\|_{\mathcal{H}}^2 d\tau \\ & + 2C\epsilon\|g\|_\infty \int_0^t \|\tilde{v}_\tau\|^2 d\tau \\ \le& C^2\left(\frac{1+\lambda_1}{\lambda_1(1-\mu C_*^2)}\right)^p \|g\|_\infty^2 \int_0^T (2\rho)^{2(p-1)} \|u_1 - u_2\|_{\mathcal{H}}^2 d\tau + \int_0^t \|\tilde{v}_\tau\|^2 d\tau \\ \le& CT\rho^{2(p-1)}\|u_1 - u_2\|_{\mathcal{H}}^2 + \int_0^t \|\tilde{v}_\tau\|^2 d\tau. \end{aligned} \tag{3.21}$$

Combining with (1.3), (3.20) and (3.21) we can deduce that

$$\begin{aligned} \left(1 - \mu C_*^2\right)\|X\tilde{v}\|^2 \le& \int_0^t \|\tilde{v}_\tau\|^2 d\tau + \|X\tilde{v}\|^2 - \int_\Omega \mu V(x)|\tilde{v}|^2 dx \\ \le& CT\rho^{2(p-1)}\|u_1 - u_2\|_{\mathcal{H}}^2. \end{aligned}$$

It follows from (3.2) that

$$\|\tilde{v}\|_{\mathcal{H}}^2 = \|\Psi(u_1) - \Psi(u_2)\|_{\mathcal{H}}^2 \leq CT\rho^{2(p-1)}\|u_1 - u_2\|_{\mathcal{H}}^2 := \delta_T\|u_1 - u_2\|_{\mathcal{H}}^2.$$

By choosing $T > 0$ such that $\delta_T = CT\rho^{2(p-1)} < 1$, we obtain that Ψ is a contraction mapping from $\mathcal{M}_T$ to itself. Thanks to the Banach fixed point theorem, we get the local existence result. The proof has been completed. □

4. Some auxiliary results of the potential wells

Under the assumptions (H), ($H_{\partial\Omega}$), (H_V), $\left(H_g\right)$ and $\left(H_p\right)$, for further discussions we construct a family of potential wells in this section, and prove some auxiliary results for it.

First, we define the potential energy functional J and Nehari functional I on $H^1_{X,0}(\Omega)$ given by

$$\begin{aligned} J(u) &= \frac{1}{2}\|Xu\|^2 - \frac{1}{2}\int_\Omega \mu V(x)|u|^2\,dx - \frac{1}{p+1}\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}, \\ I(u) &= \|Xu\|^2 - \int_\Omega \mu V(x)|u|^2\,dx - \|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}. \end{aligned} \tag{4.1}$$

It follows that

$$J(u) = \frac{p-1}{2(p+1)}\left(\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2\,dx\right) + \frac{1}{p+1}I(u). \tag{4.2}$$

Define the mountain pass level

$$d := \inf\left\{\sup_{\lambda\geq 0} J(\lambda u) \mid u \in H^1_{X,0}(\Omega), \|Xu\| \neq 0\right\}, \tag{4.3}$$

also called potential well depth. We now discuss the properties of the functionals J and I.

Lemma 4.1. *For arbitrary $u \in H^1_{X,0}(\Omega)$ and $\|Xu\| \neq 0$, we have*

(1) $\lim_{\lambda\to 0} J(\lambda u) = 0$, *and* $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$;

(2) $J(\lambda u)$ *with respect to λ is strictly decreasing on* $[\lambda_X, +\infty)$, *strictly increasing on* $[0, \lambda_X]$, *and thus attains the maximum at λ_X, where*

$$\lambda_X = \left(\frac{\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2\,dx}{\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}};$$

(3)

$$\begin{cases} I(\lambda u) > 0, & \lambda \in (0, \lambda_X), \\ I(\lambda u) = 0, & \lambda = \lambda_X, \\ I(\lambda u) < 0, & \lambda \in (\lambda_X, +\infty); \end{cases}$$

(4) $d = \frac{p-1}{2(p+1)}\left(1 - \mu C_*^2\right)^{\frac{p+1}{p-1}} C_X^{-\frac{2(p+1)}{p-1}}$, *where C_X is the best Sobolev constant defined in* (2.4).

Proof. It follows from (4.1) that

$$J(\lambda u) = \lambda^2\left(\frac{1}{2}\|Xu\|^2 - \frac{1}{2}\int_\Omega \mu V(x)|u|^2\,dx - \frac{\lambda^{p-1}}{p+1}\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}\right),$$

and

$$I(\lambda u) = \lambda^2\|Xu\|^2 - \lambda^2\int_\Omega \mu V(x)|u|^2\,dx - \lambda^{p+1}\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}.$$

Then, we have Lemma 4.1 (1) and

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\|Xu\|^2 - \lambda\int_\Omega \mu V(x)|u|^2\,dx - \lambda^p\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1} = \frac{1}{\lambda}I(\lambda u).$$

Hence we have a unique $\lambda_X := \left(\frac{\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2 dx}{\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}}$ such that $\frac{d}{d\lambda}J(\lambda u)\,|_{\lambda=\lambda_X} = 0$ and

$$\begin{aligned}J(\lambda_X u) &= \frac{\lambda_X^2}{2}\|Xu\|^2 - \frac{\lambda_X^2}{2}\int_\Omega \mu V(x)|u|^2\,dx - \frac{\lambda_X^{p+1}}{p+1}\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}\\ &= \left(\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2\,dx\right)^{\frac{p+1}{p-1}}\left(\frac{1}{2} - \frac{1}{p+1}\right)\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{-\frac{2(p+1)}{p-1}}\\ &\geq \frac{p-1}{2(p+1)}\left(1-\mu C_*^2\right)^{\frac{p+1}{p-1}}C_X^{-\frac{2(p+1)}{p-1}},\end{aligned}$$

where we used (1.3) and (2.4) in the inequality above. Together with (4.3) we immediately get remaining conclusions. □

Defining the Nehari manifold

$$\mathcal{N} := \left\{u \in H^1_{X,0}(\Omega) \mid I(u) = 0, \|Xu\| \neq 0\right\},$$

by Lemma 4.1 we get $d > 0$, and

$$d = \inf_{u\in\mathcal{N}} J(u). \tag{4.4}$$

For any $\delta > 0$, we introduce the functionals

$$I_\delta(u) = \delta\|Xu\|^2 - \delta\int_\Omega \mu V(x)|u|^2\,dx - \|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}$$

with the associated Nehari manifolds

$$\mathcal{N}_\delta = \left\{u \in H^1_{X,0}(\Omega) \mid I_\delta(u) = 0, \|Xu\| \neq 0\right\},$$

and the depth of such potential wells

$$d(\delta) := \inf_{u\in\mathcal{N}_\delta} J(u),\ r(\delta) = \left(\frac{\left(1-\mu C_*^2\right)\delta}{C_X^{p+1}}\right)^{\frac{1}{p-1}}, \tag{4.5}$$

where C_* is defined in (1.3). With these in mind we can prove

Lemma 4.2. *Assume $u \in H^1_{X,0}(\Omega)$, we obtain*

(1) *if $0 < \|Xu\| < r(\delta)$, there holds $I_\delta(u) > 0$;*
(2) *if $I_\delta(u) < 0$, there holds $\|Xu\| > r(\delta)$;*
(3) *if $I_\delta(u) = 0$, either $\|Xu\| = 0$ or $\|Xu\| \geq r(\delta)$ holds;*
(4) *if $I_\delta(u) = 0$ and $\|Xu\| \neq 0$, there hold*

$$\begin{cases} J(u) < 0, & \delta \in \left(\frac{p+1}{2}, +\infty\right), \\ J(u) = 0, & \delta = \frac{p+1}{2}, \\ J(u) > 0, & \delta \in \left(0, \frac{p+1}{2}\right). \end{cases}$$

