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Research article

On the Hamiltonian and geometric structure of Langmuir circulation

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Abstract: The Craik-Leibovich equation (CL) serves as the theoretical model for Langmuir circulation. We show that the CL equation can be reduced to the dual space of a certain Lie algebra central extension. On this space, the CL equation can be rewritten as a Hamiltonian equation corresponding to the kinetic energy. Additionally, we provide an explanation of the appearance of this central extension structure through an averaging theory for Langmuir circulation. Lastly, we prove a stability theorem for two-dimensional steady flows of the CL equation. The paper also contains two examples of stable steady CL flows.

Keywords: Langmuir circulation; Craik-Leibovich equation; Euler equation; central extension; Hamiltonian structure; stability
Mathematics Subject Classification: 76M60, 76E30, 37K30, 37K45, 37K65

1. Introduction

In 1938, Langmuir [1] reported his observation of windrows of seaweeds in the Sargasso Sea. When a wind blows over a water surface steadily, small objects, such as seaweeds or bubbles, floating on the water will align with the wind direction. This is called Langmuir circulation. Since its discovery, this fascinating phenomenon has sparked a lot of research, both experimental and theoretical. In 1976, Craik and Leibovich [2] derived the CL equation as a theoretical model for Langmuir circulation.

According to the Craik-Leibovich theory, Langmuir circulation arises due to the interaction between the flow and the fast oscillating fluid surface. The corresponding averaged system in a 3-dimensional domain with fixed boundaries is the CL equation:

$$\begin{cases} \frac{\partial v}{\partial t} + (v, \nabla) v + \operatorname{curl} v \times V_s = -\nabla p, \\ (v + V_s) \cdot \mathbf{n} = 0, \end{cases} \tag{1.1}$$

where V_s is obtained by averaging the oscillating surface and is referred to as the Stokes drift, while $\mathbf{n}$ represents the outer normal vector of the boundary.

The Hamiltonian formulation of the classical CL equation was studied in the works of Holm [3] and Vladimirov [4]. In the present paper, we will discuss the Hamiltonian structure of the CL equation from a geometric point of view based on [5]. It turns out that, on the dual space of a certain Lie algebra central extension, the CL equation can be rewritten as a Hamiltonian equation corresponding to the kinetic energy. Then, one can generalize the CL equation to any Riemannian manifold with boundaries.

The equation (1.1) was first derived by Craik and Leibovich using the averaging method. Later, Vladimirov and his coauthors [4] developed a multiscale averaging method to study Langmuir circulation. In [5], the author carried out a general averaging theory on a principal bundle related to incompressible free boundary hydrodynamics problems to obtain the CL equation. This perturbation theory also clarifies the origin of the Stokes drift.

Central extension structures also appear in other mathematical equations. We would like to mention the superconductivity equation and the $\beta-$plane equation. For details of these examples, please refer to Section 2.

The rest of the paper is organized as follows. Some preliminaries about the Euler equation and central extensions of Lie algebras are given in Section 2. In Section 3, we first generalize the CL equation to any Riemannian manifold with boundaries. Then, we prove the central extension structure of the CL equation along with its Hamiltonian structure, and we obtain a broad class of invariant functionals. The averaging theory for Langmuir circulation which explains the appearance of the central extension structure is presented in Section 4. Lastly, Section 5 discusses the stability of two-dimensional steady flows of the Craik-Leibovich equation.

2. Preliminaries: The Euler equation and central extension

In his seminal paper [6], Arnold studied the geodesics on Lie groups with one-sided invariant Riemannian metric. He showed that the geodesic equation can be reduced to the Euler equation on the dual space of Lie algebra.

More specifically, consider a Lie group G which can be finite or infinite dimensional. Let $\mathfrak{g}$ be its Lie algebra with $\mathfrak{g}^*$ being the dual of $\mathfrak{g}$. The geodesic equation can be formulated as a Hamiltonian equation on T^*G with a kinetic Hamiltonian. Use the inertia operator $K : \mathfrak{g} \to \mathfrak{g}^*$, and one can write the kinetic Hamiltonian as a function on $\mathfrak{g}^*$: $H(\mu) = -\frac{1}{2}\langle K^{-1}\mu, \mu\rangle$ for $\mu \in \mathfrak{g}^*$. The corresponding Hamiltonian equation on $\mathfrak{g}^*$ is given by

$$\frac{d\mu}{dt} = -ad^*_{K^{-1}\mu}\mu. \tag{2.1}$$

Definition 2.1. Equation (2.1) is called the **Euler equation**.

