



Research article

A short proof of cuplength estimates on Lagrangian intersections

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Abstract: In this note we give a short proof of Arnold’s conjecture for the zero section of a cotangent bundle of a closed manifold. The proof is based on some basic properties of Lagrangian spectral invariants from Floer theory.

Keywords: Arnold’s conjecture; cuplength estimates; Lagrangian intersections; spectral invariants

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1. Introduction

Let M be a n -dimensional closed manifold. We denote by ω the canonical symplectic structure on the cotangent bundle T^*M , which is given by $\omega = -d\theta$ with the Liouville one-form $\theta = pdq$. Given $H \in C^\infty([0, 1] \times T^*M)$, the Hamiltonian vector field X_H is determined by $dH = -\omega(X_H, \cdot)$. The flow of X_H is denoted by φ_H^t and its time-one map by $\varphi_H := \varphi_H^1$. Denote by $\mathcal{H}am_c(M, \omega)$ the set of all Hamiltonian diffeomorphisms with compact support. Clearly, for any asymptotically constant Hamiltonian H we have $\varphi_H \in \mathcal{H}am_c(M, \omega)$. Fix a ground field \mathbb{F} , e.g., \mathbb{Z}_2 , \mathbb{R} , or \mathbb{Q} . The singular homology of a topological space X with coefficients in \mathbb{F} is denoted by $H_*(X)$. The \mathbb{F} -cuplength $cl(M)$ of M is by definition the maximal integer k such that there exist homology classes u_1, \dots, u_{k-1} in the homology $H_*(M)$ as a ring (with the intersection product) with $\deg(u_i) < \dim(M)$ such that

$$u_1 \cap \dots \cap u_{k-1} \neq 0,$$

where \cap denotes the intersection product.

The goal of this note is to use spectral invariants from Floer theory to reprove the cuplength estimate:

Theorem 1. *Let O_M denote the zero section of T^*M . Then we have*

$$\#(\varphi(O_M) \cap O_M) \geq cl(M), \quad \forall \varphi \in \mathcal{H}am_c(M, \omega).$$

The above estimate is a special case of the Arnold conjectures [1]. This case has already been solved by different approaches, for instance, by Chaperon [2] for the cotangent bundle of torus using

variational methods given by Conley and Zehnder [4], and for general cotangent bundles by Hofer [10] applying Lyusternik-Shnirelman category theory, and Laudendach and Sikorav [13] employing a finite dimensional method of “broken extremals”, etc. All available proofs listed above were given by finite dimensional homological methods essentially. The method of our proof of Theorem 1 is making good use of the properties of Lagrangian spectral invariants [16, 17] from Floer theory which, roughly speaking, is a version of infinite dimensional Morse theory.

Remark 1. By a modification of the method used here, one can prove a slightly general case of Arnold’s conjectures: If L is a closed Lagrangian submanifold of a smooth tame symplectic manifold (P, ω) satisfying $\pi_2(M, L) = 0$, then for any $\varphi \in \mathcal{H}am_c(P, \omega)$, $L \cap \varphi(L)$ has at least $cl(L)$ points. This case was independently proved by Floer [7] and Hofer [11]. It seems to the author that the spectral method from Floer theory fits in the degenerate Arnold conjecture very well. For the case that L is a closed monotone Lagrangian submanifold of a smooth tame symplectic manifold P , we refer to [9] for partial results about the Arnold conjecture on Lagrangian intersections in degenerate (non-transversal) sense. Maybe other methods (eg Hofer’s) could also be modified to deal with this case, but we are not aware of any reference about it at the time of writing. As far as the author knows the degenerate Arnold’s conjecture in the monotone case is not understood quite well, and even for a weaker problem, ie estimating the number of intersections of a closed Lagrangian submanifold with itself under Hamiltonian flows without the nondegenerate assumption. We believe that the spectral method from Floer theory would provide further potential value for attacking these kinds of problems.