Proof. (1) As $0 < \|Xu\| < r(\delta)$, by (1.3) and (2.4) there holds

$$\begin{aligned} \delta \int_\Omega \mu V(x)|u|^2\,dx + \|g(x)^{\frac{1}{p+1}} u\|_{p+1}^{p+1} \leq & \delta\mu C_*^2\|Xu\|^2 + C_X^{p+1}\|Xu\|^{p+1} \\ < & \left(\delta\mu C_*^2 + C_X^{p+1} r^{p-1}(\delta)\right)\|Xu\|^2 \\ = & \delta\|Xu\|^2. \end{aligned}$$

By the definitions of $I_\delta(u)$ we have Lemma 4.2 (1).

(2) For $I_\delta(u) < 0$, we obtain that $\|Xu\| \neq 0$ and

$$\begin{aligned} \delta\|Xu\|^2 < & \delta \int_\Omega \mu V(x)|u|^2\,dx + \|g(x)^{\frac{1}{p+1}} u\|_{p+1}^{p+1} \\ \leq & \left(\delta\mu C_*^2 + C_X^{p+1}\|Xu\|^{p-1}\right)\|Xu\|^2. \end{aligned}$$

The conclusion (2) follows.

(3) When $I_\delta(u) = 0$, there holds

$$\begin{aligned} \delta\|Xu\|^2 = & \delta \int_\Omega \mu V(x)|u|^2\,dx + \|g(x)^{\frac{1}{p+1}} u\|_{p+1}^{p+1} \\ \leq & \left(\delta\mu C_*^2 + C_X^{p+1}\|Xu\|^{p-1}\right)\|Xu\|^2. \end{aligned}$$

Thus the conclusion (3) holds.

(4) The last conclusion follows immediately from (3) and

$$J(u) = \left(\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2\,dx\right)\left(\frac{1}{2} - \frac{\delta}{p+1}\right) + \frac{I_\delta(u)}{p+1}. \tag{4.6}$$

□

Next, we estimate the depth $d(\delta)$ and its expression as follows.

Lemma 4.3. *For the function $d(\delta)$, there hold*

(1) *for $\delta \in \left(0, \frac{p+1}{2}\right)$, $d(\delta) \geq b(\delta) r^2(\delta)$, where $b(\delta) := \left(1 - \mu C_*^2\right)\left(\frac{1}{2} - \frac{\delta}{p+1}\right)$;*
(2) *for $\delta \in \left(0, \frac{p+1}{2}\right)$, $d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u) = \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}\delta^{\frac{2}{p-1}} d$;*

(3) $\lim_{\delta\to 0} d(\delta) = 0$, $d\left(\frac{p+1}{2}\right) = 0$, *and* $d(\delta) < 0$ *for* $\delta \in \left(\frac{p+1}{2}, +\infty\right)$;

(4) $d(\delta)$ *is strictly increasing on* $0 < \delta \leq 1$, *decreasing on* $1 \leq \delta \leq \frac{p+1}{2}$ *and attains the maximum d at* $\delta = 1$.

Proof. (1) For $u \in \mathcal{N}_\delta$, we have $I_\delta(u) = 0$ and $\|Xu\| \neq 0$. It follows from Lemma 4.2 (3) that

$$\|Xu\| \geq r(\delta).$$

Together with (1.3) and (4.6) we see that

$$\begin{aligned} J(u) &= \left(\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2 dx\right)\left(\frac{1}{2} - \frac{\delta}{p+1}\right) + \frac{I_\delta(u)}{p+1} \\ &\geq \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\left(1 - \mu C_*^2\right)\|Xu\|^2 \\ &\geq b(\delta) r^2(\delta). \end{aligned}$$

By combining with (4.5) we have $d(\delta) \geq b(\delta) r^2(\delta)$.

(2) Taking $u_* \in \mathcal{N}$ as the minimizer of $d = \inf_{u\in\mathcal{N}} J(u)$, i.e., $d = J(u_*)$, we introduce $\lambda = \lambda(\delta)$ by

$$\delta\|X(\lambda u_*)\|^2 - \delta\int_\Omega \mu V(x)|\lambda u_*|^2 dx = \|g(x)^{\frac{1}{p+1}} \lambda u_*\|_{p+1}^{p+1}.$$

Then there holds

$$\lambda = \lambda(\delta) = \left(\frac{\delta\|Xu_*\|^2 - \delta\int_\Omega \mu V(x)|u_*|^2 dx}{\|g(x)^{\frac{1}{p+1}} u_*\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}} = \delta^{\frac{1}{p-1}}, \ \forall\, \delta > 0,$$

and thus $\lambda u_* \in \mathcal{N}_\delta$. Together with $I(u_*) = 0$, (4.1) and (4.5), we deduce

$$\begin{aligned} d(\delta) \leq J(\lambda u_*) &= \frac{1}{2}\left(\|Xu_*\|^2 - \int_\Omega \mu V(x)|u_*|^2 dx\right)\lambda^2 - \frac{\lambda^{p+1}}{p+1}\|g(x)^{\frac{1}{p+1}} u_*\|_{p+1}^{p+1} \\ &= \frac{1}{2}\left(\|Xu_*\|^2 - \int_\Omega \mu V(x)|u_*|^2 dx\right)\delta^{\frac{2}{p-1}} - \frac{1}{p+1}\delta^{\frac{p+1}{p-1}}\|g(x)^{\frac{1}{p+1}} u_*\|_{p+1}^{p+1} \\ &= \left(\|Xu_*\|^2 - \int_\Omega \mu V(x)|u_*|^2 dx\right)\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\delta^{\frac{2}{p-1}}. \end{aligned}$$

Note that

$$d = J(u_*) = \left(\|Xu_*\|^2 - \int_\Omega \mu V(x)|u_*|^2 dx\right)\left(\frac{1}{2} - \frac{1}{p+1}\right),$$

thus

$$d(\delta) \leq \frac{2(p+1)}{p-1}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\delta^{\frac{2}{p-1}} d \tag{4.7}$$

for any $\delta \in \left(0, \frac{p+1}{2}\right)$.