One can also write the kinetic energy $E(v) = \frac{1}{2}\langle v, Kv\rangle$ on the Lie algebra $\mathfrak{g}$. Then, by the right (or left) translation, one obtains a right (or left)-invariant metric on the Lie group G. Arnold proved in [6] that for the volume-preserving diffeomorphism group equipped with the right-invariant L^2-metric, equation (2.1) is the incompressible Euler equation in hydrodynamics.

Let $\mathfrak{h}$ be a Lie algebra and H be a vector space. One can define a bilinear, antisymmetric map: $\widehat{\omega} : \mathfrak{h} \times \mathfrak{h} \to H$. If it satisfies the identity

$$\widehat{\omega}([u,v],w) + \widehat{\omega}([v,w],u) + \widehat{\omega}([w,u],v) = 0, \ \forall u,\ v,\ w \in \mathfrak{h},$$

then $\widehat{\omega}$ is a Lie algebra 2-cocycle. One can use the 2-cocycle to define a new bracket: for $u,\ v \in \mathfrak{h}$ and $a,\ b \in H$,

$$[(u,a),(v,b)]^{\wedge} = ([u,v],\ \widehat{\omega}(u,v)).$$

Definition 2.2. A new Lie algebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus H$ with the Lie bracket $[\cdot,\cdot]^{\wedge}$ is called a **central extension** of $\mathfrak{h}$ by H.

Example 2.1. Consider a 3-dimensional compact manifold M, $dvol$ is a volume form on it, and B is a divergence-free vector field. It is known that the Lie algebra of the volume-preserving diffeomorphism group $\mathrm{Diff}_{dvol}(M)$ is formed by all the divergence-free vector fields [7]. We denote the Lie algebra by $\mathscr{X}_{dvol}(M)$. If M has boundaries, then the elements in $\mathscr{X}_{dvol}(M)$ should be tangent to the boundaries. One can define a 2-cocycle $\widehat{\omega}_B$ by

$$\widehat{\omega}_B(v,w) = \int_M (i_v i_w i_B\, dvol)\, dvol \ \text{ for } v,\ w \in \mathscr{X}_{dvol}(M).$$

Let $\widehat{\mathscr{X}_{dvol}}(M)$ be the central extension of $\mathscr{X}_{dvol}(M)$ through the 2-cocycle $\widehat{\omega}_B(v,w)$, and let $\widehat{\mathscr{X}_{dvol}}(M)$ be its dual space. Then, the Euler equation on $\widehat{\mathscr{X}_{dvol}}(M)$ coincides with the 3-dimensional superconductivity equation [8–10]

$$\frac{\partial u}{\partial t} + \nabla_u u + u \times B = -\nabla p,\ \ \nabla \cdot u = 0. \tag{2.2}$$

Example 2.2. Another interesting equation appears in the study of rotating 2D fluids. Let u be the velocity field, and in the 2D case, its vorticity ω is a function. The stream function ϕ satisfies $u = (\partial_y \phi, -\partial_x \phi)$. Then, the motion of fluids is governed by the β-plane equation:

$$\partial_t \omega = -\{\phi, \omega\} - \beta \partial_x \phi, \tag{2.3}$$

where $\beta \in \mathbb{R}$, and the last term in the equation is the effect of the Coriolis force.

All the symplectic vector fields on a 2-dimensional manifold D form a Lie algebra $\mathscr{X}_{\mathrm{symp}}(D)$. Zeitlin [11] considered a certain central extension $\widehat{\mathscr{X}_{\mathrm{symp}}}(D)$, whose dual is $\widehat{\mathscr{X}^*_{\mathrm{symp}}}(D)$, and demonstrated that equation (2.3) is the Euler equation on $\widehat{\mathscr{X}^*_{\mathrm{symp}}}(D)$.

3. The central extension and Hamiltonian structure related to the Craik-Leibovich theory of Langmuir circulation

3.1. A special central extension

In this section, we will define a central extension related to the geometric structure of Langmuir circulation.

Definition 3.1. On a Lie algebra $\mathfrak{g}$, we first define the **shifted 2-cocycle** $\widehat{\omega}_{V_s} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ for a fixed vector $V_s \in \mathfrak{g}$ by

$$\widehat{\omega}_{V_s}(u,v) = -\left\langle ad^*_u\, K(V_s), v \right\rangle, \tag{3.1}$$

where $u,\ v \in \mathfrak{g}$, and K is a map from $\mathfrak{g}$ to $\mathfrak{g}^*$.

Remark 3.1. Because $\langle ad^*_u\, K(V_s), v\rangle = -\langle K(V_s), [u,v]\rangle$, the shifted 2-cocycle $\widehat{\omega}_{V_s}$ is a 2-coboundary.