Now, by taking $u^* \in \mathcal{N}_\delta$ as the minimizer of $d(\delta) = \inf_{u\in\mathcal{N}_\delta} J(u)$, i.e., $J(u^*) = d(\delta)$, we determine $\lambda = \lambda(\delta)$ by

$$\|X(\lambda u^*)\|^2 - \int_\Omega \mu V(x)|\lambda u^*|^2\,dx = \|g(x)^{\frac{1}{p+1}}\lambda u^*\|_{p+1}^{p+1}.$$

Therefore, we obtain

$$\lambda = \lambda(\delta) = \left(\frac{\|Xu^*\|^2 - \int_\Omega \mu V(x)|u^*|^2\,dx}{\|g(x)^{\frac{1}{p+1}}u^*\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}} = \delta^{\frac{1}{1-p}},\ \forall\,\delta > 0,$$

and thus $\lambda u^* \in \mathcal{N}$. Combining with (4.1), (4.4) and $I_\delta(u^*) = 0$, we have

$$\begin{aligned} d \le J(\lambda u^*) &= \frac{1}{2}\left(\|Xu^*\|^2 - \int_\Omega \mu V(x)|u^*|^2\,dx\right)\lambda^2 - \frac{\lambda^{p+1}}{p+1}\|g(x)^{\frac{1}{p+1}}u^*\|_{p+1}^{p+1} \\ &= \left(\frac{\lambda^2}{2} - \frac{\lambda^{p+1}}{p+1}\delta\right)\left(\|Xu^*\|^2 - \int_\Omega \mu V(x)|u^*|^2\,dx\right) \\ &= \delta^{-\frac{2}{p-1}}\left(\|Xu^*\|^2 - \int_\Omega \mu V(x)|u^*|^2\,dx\right)\left(\frac{1}{2} - \frac{1}{p+1}\right). \end{aligned}$$

Together with

$$d(\delta) = J(u^*) = \left(\|Xu^*\|^2 - \int_\Omega \mu V(x)|u^*|^2\,dx\right)\left(\frac{1}{2} - \frac{\delta}{p+1}\right),$$

we deduce

$$d \le \left(\frac{1}{2} - \frac{\delta}{p+1}\right)^{-1}\left(\frac{1}{2} - \frac{1}{p+1}\right)\delta^{-\frac{2}{p-1}}d(\delta),$$

which shows

$$d(\delta) \ge \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}\delta^{\frac{2}{p-1}}d,\quad \delta \in \left(0, \frac{p+1}{2}\right). \tag{4.8}$$

By (4.7) and (4.8) we have Lemma 4.3 (2).

The conclusions of (3) and (4) follow immediately from (2) and

$$d'(\delta) = \frac{2(p+1)}{(p-1)^2}(1-\delta)\delta^{\frac{3-p}{p-1}}d,\ \delta \in \left(0, \frac{p+1}{2}\right).$$

□

Lemma 4.4. *Assume that* $u \in H^1_{X,0}(\Omega)$, $J(u) \le d(\delta)$ *with* $\delta \in \left(0, \frac{p+1}{2}\right)$.

(1) *For* $I_\delta(u) > 0$, *there holds* $\|Xu\|^2 < d(\delta)/b(\delta)$.
(2) *For* $I_\delta(u) = 0$, *there holds* $\|Xu\|^2 \le d(\delta)/b(\delta)$.
(3) *For* $\|Xu\|^2 > d(\delta)/b(\delta)$, *there holds* $I_\delta(u) < 0$.

Proof. As $\delta \in \left(0, \frac{p+1}{2}\right)$, we can see from (4.6), (1.3) and $J(u) \le d(\delta)$ that

$$\begin{aligned} d(\delta) &\ge \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\left(\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2 dx\right) + \frac{I_\delta(u)}{p+1} \\ &\ge \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\left(1 - \mu C_*^2\right)\|Xu\|^2 + \frac{I_\delta(u)}{p+1} \\ &= b(\delta)\|Xu\|^2 + \frac{I_\delta(u)}{p+1}. \end{aligned} \tag{4.9}$$

Then, the corresponding conclusions in Lemma 4.4 follow from the assumption of (1)-(3), respectively. □

Lemma 4.5. *Suppose that* $0 < J(u) < d$ *for any given* $u \in H^1_{X,0}(\Omega)$. *Denote by* δ_1, δ_2 *the two roots of* $d(\delta) = J(u)$ *with* $\delta_1 < 1 < \delta_2$. *Then the sign of* $I_\delta(u)$ *is unchangeable on* $\delta \in (\delta_1, \delta_2)$.

Proof. Otherwise, we assume that $I_{\tilde\delta}(u) = 0$ for some $\tilde\delta \in (\delta_1, \delta_2)$. Note from the assumption $J(u) > 0$ that $\|Xu\| \neq 0$. It follows from (4.5) that $d(\tilde\delta) \le J(u)$, which contradicts $J(u) = d(\delta_1) = d(\delta_2) < d(\tilde\delta)$. □

Now, we introduce the potential well

$$W = \left\{u \in H^1_{X,0}(\Omega) \mid J(u) < d, I(u) > 0\right\} \cup \{0\},$$

and the outer of the potential well

$$V = \left\{u \in H^1_{X,0}(\Omega) \mid J(u) < d, I(u) < 0\right\}.$$

For each $\delta \in \left(0, \frac{p+1}{2}\right)$, by the ideas of [23] we can extend W and V respectively to the more general family of potential wells

$$W_\delta = \left\{u \in H^1_{X,0}(\Omega) \mid J(u) < d(\delta), I_\delta(u) > 0\right\} \cup \{0\},$$

and its outsider

$$V_\delta = \left\{u \in H^1_{X,0}(\Omega) \mid J(u) < d(\delta), I_\delta(u) < 0\right\}.$$

From Lemma 4.3 we get the following result.

Lemma 4.6. *There hold that*

(1) $W_{\delta_*} \subset W_{\delta^*}$ *for any* $0 < \delta_* < \delta^* \le 1$;
(2) $V_{\delta^*} \subset V_{\delta_*}$ *for any* $1 \le \delta_* < \delta^* < \frac{p+1}{2}$.

Moreover, by introducing

$$\begin{aligned} B_{r(\delta)} &= \left\{u \in H^1_{X,0}(\Omega) \mid \|Xu\| < r(\delta)\right\}, \\ \bar B_{r(\delta)} &= \left\{u \in H^1_{X,0}(\Omega) \mid \|Xu\| \le r(\delta)\right\}, \\ B^c_{r(\delta)} &= \left\{u \in H^1_{X,0}(\Omega) \mid \|Xu\| \ge r(\delta)\right\}, \end{aligned}$$

we can prove the following result.

Lemma 4.7. *For* $0 < \delta < \frac{p+1}{2}$, *we have*

$$B_{r_1(\delta)} \subset W_\delta \subset B_{r_2(\delta)},\ \ V_\delta \subset \bar{B}^c_{r(\delta)},$$

where $r_1(\delta) = \min\{r(\delta), \sqrt{2d(\delta)}\}$ *and* $r_2(\delta) = \sqrt{d(\delta)/b(\delta)}$.