By means of the shifted 2-cocycle $\widehat{\omega}_{V_s}$, one can get a new Lie algebra $\hat{\mathfrak{g}}_{V_s}$, which is the central extension of $\mathfrak{g}$ by $\mathbb{R}$. We derive the Euler equation on its dual $\hat{\mathfrak{g}}^*_{V_s}$.

Proposition 3.1. *The Euler equation on $\hat{\mathfrak{g}}^*_{V_s}$ is*

$$\frac{d}{dt}\,\mu = -ad^*_{K^{-1}\mu}\,\{\mu - aK(V_s)\}. \tag{3.2}$$

Proof. Since

$$\begin{aligned}\left\langle ad^*_{(X,b)}(\mu,a),(Y,c)\right\rangle &= \langle(\mu,a),([X,Y],\widehat{\omega}_{V_s}(X,Y))\rangle = \langle\mu,[X,Y]\rangle + a\,\widehat{\omega}_{V_s}(X,Y)\\ &= \langle ad^*_X\,\mu,Y\rangle - \langle a\,ad^*_X K(V_s),Y\rangle = \langle ad^*_X\,\mu - a\,ad^*_X K(V_s),Y\rangle\,,\end{aligned}$$

take $X = K^{-1}\mu$ to get the Euler equation on $\hat{\mathfrak{g}}^*_{V_s}$

$$\frac{d}{dt}\,\mu = -ad^*_{K^{-1}\mu}\,\{\mu - aK(V_s)\}.$$

Remark 3.2. As mentioned in Section 2, the Euler equation is Hamiltonian, and the corresponding Hamiltonian function is $H(\mu) = -\frac{1}{2}\langle K^{-1}\mu,\,\mu\rangle$.

3.2. Hamiltonian structure and the generalized CL equation

Let M be an n-dimensional Riemannian manifold with boundary ∂M. The group $\mathrm{Diff}_{dvol}(M)$ is the group of all volume-preserving diffeomorphisms on M. Its Lie algebra $\mathscr{X}_{dvol}(M)$ consists of all the divergence-free vector fields on M that are tangent to the boundary ∂M. The dual space $\mathscr{X}^*_{dvol}(M)$ of the Lie algebra is the space of 1-forms on M modulo the exact 1-forms. We now introduce the generalized CL equation.

Theorem 3.1. ([5]) *The* ***generalized CL equation*** *on the space $\mathscr{X}^*_{dvol}(M)$ is*

$$\frac{d}{dt}\,[u] = -\mathscr{L}_{v+V_s}\,[u], \tag{3.3}$$

where $v+V_s \in \mathscr{X}_{dvol}(M)$, and $[u] = [v^b] \in \mathscr{X}^*_{dvol}(M)$.

Proof. Let $u = v^b$, and the equation (3.3) becomes

$$\frac{d}{dt}\,u = -\mathscr{L}_{v+V_s}\,u + df.$$

From the identities

$$\mathscr{L}_v(v^b) = (\nabla_v v)^b + \frac{1}{2}d\langle v,\,v\rangle$$

and

$$*(\mathrm{curl}\,v \wedge V_s) = i_{V_s} i_{\mathrm{curl}\,v}\mu = i_{V_s} dv^b = \mathscr{L}_{V_s}u,$$

we obtain an equation which can be seen as the generalization of the CL equation on M,

$$v_t + \nabla_v v + \nabla p = V_s \times \mathrm{curl}\,v. \tag{3.4}$$

Also, the condition $v+V_s \in \mathscr{X}_{dvol}(M)$ gives us the boundary condition. In dimension 3, the equation (3.4) is the classical CL equation.

Remark 3.3. Note that velocity fields v and V_s do not have to be elements of $\mathscr{X}_{dvol}(M)$, but their sum $v+V_S$ is an element of $\mathscr{X}_{dvol}(M)$, which is the boundary condition.

We show that the CL equation (3.3) is also Hamiltonian on the dual space $\mathscr{X}^*_{dvol}(M)$.

Corollary 3.1. *Consider the function* $H(u) = -\frac{1}{2}([u+V_s^b],\ K^{-1}[u+V_s^b])$ *defined on* $\mathscr{X}^*_{dvol}(M)$. *The Hamiltonian equation for this function is the CL equation (3.3).*

Proof. The Hamiltonian equation on $\mathscr{X}^*_{dvol}(M)$ is

$$\frac{d}{ds}\,[u] = -\mathscr{L}_{\frac{\delta H}{\delta[u]}}\,[u].$$

For the Hamiltonian function $H = -\frac{1}{2}([u+V_s^b],\ K^{-1}[u+V_s^b])$, the functional derivative is $\frac{\delta H}{\delta[u]} = K^{-1}[u+V_s^b]) = v+V_s$. Therefore, we obtain equation (3.3).

Now, set $[u]' = [u+V_s^b]$, and equation (3.3) becomes

$$\frac{d}{dt}\,[u]' = -\mathscr{L}_{K^{-1}[u]'}\,\{[u]' - [V_s^b]\}. \tag{3.5}$$

Consider a 2-cocycle $\widehat{\omega}_{V_s}$ on $\mathscr{X}_{dvol}(M)$ defined by

$$\widehat{\omega}_{V_s}(u,v) = -\left\langle \mathscr{L}_u\, V_s^b, v\right\rangle.$$

Then, using it, one can define a Lie algebra $\widehat{\mathscr{X}_{dvol}}(M)$, which is a central extension of $\mathscr{X}_{dvol}(M)$ by $\mathbb{R}$.

Theorem 3.2. ([5]) *Equation (3.5) is the Euler equation on* $\widehat{\mathscr{X}^*_{dvol}}(M)$.

Proof. Let $a=1$ in equation (3.2), and we obtain equation (3.5).

Remark 3.4. One can also express the 2-cocycle $\widehat{\omega}_{V_s}$ as the integral of a 2-form dV_s^b on M. Indeed,

$$\begin{aligned}
&\int_M -dV_s^b(u,v)\,dvol = -\int_M \langle i_u dV_s^b, v\rangle\,dvol \\
&= -\int_M \langle \mathscr{L}_u V_s^b, v\rangle\,dvol + \int_M \langle d i_u V_s^b, v\rangle\,dvol = -\left\langle \mathscr{L}_u\, V_s^b, v\right\rangle = \widehat{\omega}_{V_s}(u,v),
\end{aligned}$$

where the equation $\int_M \langle d i_u V_s^b, v\rangle\,dvol = 0$ holds since $v \in \mathscr{X}_{dvol}(M)$.

Next, we present the first integrals of the CL equation. The following corollary follows immediately from the fact that the action of the group $\mathrm{Diff}_{dvol}(M)$ can be viewed as a change of variables that preserves $dvol$ on M.

Corollary 3.2. *(1) For a* $(2k+1)$*-dimensional manifold M, the first integral of Equation (3.3) is* $I([u]) = \int_M u\wedge(du)^k$.

(2) For a $2k$*-dimensional manifold M and an arbitrary function h, the first integral of Equation (3.3) is*

$$I_h([u]) = \int_M h\left(\frac{(du)^k}{dvol}\right)\,dvol.$$

Remark 3.5. Note that in dimension 3 (i.e., $k=1$), the integral $I([u])=\int_M u\wedge du$ is the Eulerian mean helicity discussed in [3].

Remark 3.6. The moment $[u]=[v^b]$ is transferred by the flow corresponding to the velocity field $v+V_s$ (Kelvin's theorem for the CL equation). For the CL equation, we provide two equivalent definitions of isovorticed fields, corresponding to equation (3.3) and (3.5), respectively.

Definition 3.2. For equation (3.3), two vector fields u_1 and u_2 are **isovorticed** if curl u_1 can be transferred to curl u_2 by a volume-preserving diffeomorphism, and they satisfy the same boundary condition: $(u_1+V_s)\cdot\mathbf{n}=(u_2+V_s)\cdot\mathbf{n}=0$.

Definition 3.2′. For equation (3.5), two vector fields u_1, $u_2\in\mathscr{X}_{dvol}(M)$ are **isovorticed** if curl (u_1-V_s) can be transferred to curl (u_2-V_s) by a volume-preserving diffeomorphism.

4. Averaging theory for Langmuir circulation

The central extension structure of the CL equation arises from the process of averaging. Specifically, let $M\to H$ be a circle bundle with the base H being a Lie group. Consider a Hamiltonian function $\mathscr{H}(x,y)$ for $(x,y)\in T_x^*M$. After averaging with respect to the circle action S^1, we obtain the averaged system on the reduced manifold, which is T^*H equipped with a reduced symplectic form ω_σ. The averaged system is Hamiltonian. We denote the averaged Hamiltonian function by $\bar{\mathscr{H}}(\bar{x},\bar{y})$ for $(\bar{x},\bar{y})\in T_{\bar{x}}^*H$. The reduced symplectic form ω_σ is

$$\omega_\sigma=\omega_0-\sigma\,\pi_H^*\,d\bar{a},$$

where $\omega_0=d\bar{x}\wedge d\bar{y}$, and π_H^* is the pullback of the projection $\pi_H:T^*H\to H$. Here, $\sigma\in\mathbb{R}$ is a value of the momentum map of the circle action. The 1-form $\bar{a}$ is the averaged connection 1-form on H (see Theorem 6.7 in [12] for more details).