Proof. For arbitrary $u \in B_{r_1(\delta)}$, we have $\|Xu\| < r(\delta)$. Together with 4.2 (1) we deduce that either $I_\delta(u) > 0$ or $\|Xu\| = 0$ holds. In addition, by (4.1) there holds $J(u) \leq \frac{1}{2}\|Xu\|^2$. By combining with $\|Xu\|^2 < 2d(\delta)$ we have $J(u) < d(\delta)$. Then $u \in W_\delta$, and thus $B_{r_1(\delta)} \subset W_\delta$. By Lemmas 4.2 and 4.4 the other conclusion follows. □

By Definition 2.3 we see that the weak solution u satisfies the energy equality

$$\int_0^t \|u_\tau\|^2 d\tau + J(u) = J(u_0),\ \forall\, t \in [0, T). \tag{4.10}$$

Next, we consider the invariance of W_δ, V_δ as follows.

Proposition 4.1. *Assume that* $u_0 \in H^1_{X,0}(\Omega)$, $0 < \mu < d$. *Denote by* δ_1, δ_2 *the two solutions of* $d(\delta) = \mu$ *for* $\delta_1 < 1 < \delta_2$. *For any weak solution u of problem* (1.1) *satisfying* $J(u_0) \in (0, \mu]$, *there hold that for arbitrary* $t \in [0, T)$, $\delta \in (\delta_1, \delta_2)$,

(1) *if* $I(u_0) > 0$, *then* $u \in W_\delta$;
(2) *if* $I(u_0) < 0$, *then* $u \in V_\delta$.

Proof. (1) First, we claim $u_0 \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$. In fact, if $J(u_0) \leq \mu$ and $I(u_0) > 0$, we see from Lemma 4.5 that $J(u_0) < d(\delta)$ and $I_\delta(u_0) > 0$, and the claim follows.

Now, for arbitrary $\delta \in (\delta_1, \delta_2)$, $t \in (0, T)$ we claim $u(x, t) \in W_\delta$. Otherwise, there exist a first time $t_0 \in (0, T)$ and $\delta_0 \in (\delta_1, \delta_2)$ such that $u(x, t_0) \in \partial W_{\delta_0}$. This implies that either $I_{\delta_0}(u(t_0)) = 0$, $\|Xu(t_0)\| \neq 0$ or $J(u(t_0)) = d(\delta_0)$ holds. By (4.10) we obtain

$$\int_0^t \|u_\tau\|^2 d\tau + J(u) = J(u_0) < d(\delta),\ \forall\, t \in [0, T),\ \delta \in (\delta_1, \delta_2), \tag{4.11}$$

which implies $J(u(t_0)) \neq d(\delta_0)$. Thus $I_{\delta_0}(u(t_0)) = 0$ and $\|Xu(t_0)\| \neq 0$, by (4.5) we get $J(u(t_0)) \geq d(\delta_0)$, which contradicts (4.11).

(2) First, we claim $u_0 \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$. By $J(u_0) \leq \mu$, $I(u_0) < 0$ and Lemma 4.5 we get $J(u_0) < d(\delta)$ and $I_\delta(u_0) < 0$, and thus the claim follows.

Next, for arbitrary $\delta \in (\delta_1, \delta_2)$ and $t \in (0, T)$ we claim $u(x, t) \in V_\delta$. Otherwise, there exist a first time $t_0 \in (0, T)$ and $\delta_0 \in (\delta_1, \delta_2)$ such that $I_{\delta_0}(u(t)) < 0$ for $t \in [0, t_0)$, and $u(x, t_0) \in \partial V_{\delta_0}$. This implies that

$$I_{\delta_0}(u(t_0)) = 0 \ \text{ or } \ J(u(t_0)) = d(\delta_0).$$

It follows from (4.11) that $J(u(t_0)) \neq d(\delta_0)$, and thus $I_{\delta_0}(u(t_0)) = 0$. Together with Lemma 4.2 there holds $\|Xu(t)\| \geq r(\delta_0)$ for $0 \leq t \leq t_0$. Hence, we see from (4.5) that $J(u(t_0)) \geq d(\delta_0)$, which contradicts (4.11). □

Now, by Proposition 4.1 and Lemma 4.3 we have the corollary as follows.

Corollary 4.1. *Assume that* $u_0 \in H^1_{X,0}(\Omega)$, $0 < J(u_0) \le \mu < d$. *Denote by* δ_1, δ_2 *the two solutions of* $d(\delta) = \mu$ *for* $\delta_1 < 1 < \delta_2$. *Then, both* W_δ *and* V_δ *are invariant for arbitrary* $\delta \in (\delta_1, \delta_2)$, *and thus*

$$W_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} W_\delta, \qquad V_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} V_\delta$$

are invariant under the flow of problem (1.1).

Furthermore, we discuss the invariant manifolds of the solutions with non-positive level energy by the following results.

Proposition 4.2. *For any nontrivial solutions* u *of problem* (1.1) *satisfying* $J(u_0) = 0$, *we have* $u \in B^c_{r_0}$, *where*

$$B^c_{r_0} = \left\{u \in H^1_{X,0}(\Omega) \mid \|Xu\| \ge r_0\right\}, \ r_0 := \left(\frac{p+1}{2C_X^{p+1}}\left(1 - \mu C_*^2\right)\right)^{\frac{1}{p-1}}.$$

Proof. It follows from (4.10) that $J(u) \le 0$ for $0 \le t < T$. Then

$$\begin{aligned}\frac{1}{2}\|Xu\|^2 \le& \frac{1}{2}\int_\Omega \mu V(x)|u|^2\,dx + \frac{1}{p+1}\|g(x)^{\frac{1}{p+1}} u\|^{p+1}_{p+1} \\ \le& \left(\frac{\mu}{2}C_*^2 + \frac{1}{p+1}C_X^{p+1}\|Xu\|^{p-1}\right)\|Xu\|^2, \ \forall\, t \in [0, T),\end{aligned}$$

which implies that either $\|Xu\| = 0$ or $\|Xu\| \ge r_0$ holds. We claim $\|Xu\| \equiv 0$ for any $t \in [0, T)$ if $\|Xu_0\| = 0$. If it is false, there holds $0 < \|Xu(t_0)\| < r_0$ for some $t_0 \in (0, T)$, a contradiction appears. Similarly, for the case $\|Xu_0\| \ge r_0$ we can prove $\|Xu\| \ge r_0$ for $t \in [0, T)$. The conclusion follows. □

Proposition 4.3. *Let* $u_0 \in H^1_{X,0}(\Omega)$. *If either* $J(u_0) < 0$ *or* $J(u_0) = 0$, $\|Xu_0\| \ne 0$ *occurs, then* $u \in V_\delta$ *for any* $\delta \in \left(0, \frac{p+1}{2}\right)$, *where* u *is a weak solution of problem* (1.1).