Furthermore, if the original Hamiltonian $\mathscr{H}(x,y)$ is H-invariant, then we can perform another reduction on the cotangent bundle (T^*H,ω_σ) ("the reduction by stages"). Let $\mathfrak{h}^*$ be the dual of the Lie algebra $\mathfrak{h}$ of H, and then this reduction gives us a certain Poisson structure on $\mathfrak{h}^*$, given by (see Theorem 7.2.1 in [13])

$$\{f,g\}_\sigma(m)=-\left\langle m,\left[\frac{\delta f}{\delta m},\frac{\delta g}{\delta m}\right]\right\rangle-c_\sigma(e)\left(\frac{\delta f}{\delta m},\frac{\delta g}{\delta m}\right)\tag{4.1}$$

for $m\in\mathfrak{h}^*$, where f and g are smooth functions on $\mathfrak{h}^*$. The 2-form c_σ on H satisfies $\omega_\sigma=\pi_H^*c_\sigma$, and $c_\sigma(e)$ represents taking the value at the identity element $e\in H$.

For a nonzero value σ of the momentum map of the circle action, we can define a 2-cocycle $c:\mathfrak{h}\times\mathfrak{h}\to\mathbb{R}$ by $c:=\frac{1}{\sigma}c_\sigma(e)$. By means of c, we obtain the central extension $\widehat{\mathfrak{h}}$ of the Lie algebra $\mathfrak{h}$. The Poisson bracket (4.1) can be regarded as the natural Poisson bracket on the dual of $\widehat{\mathfrak{h}}$.

This explains the origin of the central extension structure that appears in the Hamiltonian formulation of the CL equation.

5. Stability of 2-dimensional steady flows of the Craik-Leibovich equation

5.1. Steady Craik-Leibovich flows

The vorticity equation of the incompressible CL flow corresponding to equation (3.5) is

$$\frac{\partial \omega}{\partial t} + \{v,\ \omega - \mathrm{curl}\, V_s\} = 0, \tag{5.1}$$

where $\omega = \mathrm{curl}\, v$ is the vorticity. Note that the velocity field v is divergence-free, and $(v - V_s)$ satisfies the CL equation (1.1). Therefore, the steady solution of equation (5.1) is given by

$$\{v, \mathrm{curl}\,(v - V_s)\} = 0. \tag{5.2}$$

Let us consider the following variational problem:

Problem 1. Suppose that the central extension group $\widehat{\mathrm{Diff}}_{dvol}(M)$, which corresponds to the central extension of the Lie algebra $\mathscr{X}_{dvol}(M)$ described in Section 3.2, exists. Given a vector field $u_0 \in \mathscr{X}_{dvol}(M)$, we aim to find the critical points of the kinetic function $K(u) = \frac{1}{2}\langle u, u\rangle$ on the set $S = \{u \in \mathscr{X}_{dvol}(M) \mid (u,1) = \mathrm{Ad}_g(u_0, 1),\ g \in \widehat{\mathrm{Diff}}_{dvol}(M)\}$, where Ad is the group adjoint action.

It turns out that the steady CL flows are the critical points of this variational problem.

Theorem 5.1. *The steady CL flows that satisfy equation (5.2) coincide with the critical points of variational problem 1.*

Proof. Let $(u,b) \in \mathscr{X}_{dvol}(M) \oplus \mathbb{R}$. The variation $\delta(v,a)$ of a field (v,a) under the adjoint action of (u,b) is given by

$$\delta(v,a) = [(u,b),\ (v,a)]^\wedge = ([u,\ v], -\langle V_s, [u,\ v]\rangle).$$

Suppose $v \in \mathscr{X}_{dvol}(M)$ is a critical point of Problem 1. Then, the first variation of E taken at v should be 0, so we have

$$\begin{aligned} 0 = \delta E &= \langle (v,1), \delta(v,1)\rangle = \langle (v,1), (\{v,\ u\}, -\langle V_s, [u,\ v]\rangle)\rangle \\ &= \langle (v,1), \{v,\ u\}\rangle - \langle V_s, \mathrm{curl}(u\times v)\rangle = \langle u, v\times \mathrm{curl}\, u\rangle - \langle \mathrm{curl}\, V_s, u\times v\rangle \\ &= \langle u, v\times \mathrm{curl}\, v\rangle - \langle u, v\times \mathrm{curl}\, V_s\rangle = \langle u, v\times \mathrm{curl}\,(v - V_s)\rangle. \end{aligned}$$

So, we have $\{v, \mathrm{curl}\,(v - V_s)\} = 0$.