Proof. It follows from (4.10) and (4.9) that

$$J(u_0) \ge J(u) \ge b(\delta)\|Xu\|^2 + \frac{I_\delta(u)}{p+1}, \ \forall\, \delta \in \left(0, \frac{p+1}{2}\right). \tag{4.12}$$

If $J(u_0) < 0$, there holds

$$J(u) < 0 < d(\delta), \ I_\delta(u) < 0, \ \forall\, \delta \in \left(0, \frac{p+1}{2}\right). \tag{4.13}$$

This shows that

$$u \in V_\delta, \ \forall\, \delta \in \left(0, \frac{p+1}{2}\right), \ t \in [0, T). \tag{4.14}$$

On the other hand, if $J(u_0) = 0$ and $\|Xu_0\| \ne 0$ occur, by Proposition 4.2 we have $\|Xu\| \ge r_0$ for $t \in [0, T)$. By combining with (4.12), we obtain (4.13), and thus (4.14). The conclusion follows. □

Corollary 4.2. *Let* $u_0 \in H^1_{X,0}(\Omega)$. *If either* $J(u_0) < 0$ *or* $J(u_0) = 0$, $\|Xu_0\| \ne 0$ *occurs, then* $u \in B^c_{r\left(\frac{p+1}{2}\right)}$, *where* u *is a weak solution of problem* (1.1).

Proof. For any $\delta \in \left(0, \frac{p+1}{2}\right)$, by Proposition 4.3 and Lemma 4.2 we see that

$$\|Xu\| > r(\delta), \; t \in [0, T).$$

Letting $\delta \to \frac{p+1}{2}$, we obtain $\|Xu\| \geq r\left(\frac{p+1}{2}\right)$. The conclusion follows. □

Finally, for $J(u_0) < d$ we discuss the vacuum isolating of solutions.

Proposition 4.4. *Let $u_0 \in H^1_{X,0}(\Omega)$, $\mu \in (0, d)$. Denote by δ_1, δ_2 the two solutions of $d(\delta) = \mu$ for $\delta_1 < 1 < \delta_2$, we have a vacuum region*

$$U_\mu = \mathcal{N}_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} \mathcal{N}_\delta = \left\{ w \in H^1_{X,0}(\Omega) \mid \|Xw\| \neq 0, \; I_\delta(w) = 0, \; \delta_1 < \delta < \delta_2 \right\}$$

for given $\mu \geq J(u_0)$, such that any weak solution u of problem (1.1) *is outside of U_μ. Moreover, U_μ becomes larger and larger if μ is decreasing, and U_μ approximates U_0 as $\mu \to 0$, where*

$$U_0 = \left\{ w \in H^1_{X,0}(\Omega) \mid \|Xw\| \neq 0, \; I_\delta(w) = 0, \; \delta \in \left(0, \frac{p+1}{2}\right) \right\}.$$

Proof. For any weak solution u of problem (1.1) with $J(u_0) \leq \mu$, it is sufficient to prove that if $\|Xu\| \neq 0$, for any $\delta \in (\delta_1, \delta_2)$ there holds $u(t) \notin \mathcal{N}_\delta$, equivalently, $I_\delta(u(t)) \neq 0$ for $t \in [0, T)$.

We claim $I_\delta(u_0) \neq 0$. If it is false, then $I_\delta(u_0) = 0$. Together with Lemma 4.3 and (4.5) we have $d(\delta_1) = d(\delta_2) = \mu < d(\delta) \leq J(u_0)$, which contradicts $J(u_0) \leq \mu$.

Now, assume that there exists $t_1 > 0$ such that $u(t_1) \in U_\mu$. This shows that $u(t_1) \in \mathcal{N}_{\delta_0}$ for some $\delta_0 \in (\delta_1, \delta_2)$. Then we see from (4.11) and (4.5) that $J(u_0) < d(\delta_0) \leq J(u(t_1)) \leq J(u_0)$, which is a contradiction. Proposition 4.4 has been proved. □

5. Global existence and blow-up in finite time of solutions

In this section, we establish the global existence, the asymptotic behavior and the finite time blow-up of solutions for problem (1.1) with subcritical or critical initial energy.

5.1. Global existence of solutions

By the potential well method and the Galerkin method, we will show the following theorem.

Theorem 5.1 (Global existence)**.** *Under the assumptions* (H), $(\mathrm{H}_{\partial\Omega})$, (H_V), $\left(\mathrm{H}_g\right)$ *and* $\left(\mathrm{H}_p\right)$, *for any $u_0 \in H^1_{X,0}(\Omega)$ satisfying $J(u_0) \leq d$ and $I(u_0) \geq 0$, there exists a global weak solution u for the problem* (1.1) *such that $u(x,t) \in L^\infty\left(0, +\infty; H^1_{X,0}(\Omega)\right)$ with $u_t \in L^2\left(0, +\infty; L^2(\Omega)\right)$. Moreover,*

- *if $J(u_0) < d$, there holds*

$$\|Xu(\cdot, t)\| \leq \|Xu_0\| e^{\frac{1}{2} - \xi\lambda_1 t}, \quad t \in [0, +\infty), \tag{5.1}$$

where

$$\xi = 1 - \mu C_*^2 - C_X^{p+1} \left(\frac{2(p+1)}{p-1} \left(1 - \mu C_*^2\right)^{-1} J(u_0) \right)^{\frac{p-1}{2}} > 0;$$

- *if $J(u_0) = d$ and $I(u_0) > 0$, for any $\varepsilon \in (0, d)$ small enough, there exists $t_\varepsilon > 0$ such that*

$$\|Xu(\cdot, t)\| \leq \|Xu(t_\varepsilon)\| e^{\frac{1}{2} - \zeta\lambda_1 t}, \ t \in [t_\varepsilon, +\infty), \tag{5.2}$$

where

$$\zeta = 1 - \mu C_*^2 - C_X^{p+1} \left(\frac{2(p+1)}{p-1} \left(1 - \mu C_*^2\right)^{-1} (d - \varepsilon) \right)^{\frac{p-1}{2}} > 0.$$

For later use, we recall the following estimation.

Lemma 5.1 (cf. [18] Theorem 8.1)**.** *Denote by $\varphi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ a non-increasing function. If*

$$\int_s^{+\infty} \varphi(t)\, dt \leq C\varphi(s), \ s \in [0, +\infty)$$

for some constant $C > 0$, then $\varphi(t) \leq \varphi(0) e^{1 - t/C}$ for all t.

Proof of Theorem 5.1. We divide our proof into the four steps as follows.

Step 1: Global existence for $J(u_0) < d$.