5.2. Stability theorem for equilibrium points on central extensions of Lie algebras

Let $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ be a one-dimensional central extension of an arbitrary Lie algebra with 2-cocycle $\hat{\omega}$. We introduce the bilinear operation $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$\langle [v_1, v_2], v_3\rangle = \langle B(v_3, v_1), v_2\rangle, \tag{5.3}$$

where $v_i \in \mathfrak{g},\ i = 1,2,3$. Using this operation B, one can rewrite the Euler equation in its Lie algebra form [7]:

$$\frac{dv}{dt} = B(v,v), \tag{5.4}$$

where $v \in \mathfrak{g}$.

Then, we define an operator $w : \mathfrak{g} \to \mathfrak{g}$ induced from the 2-cocycle $\hat{\omega}$ by $\hat{\omega}(u,v) = \langle w(u), v\rangle$ for any $u,\ v \in \mathfrak{g}$.

Proposition 5.1. *On the central extension $\hat{\mathfrak{g}}$, the Euler equation is given by*

$$\frac{dv}{dt} = B(v,v) + a\,w(v). \tag{5.5}$$

Thus, the equilibrium point $(v_e, a_e) \in \hat{\mathfrak{g}}$ satisfies

$$B(v_e, v_e) + a_e\,w(v_e) = 0. \tag{5.6}$$

Proof. We can compute the bilinear operation $\hat{B} : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ of the central extension $\hat{\mathfrak{g}}$ as follows:

$$\begin{aligned}\left\langle \hat{B}((v_3,a_3),(v_1,a_1)),(v_2,a_2)\right\rangle &= \langle [(v_1,a_1),(v_2,a_2)],(v_3,a_3)\rangle = \langle ([v_1,v_2],\hat{\omega}(v_1,v_2)),(v_3,a_3)\rangle \\ &= \langle [v_1,v_2],v_3\rangle + a_3\hat{\omega}(v_1,v_2) = \langle B(v_3,v_1)+a_3 w(v_1),v_2\rangle\,,\end{aligned}$$

where $(v_i,a_i) \in \hat{\mathfrak{g}},\ i = 1,2,3$. So, the Euler equation on $\hat{\mathfrak{g}}$ takes the form

$$\frac{d(v,a)}{dt} = \hat{B}((v,a),(v,a))$$

and becomes equation (5.5). (For brevity, here we omit the second equation $\frac{da}{dt} = 0$.)

The Lie algebra $\hat{\mathfrak{g}}$ is foliated by the coadjoint orbits. Next, we prove a stability theorem for the equilibrium points on $\hat{\mathfrak{g}}$.

Theorem 5.2. *Assume that the equilibrium point $(v_e,a_e) \in \hat{\mathfrak{g}}$ is a regular point of the coadjoint foliation. Consider a test quadratic form $T\mid_{(v_e,a_e)}$:*

$$T\mid_{(v_e,a_e)}(\xi) = \langle B(v_e,\zeta)+a_e\,w(\zeta),\ B(v_e,\zeta)+a_e\,w(\zeta)\rangle + \langle [\zeta,v_e],\ B(v_e,\zeta)+a_e\,w(\zeta)\rangle, \tag{5.7}$$

where $\xi = B(v_e,\zeta)+a_e\,w(\zeta) \in \mathfrak{g}$. If for all nonzero $\xi \in \mathfrak{g}$ we have $T\mid_{(v_e,a_e)}(\xi) > 0$ or $T\mid_{(v_e,a_e)}(\xi) < 0$, then the equilibrium solution $(v_e,a_e) \in \hat{\mathfrak{g}}$ of equation (5.5) *is Lyapunov stable.*

Proof. To prove this, we use the second variation of the kinetic function $K(u) = \frac{1}{2}\langle u,\ u\rangle$ on the leaf of this coadjoint foliation of $\hat{\mathfrak{g}}$. As shown by Arnold (see, e.g., [7]), the second variation is given by

$$2\delta^2 K\mid_{(v_e,a_e)}(\xi) = \langle \hat{B}(v_e,\zeta),\ \hat{B}(v_e,\zeta)\rangle + \langle [\zeta,v_e],\ \hat{B}(v_e,\zeta)\rangle, \tag{5.8}$$

where $\xi = \hat{B}(v_e,\zeta) \in \mathfrak{g}$. Note that the quadratic form $\delta^2 K$ does not depend on the choice of ζ but only on $\xi = \hat{B}(v_e,\zeta)$.

Thanks to the computation in Proposition 5.1, we have $\hat{B}(v_e,\zeta) = B(v_e,\zeta)+a_e\,w(\zeta)$. Substituting this into (5.8), we obtain the test quadratic form (5.7). The Lyapunov stability of the equilibrium point (v_e,a_e) then follows from a revised Lagrange's theorem in chapter §II.3 of [7].