Let $\{\phi_k(x)\}_{k \geq 1}$ be a base of $H^1_{X,0}(\Omega)$ in Proposition 2.5. Then we can construct the approximate solutions of problem (1.1) as follows

$$u_m(x, t) = \sum_{k=1}^{m} a_{km}(t)\, \phi_k(x), \ m = 1, 2, \cdots,$$

such that

$$\left(u_{mt}, \phi_j\right) + \left(Xu_m, X\phi_j\right) - \left(\mu V(x)\, u_m, \phi_j\right) = \left(g(x)\, |u_m|^{p-1} u_m, \phi_j\right), j = 1, \cdots, m, \tag{5.3}$$

and as $m \to \infty$,

$$u_m(x, 0) = \sum_{k=1}^{m} a_{km}(0)\, \phi_k(x) \to u_0(x) \ \text{ in } H^1_{X,0}(\Omega). \tag{5.4}$$

Now, multiply (5.3) by $a'_{jm}(t)$, sum for j, integrate with respect to t, we get

$$\int_0^t \|u_{m\tau}\|^2 d\tau + J(u_m(t)) = J(u_m(0)), \ t \in [0, T). \tag{5.5}$$

Together with (5.4) we obtain $J(u_m(0)) \to J(u_0)$ as $m \to \infty$, and thus

$$\int_0^t \|u_{m\tau}\|^2 d\tau + J(u_m(t)) = J(u_m(0)) < d, \ t \in [0, T) \tag{5.6}$$

for m large enough.

Similar to the proof of Proposition 4.1 (1), for m large enough and $t \in [0, T)$, by (5.6) we have $u_m(x, t) \in W$. Together with (1.3), (4.2) and (5.6) we conclude that

$$\int_0^t \|u_{m\tau}\|^2 d\tau + \frac{p-1}{2(p+1)} \left(1 - \mu C_*^2\right) \|Xu_m\|^2 < d, \ t \in [0, T),$$

which shows that $T = +\infty$,

$$\begin{aligned}
&\int_0^t \|u_{m\tau}\|^2 d\tau < d,\\
&\|Xu_m\|^2 < \frac{2(p+1)}{p-1}\left(1-\mu C_*^2\right)^{-1} d,\\
&\int_\Omega V(x)|u_m|^2\,dx \le C_*^2\|Xu_m\|^2 < \frac{2(p+1)}{p-1}C_*^2\left(1-\mu C_*^2\right)^{-1} d,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\int_\Omega \left|g(x)^{\frac{p}{p+1}}|u_m|^{p-1}u_m\right|^{\frac{p+1}{p}} dx &= \|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}\\
&\le C_X^{p+1}\|Xu_m\|^{p+1}\\
&< C_X^{p+1}\left(\frac{2(p+1)}{p-1}\left(1-\mu C_*^2\right)^{-1} d\right)^{\frac{p+1}{2}},
\end{aligned} \tag{5.8}$$

where we used (2.4) for the penultimate inequality.

Let $\stackrel{w^*}{\longrightarrow}$ be the weakly star convergence. By (5.7) and (5.8) we have a subsequence, also denoted by $\{u_m\}$, satisfying as $m \to \infty$,

$$\begin{aligned}
&u_{mt} \stackrel{w^*}{\longrightarrow} u_t \text{ in } L^2\left(0,\infty;L^2(\Omega)\right),\\
&u_m \stackrel{w^*}{\longrightarrow} u \text{ in } L^\infty\left(0,\infty;H^1_{X,0}(\Omega)\right),\\
&g(x)^{\frac{p}{p+1}}|u_m|^{p-1}u_m \stackrel{w^*}{\longrightarrow} g(x)^{\frac{p}{p+1}}|u|^{p-1}u \text{ in } L^\infty\left(0,\infty;L^{\frac{p+1}{p}}(\Omega)\right).
\end{aligned}$$

Then, fix j and let $m \to \infty$ in (5.3), we deduce

$$\left(u_t,\phi_j\right) + \left(Xu, X\phi_j\right) - \left(\mu V(x)u,\phi_j\right) = \left(g(x)|u|^{p-1}u,\phi_j\right),\ j = 1,2,\ldots.$$

As $\{\phi_k(x)\}_{k\ge1}$ is a base of $H^1_{X,0}(\Omega)$, and thus for any $w \in H^1_{X,0}(\Omega)$ there holds

$$(u_t,w) + (Xu,Xw) - (\mu V(x)u,w) = \left(g(x)|u|^{p-1}u,w\right),\ t>0.$$

Moreover, it follows from (5.4) that $u(x,0) = u_0(x)$ in $H^1_{X,0}(\Omega)$. Therefore, we have a global weak solution $u(x,t) \in L^\infty\left(0,+\infty;H^1_{X,0}(\Omega)\right)$ satisfying $u_t(x,t) \in L^2\left(0,+\infty;L^2(\Omega)\right)$.

Step 2: Asymptotic behavior for $J(u_0) < d$.

Now, we only need to discuss the case that $0 < J(u_0) < d$ and $I(u_0) > 0$. We see from Proposition 4.1 that $u \in W$ for $t \ge 0$, which gives $I(u) \ge 0$ for $t \ge 0$. It follows from (1.3), (4.2) and (4.10) that

$$\begin{aligned}
J(u_0) \ge J(u) &= \left(\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2\,dx\right)\left(\frac{1}{2}-\frac{1}{p+1}\right) + \frac{1}{p+1}I(u)\\
&\ge \frac{p-1}{2(p+1)}\left(1-\mu C_*^2\right)\|Xu\|^2.
\end{aligned} \tag{5.9}$$

Then by (2.4) there holds

$$\begin{aligned}\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1} &\le C_X^{p+1}\|Xu\|^{p+1}\\ &\le C_X^{p+1}\left(\frac{2(p+1)}{p-1}\left(1-\mu C_*^2\right)^{-1}J(u_0)\right)^{\frac{p-1}{2}}\|Xu\|^2.\end{aligned} \tag{5.10}$$

Inserting (5.10) into (4.1), by (1.3) we conclude that

$$\begin{aligned}I(u) =&\|Xu\|^2-\int_\Omega \mu V(x)|u|^2\,dx-\|g(x)^{\frac{1}{p+1}}u\|_{p+1}^{p+1}\\ \ge&\left(1-\mu C_*^2-C_X^{p+1}\left(\frac{2(p+1)}{p-1}\left(1-\mu C_*^2\right)^{-1}J(u_0)\right)^{\frac{p-1}{2}}\right)\|Xu\|^2\\ =&\,\xi\|Xu\|^2,\end{aligned} \tag{5.11}$$

where

$$\xi := 1-\mu C_*^2-C_X^{p+1}\left(\frac{2(p+1)}{p-1}\left(1-\mu C_*^2\right)^{-1}J(u_0)\right)^{\frac{p-1}{2}}.$$

Note from $J(u_0) < d$ and Lemma 4.1 (4) that $\xi > 0$.

Furthermore, by taking $w = u$ in (2.7), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + I(u) = 0,\ t\in[0,+\infty).$$

This gives that

$$\int_t^T I(u(\tau))\,d\tau = \frac{1}{2}\|u(t)\|^2-\frac{1}{2}\|u(T)\|^2 \le \frac{1}{2}\|u(t)\|^2,\ t\in[0,T). \tag{5.12}$$

Then, by (5.11), (5.12) and the Poincaré inequality (2.3) we get

$$\int_t^T \|Xu(\cdot,\tau)\|^2 d\tau \le \frac{1}{2\xi\lambda_1}\|Xu(t)\|^2,\ t\in[0,T).$$

Let $T \to +\infty$, by Lemma 5.1 we obtain (5.1).

Step 3: Global existence for $J(u_0) = d$.

Let $u_{0m} = \theta_m u_0$ for $m > 1$ and $\theta_m = 1-\frac{1}{m}$. We discuss the problem (1.1) with the initial condition

$$u(x,0) = u_{0m}(x). \tag{5.13}$$

From Lemma 4.1 (3) and $I(u_0) \ge 0$ we have

$$\begin{aligned}&\lambda_X = \lambda_X(u_0) \ge 1,\\ &I(u_{0m}) = I(\theta_m u_0) > 0,\\ &J(u_{0m}) = J(\theta_m u_0) < J(u_0) = d.\end{aligned}$$

The remaining proof follows from the similar proof of step 1.

Step 4: Asymptotic behavior for $J(u_0) = d$ and $I(u_0) > 0$.

It follows from the discussions above that $I(u) \geq 0$ for $t \geq 0$. Therefore, we only need to discuss the following two cases.

(1) $I(u) = -(u_t, u) > 0$ for $t \geq 0$. It follows that $\|u_t\| > 0$, and thus $\int_0^t \|u_\tau\|^2 d\tau$ is increasing for t on $[0, +\infty)$. Then, for any given $\varepsilon \in (0, d)$ small enough, by (4.10) there holds

$$d - \varepsilon = J(u(t_\varepsilon)) = J(u_0) - \int_0^{t_\varepsilon} \|u_\tau\|^2 d\tau$$

for some $t_\varepsilon > 0$. Letting the initial time $t = t_\varepsilon$, by similar proof of step 2 we obtain (5.2).

(2) For some $t_1 > 0$ there hold $I(u(t_1)) = 0$ and $I(u) > 0$ for $t \in [0, t_1)$. It follows that $\|u_t\| > 0$, and thus $\int_0^t \|u_\tau\|^2 d\tau$ is strictly increasing for $0 \leq t < t_1$. By (4.10) we conclude that

$$J(u(t_1)) = d - \int_0^{t_1} \|u_\tau\|^2 d\tau < d.$$

Together with (4.4) we deduce that $\|Xu(t_1)\| = 0$. Then by $I(u(t_1)) = 0$ we get $J(u(t_1)) = 0$. By combining with

$$\int_{t_1}^{t} \|u_\tau\|^2 d\tau + J(u) = J(u(t_1)), \quad t \in [t_1, +\infty),$$

we obtain $J(u(t)) \leq 0$ for $t \geq t_1$. Together with (1.3), (2.4) and (4.1) we conclude

$$\begin{aligned} \frac{1}{2}\|Xu\|^2 \leq& \frac{1}{2}\int_\Omega \mu V(x)|u|^2 dx + \frac{1}{p+1}\|g(x)^{\frac{1}{p+1}} u\|_{p+1}^{p+1} \\ \leq& \left(\frac{\mu}{2}C_*^2 + \frac{1}{p+1}C_X^{p+1}\|Xu\|^{p-1}\right)\|Xu\|^2, \ t \in [t_1, +\infty). \end{aligned}$$

This shows that either $\|Xu\| \geq \left(\frac{p+1}{2C_X^{p+1}}\left(1 - \mu C_*^2\right)\right)^{\frac{1}{p-1}}$ or $\|Xu\| = 0$ for $t \geq t_1$ holds. The former doesn't occur as $\|Xu(t_1)\| = 0$, thus $\|Xu\| \equiv 0$ for $t \geq t_1$. The decay estimate (5.2) follows.

Theorem 5.1 has been proved. □

Remark 5.1. *If one replace the assumption " $J(u_0) \leq d$, $I(u_0) \geq 0$" in Theorem 5.1 by " $0 < J(u_0) < d$, $I_{\delta_2}(u_0) > 0$" for δ_1, δ_2 being the two solutions of $d(\delta) = J(u_0)$ with $\delta_1 < 1 < \delta_2$, by Proposition 4.1 one can deduce that the problem* (1.1) *has a global weak solution $u \in L^\infty\left(0, +\infty; H^1_{X,0}(\Omega)\right)$ satisfying $u_t \in L^2\left(0, +\infty; L^2(\Omega)\right)$ and $u \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$, $t \in [0, +\infty)$.*

Remark 5.2. *If one replace the assumption " $I_{\delta_2}(u_0) > 0$" in Remark 5.1 by " $\|Xu_0\| < r(\delta_2)$", by Lemmas 4.2, 4.4 and Proposition 4.1 one can deduce that the problem* (1.1) *has a global weak solution $u \in L^\infty\left(0, +\infty; H^1_{X,0}(\Omega)\right)$ satisfying $u_t \in L^2\left(0, +\infty; L^2(\Omega)\right)$ and*

$$\|Xu\|^2 < \frac{d(\delta)}{b(\delta)}, \ \delta \in (\delta_1, \delta_2), \ t \in [0, +\infty).$$

Furthermore, there holds $\|Xu\|^2 \leq \frac{d(\delta_1)}{b(\delta_1)}$, $t \in [0, +\infty)$.

5.2. Blow-up in finite time of solutions

In this subsection, we mainly prove the following result.

Theorem 5.2 (Blow-up)**.** *Under the assumptions* (H), (H$_{\partial\Omega}$), (H$_V$), $\left(\mathrm{H}_g\right)$ *and* $\left(\mathrm{H}_p\right)$, *for* $u_0 \in H^1_{X,0}(\Omega)$ *satisfying* $J(u_0) \le d$ *and* $I(u_0) < 0$*, the weak solution* $u(x,t)$ *of problem* (1.1) *is finite time blow-up, i.e., for some* $T > 0$ *there holds*

$$\lim_{t\to T^-}\int_0^t \|u(\cdot,\tau)\|^2 d\tau = +\infty. \tag{5.14}$$

Proof. According to Theorem 3.1 we see that the problem (1.1) has a local weak solution $u \in C\left([0,T];H^1_{X,0}(\Omega)\right)$. We will complete the proof of Theorem 5.2 by two steps as follows.

Step 1: Blow-up for $J(u_0) < d$**.**

By introducing

$$F(t) := \int_0^t \|u(\tau)\|^2 d\tau,\ t \in [0,T],$$

we obtain

$$\begin{aligned}\dot F(t) &= \|u(t)\|^2,\\ \ddot F(t) &= 2(u_t,u) = -2I(u).\end{aligned} \tag{5.15}$$

Combining with (1.3), the Poincaré inequality (2.3), (4.2) and (4.10) we obtain

$$\begin{aligned}\ddot F(t) &= (p-1)\left(\|Xu\|^2 - \int_\Omega \mu V(x)|u|^2\,dx\right) - 2(p+1)J(u)\\ &\ge (p-1)\left(1-\mu C_*^2\right)\lambda_1 \dot F(t) - 2(p+1)J(u_0) + 2(p+1)\int_0^t \|u_\tau\|^2 d\tau.\end{aligned} \tag{5.16}$$