5.3. An a priori estimate for 2-dimensional steady flows of the CL equation

Let D be a 2-dimensional domain with boundary, and dA is an area form. The velocity field $v_e = \nabla^{\perp}\psi_e$ is a stationary solution of equation (5.1), and ψ_e^* stands for the stream function of the shifted velocity field $v_e - V_s$.

We can now prove the following theorem, which provides an a priori estimate for 2-dimensional steady flows of the CL equation:

Theorem 5.3. *Consider a 2-dimensional domain D with an area form dA. Assume that (i)* $\psi_e = F(\Delta\psi_e^*)$ *for some function F, and (ii) there exist two constants* c_1 *and* c_2 *such that*

$$0 < c_1 \leq \frac{\nabla\psi_e}{\nabla\Delta\psi_e^*} \leq c_2 < \infty. \tag{5.9}$$

Let $\psi(x,y,t) = \psi_e + h(x,y,t)$ *be the stream function corresponding to a different solution of the CL equation such that* $\oint_{\partial D} \nabla^{\perp}\psi \cdot dl = \oint_{\partial D} \nabla^{\perp}\psi_e \cdot dl$. *Then, for the perturbation* $h = h(x,y,t)$, *we have the following inequality:*

$$\|\nabla h\|_2^2 + c_1\|\Delta h\|_2^2 \leq \|\nabla h_0\|_2^2 + c_2\|\Delta h_0\|_2^2, \tag{5.10}$$

where $h_0 = h(x,y,0)$, *and* $\|\cdot\|_2^2$ *denotes the square of the* L^2-*norm, which is given by* $\|u\|_2^2 = \iint_D (u,u)\, dA$ *for a vector field u and* $\|f\|_2^2 = \iint_D f^2\, dA$ *for a function f.*

Proof. By the assumption of Theorem 5.3, we have $\psi_e = F(\Delta\psi_e^*)$. Let the function P be the primitive of F, i.e., $P' = F$. Then, $P''(\Delta\psi_e^*) = \frac{\nabla\psi_e}{\nabla\Delta\psi_e^*}$. Again by the assumption, we have $c_1 \leq P''(\omega) \leq c_2$, which gives

$$c_1\frac{\eta^2}{2} \leq P(\omega+\eta) - P(\omega) - P'(\omega)\eta \leq c_2\frac{\eta^2}{2}.$$

This implies

$$\|\nabla h\|_2^2 + 2\iint_D (P(\Delta\psi_e^* + \Delta h) - P(\Delta\psi_e^*) - P'(\Delta\psi_e^*)\Delta h)\, dA \geq \|\nabla h\|_2^2 + c_1\|\Delta h\|_2^2, \tag{5.11}$$

$$\|\nabla h_0\|_2^2 + 2\iint_D (P(\Delta\psi_e^* + \Delta h_0) - P(\Delta\psi_e^*) - P'(\Delta\psi_e^*)\Delta h_0)\, dA \leq \|\nabla h_0\|_2^2 + c_2\|\Delta h_0\|_2^2. \tag{5.12}$$

Introduce a functional

$$C(h) = \frac{\|\nabla h\|_2^2}{2} + \iint_D \left(P(\Delta\psi_e^* + \Delta h) - P(\Delta\psi_e^*) - P'(\Delta\psi_e^*)\Delta h\right)\, dA.$$

Then, the left-hand sides of (5.11) and (5.12) are $2C(h(t))$ and $2C(h(0))$, respectively. Therefore, if we could prove

$$C(h(t)) = C(h(0)), \tag{5.13}$$

then the theorem will follow immediately from (5.11), (5.12) and (5.13).

To prove equation (5.13), we construct the following invariant functional according to the conservation of kinetic energy and vorticity:

$$\Gamma(\psi) = \frac{\|\nabla\psi\|_2^2}{2} + \iint_D P(\Delta\psi^*)\, dA,$$

where $\nabla^{\perp}\psi^* + V_s = \nabla^{\perp}\psi$. The first variation of Γ at the equilibrium solution ψ_e is

$$\begin{aligned}\delta\,\Gamma\,|_{\psi_e}(h) &= \iint_D ((\nabla h,\, \nabla\psi_e) + P'(\Delta\psi_e^*)\Delta h)\, dA \\ &= \iint_D (-\psi_e\Delta h + P'(\Delta\psi_e^*)\Delta h)\, dA + \oint_{\partial D} \psi_e\frac{\partial h}{\partial n}\, dl.\end{aligned}$$

Since $P'(\Delta\psi_e^*) = F(\Delta\psi_e^*) = \psi_e$, and $\oint_{\partial D} \psi_e \frac{\partial h}{\partial n}\, dl = 0$, we obtain $\delta\,\Gamma\,|_{\psi_e}(h) = 0$.

For another functional $\tilde{\Gamma}(h) := \Gamma(\psi_e + h) - \Gamma(\psi_e)$, we have

$$\tilde{\Gamma}(h(t)) = \tilde{\Gamma}(h(0))$$

and

$$\tilde{\Gamma}(h) = \delta\,\Gamma\,|_{\psi_e}(h) + C(h),$$

and these two equalities imply (5.13). This completes the proof of the theorem.

Remark 5.1. Consider the leaf of the coadjoint foliation of $\widehat{\mathscr{X}_{dvol}}(D)$ which contains the equilibrium point $(v_e, 1)$. Then, $(v,1) \in \widehat{\mathscr{X}_{dvol}}(D)$ is on this leaf if and only if v is isovorticed to the equilibrium field v_e in the sense of Definition 3.2$'$. The second variation of $K(v) = \frac{1}{2}\iint_D (v,v)\, dA$ on the leaf is

$$\delta^2 K\,|_{v_e}(\xi) = \frac{1}{2}\iint_D \left((\xi,\, \xi) + \frac{\nabla\psi_e}{\nabla\Delta\psi_e^*}(\mathrm{curl}\;\xi)^2 \right) dA, \tag{5.14}$$

where ξ stands for a variation field at v_e.

Next, we prove equation (5.14). According to equation (5.8), we have

$$2\delta^2 K\,|_{v_e}(\xi) = \iint_D ((\xi,\, \xi) + (\xi, [\zeta, v_e]))\, dA, \tag{5.15}$$

where $\xi = B(v_e, \zeta) + w(\zeta) = B(v_e - V_s, \zeta)$. The last term of this equation is

$$\iint_D (\xi, [\zeta, v_e])\, dA = \iint_D (\xi, \mathrm{curl}\;(\zeta \times v_e))\, dA = \iint_D (\mathrm{curl}\;\xi, (\zeta \times v_e))\, dA. \tag{5.16}$$

Since $v_e = \nabla^{\perp}\,\psi_e$, and $v_e - V_s = \nabla^{\perp}\,\psi_e^*$, one gets

$$\mathrm{curl}\;\xi = \mathscr{L}_{\zeta}\Delta\psi_e^* = (\zeta, \nabla\Delta\psi_e^*),$$

$$\zeta \times v_e = \zeta \times (\nabla^{\perp}\,\psi_e) = (\zeta, \nabla\psi_e).$$

Thus,

$$\zeta \times v_e = \frac{\nabla\psi_e}{\nabla\Delta\psi_e^*}\mathrm{curl}\;\xi. \tag{5.17}$$

By equation (5.15), (5.16) and (5.17), we prove the required equation (5.14).

It is evident from the expression of (5.14) that the assumption (ii) in Theorem 5.3 guarantees that the second variation $\delta^2 K$ is positive definite.

In the following, we give two examples of stable steady CL flows.

Example 5.1. Let $D = \{(x,y)|\ 0 \le y \le 1\}$ be the domain, and let $u_s = (u_s(y), 0)$ be the Stokes drift velocity. A shear flow in this domain has velocity $u_e = (u(y), 0)$. It is easy to verify that this is a stationary flow. Assume that $u_e = \nabla^{\perp}\psi_e$, $u_e - u_s = \nabla^{\perp}\psi_e^*$, and we have

$$\frac{\nabla\psi_e}{\nabla\Delta\psi_e^*} = \frac{u}{(u - u_s)''}.$$

Hence, this shear flow is stable if there exist two constants c_1 and c_2 such that $0 < c_1 \le \frac{\nabla\psi_e}{\nabla\Delta\psi_e^*} \le c_2 < \infty$. This implies $u(y) - u_s(y)$ has no inflection point.

Example 5.2. Let $A = \{1 \leq r \leq 2\}$ be the domain, where $r = \sqrt{x^2+y^2}$. Consider a velocity field u_e and a drift velocity field u_s. If there are two functions f and g such that the stream functions ψ_e and ψ_e^* of u_e and $u_e - u_s$ satisfy $\psi_e = f(r)$ and $\psi_e^* = g(r)$, then the flow is steady, and we have

$$\frac{\nabla \psi_e}{\nabla \Delta \psi_e^*} = \frac{f'}{(g'' + g'/r)'}.$$

By Theorem 5.3, this flow is stable if $\frac{f'}{(g''+g'/r)'}$ is positive defined.

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Conflict of interest

The authors declare there is no conflict of interest.

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