We deduce from

$$\left(\int_0^t (u_\tau,u)\,d\tau\right)^2 = \frac14\left(\int_0^t \frac{d}{d\tau}\|u\|^2 d\tau\right)^2 = \frac14\left(\dot F^2(t) - 2\|u_0\|^2\dot F(t) + \|u_0\|^4\right)$$

that

$$\dot F^2(t) = 2\|u_0\|^2\dot F(t) - \|u_0\|^4 + 4\left(\int_0^t (u_\tau,u)\,d\tau\right)^2.$$

Together with (5.15), (5.16) and the Hölder inequality we see that

$$\begin{aligned}&F(t)\ddot F(t) - \frac{p+1}{2}\dot F^2(t)\\ \ge&\left((p-1)\left(1-\mu C_*^2\right)\lambda_1\dot F(t) - 2(p+1)J(u_0) + 2(p+1)\int_0^t\|u_\tau\|^2 d\tau\right)F(t)\\ &-\frac{p+1}{2}\left(2\|u_0\|^2\dot F(t) - \|u_0\|^4 + 4\left(\int_0^t (u_\tau,u)\,d\tau\right)^2\right)\\ =&2(p+1)\left(\int_0^t\|u\|^2 d\tau\int_0^t\|u_\tau\|^2 d\tau - \left(\int_0^t (u_\tau,u)\,d\tau\right)^2\right) + \frac{p+1}{2}\|u_0\|^4\\ &+(p-1)\left(1-\mu C_*^2\right)\lambda_1\dot F(t)F(t) - (p+1)\|u_0\|^2\dot F(t) - 2(p+1)J(u_0)F(t)\\ \ge&(p-1)\left(1-\mu C_*^2\right)\lambda_1\dot F(t)F(t) - (p+1)\|u_0\|^2\dot F(t) - 2(p+1)J(u_0)F(t).\end{aligned} \tag{5.17}$$

Next, we will prove

$$F(t)\ddot{F}(t)-\frac{p+1}{2}\dot{F}^2(t)>0 \tag{5.18}$$

in the following two cases, respectively.

(1) $J(u_0)\le 0$. It follows from (5.17) that

$$F(t)\ddot{F}(t)-\frac{p+1}{2}\dot{F}^2(t)\ge (p-1)\left(1-\mu C_*^2\right)\lambda_1\dot{F}(t)F(t)-(p+1)\|u_0\|^2\dot{F}(t). \tag{5.19}$$

We claim $I(u(t))<0$ for $t>0$. Otherwise, for some $t_0>0$ there hold $I(u(t_0))=0$ and $I(u(t))<0$ for $t\in[0,t_0)$. Then we see from Lemma 4.2 that $\|Xu(t)\|\ge r(1)$ for $0\le t\le t_0$. Together with (4.4) there holds $J(u(t_0))\ge d$, which contradicts (4.10).

Next, by (5.15) we have $\ddot{F}(t)>0$ for $t\ge 0$, which shows that

$$F(t)\ge F(0)+t\dot{F}(0)=t\dot{F}(0),\ t\ge 0.$$

Then, for t large enough we get

$$(p-1)\left(1-\mu C_*^2\right)\lambda_1 F(t)>(p+1)\|u_0\|^2,$$

which together with (5.19) implies that (5.18).

(2) $0<J(u_0)<d$. It follows from Proposition 4.1 that $u(x,t)\in V_\delta$, and thus $I_\delta(u)<0$ for $t\ge 0$ and $\delta\in[1,\delta_2)$. By combining with its continuity and Lemma 4.2 we see that $\|Xu(t)\|\ge r(\delta_2)$ and $I_{\delta_2}(u(t))\le 0$ for $t\ge 0$, where δ_2 is taken to be the bigger solution of $d(\delta)=J(u_0)$. Then, by (5.15) we deduce that for $t\ge 0$ there hold

$$\begin{aligned}
\ddot{F}(t)&=2(\delta_2-1)\left(\|Xu\|^2-\mu\int_\Omega V(x)|u|^2\,dx\right)-2I_{\delta_2}(u)\\
&\ge 2(\delta_2-1)\left(1-\mu C_*^2\right)r^2(\delta_2),\\
\dot{F}(t)&\ge 2(\delta_2-1)\left(1-\mu C_*^2\right)r^2(\delta_2)t+\dot{F}(0)\ge 2(\delta_2-1)\left(1-\mu C_*^2\right)r^2(\delta_2)t,\\
F(t)&\ge(\delta_2-1)\left(1-\mu C_*^2\right)r^2(\delta_2)t^2+F(0)=(\delta_2-1)\left(1-\mu C_*^2\right)r^2(\delta_2)t^2.
\end{aligned}$$

Then for t large enough we obtain

$$\begin{aligned}
&\frac{1}{2}(p-1)\left(1-\mu C_*^2\right)\lambda_1 F(t)>(p+1)\|u_0\|^2,\\
&\frac{1}{2}(p-1)\left(1-\mu C_*^2\right)\lambda_1\dot{F}(t)>2(p+1)J(u_0).
\end{aligned}$$

Together with (5.17) we get (5.18) again.

Finally, for any $\beta>0$ a directly calculation shows that

$$\left(F^{-\beta}(t)\right)'=-\beta F^{-\beta-1}(t)\dot{F}(t),$$

$$\left(F^{-\beta}(t)\right)''=-\beta F^{-\beta-2}(t)\left(F(t)\ddot{F}(t)-(\beta+1)\dot{F}^2(t)\right).$$

Taking $\beta = \frac{p-1}{2}$, by (5.18) we obtain $\left(F^{-\frac{p-1}{2}}(t)\right)'' < 0$ for t large enough, which implies that

$$F^{-\frac{p-1}{2}}(t) \le F^{-\frac{p-1}{2}}(t_1)\left(1 - \frac{p-1}{2}\frac{\dot{F}(t_1)}{F(t_1)}(t-t_1)\right),\ \forall\, t > t_1$$

for some $t_1 > 0$ large enough. Therefore, for some $T \in (0, +\infty)$ there holds

$$\lim_{t\to T^-} F^{-\frac{p-1}{2}}(t) = 0,\ \text{i.e.,}\ \lim_{t\to T^-} F(t) = +\infty.$$

This is exactly (5.14).

Step 2: Blow-up for $J(u_0) = d$.

By the continuities of $J(u)$ and $I(u)$ with respect to t, there exists a $t_0 \in (0, T)$ small enough such that $J(u(t_0)) > 0$ and $I(u) < 0$ for $t \in [0, t_0]$. Then we have $(u_t, u) = -I(u) > 0$, and thus $\|u_t(t)\| > 0$, i.e., $\int_0^t \|u_\tau\|^2 d\tau$ is strictly positive for $t \in [0, t_0]$. By (4.10) we further obtain

$$0 < J(u(t_0)) = J(u_0) - \int_0^{t_0} \|u_\tau\|^2 d\tau < d.$$

Let t_0 be the initial time. By following the similar proof of step 1, we conclude that u is finite time blow-up.

Theorem 5.2 has been proved. □

Acknowledgments

This work is supported by National Nature Science Foundation of China, Grant No. 12101194.

Conflict of interest

The authors declare there is no conflict of interest.

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