2. Spectral invariants

2.1. The minmax critical values

Let X be a closed n -dimensional manifold X and let $f \in C^\infty(X)$. For any $\mu \in \mathbb{R}$ we put

$$X^\mu := \{x \in X | f(x) < \mu\}.$$

To a non-zero singular homology class $\alpha \in H_*(X)$, we associate a numerical invariant by

$$c_{LS}(\alpha, f) = \inf\{\mu \in \mathbb{R} | \alpha \in \text{Im}(i_*^\mu)\},$$

where $i_*^\mu : H_*(X^\mu) \rightarrow H_*(X)$ is the map induced by the natural inclusion $i^\mu : X^\mu \rightarrow X$. This number is a critical value of f . The function $c_{LS} : H_*(X) \setminus \{0\} \times C^\infty(X)$ is often called a *minmax critical value selector*. The following proposition summarizing the properties of the resulting function, which can be easily extracted from the classical Ljusternik–Schnirelman theory, see, e.g., [3, 5, 8, 12, 19].

Proposition 2. *The minmax critical value selector c_{LS} satisfies the following properties.*

- (1) $c_{LS}(\alpha, f)$ is a critical value of f , and $c_{LS}(k\alpha, f) = c_{LS}(\alpha, f)$ for any nonzero $k \in \mathbb{F}$.
- (2) $c_{LS}(\alpha, f)$ is Lipschitz in f with respect to the C^0 -topology.
- (3) Let $[pt]$ and $[X]$ denote the class of a point and the fundamental class respectively. Then

$$c_{LS}([pt], f) = \min f \leq c_{LS}(\alpha, f) \leq \max f = c_{LS}([X], f).$$

- (4) $c_{LS}(\alpha \cap \beta, f) \leq c_{LS}(\alpha, f)$ for any $\beta \in H_*(X)$ with $\alpha \cap \beta \neq 0$.

(5) If $\beta \neq k[X]$ for some $k \in \mathbb{F}$ and $c_{LS}(\alpha \cap \beta, f) = c_{LS}(\alpha, f)$, then the set $\Sigma = \{x \in \text{Crit}(f) \mid f(x) = c_{LS}(\alpha, f)\}$ is homologically non-trivial.

Here a subset S of a topological space X is called *homologically non-trivial* in X if for every open neighborhood U of S the map $i_* : H_k(U) \rightarrow H_k(X)$ induced by the inclusion $i : U \hookrightarrow X$ is non-trivial.

2.2. The Lagrangian spectral invariants

In this subsection we briefly recall the construction of Lagrangian spectral invariants for Hamiltonian diffeomorphism mainly following Oh [16, 17], see also [15, 18]. Denote \mathcal{H}_{ac} the set of Hamiltonians $H \in C^\infty([0, 1] \times T^*M)$ which are asymptotically constant at infinity.

For $H \in \mathcal{H}_{ac}$, the action functional is defined as

$$\mathcal{A}_H(\gamma) = \int_0^1 H(t, \gamma(t)) dt - \int \gamma^* \theta$$

on the space of paths in T^*M

$$\mathcal{P} = \{\gamma : [0, 1] \rightarrow T^*M \mid \gamma(0), \gamma(1) \in O_M\}.$$

Set $L = \varphi_H^1(O_M)$. We define the *Lagrangian action spectrum* of H on T^*M by

$$\text{Spec}(L, H) = \{\mathcal{A}_H(\gamma) \mid \gamma \in \text{Crit}(\mathcal{A}_H)\}.$$

This is a compact subset of \mathbb{R} of measure zero, see for instance [16].

Given a generic $H \in \mathcal{H}_{ac}$, the intersection $\varphi_H^1(O_M) \cap O_M$ is transverse and hence $\text{Crit}(\mathcal{A}_H)$ is finite. There is an integer-valued index, called the *Maslov-Viterbo index*, $\mu_{MV} : \text{Crit}(\mathcal{A}_H) \rightarrow \mathbb{Z}$ which is normalized so that if $H : T^*M \rightarrow \mathbb{R}$ is a lift of a Morse function f then μ_{MV} coincides with the Morse index of f .

Denote by $CF_k^{<a}(L, H)$, where $a \in (-\infty, \infty]$ is not in $\text{Spec}(L, H)$, the vector space of formal sums

$$\sum_{x_i \in \mathcal{P}} \sigma_i x_i,$$

where $\sigma_i \in \mathbb{F}$, $\mu_{MV}(x_i) = k$ and $\mathcal{A}_H(x_i) < a$. The graded \mathbb{F} -vector space $CF_k^{<a}(L, H)$ has the Floer differential counting the anti-gradient trajectories of the action functional in the standard way whenever a time-dependent almost complex structure compatible with ω is fixed and the regularity requirements are satisfied, see for instance [6, 16]. As a consequence, we have a filtration of the total Lagrangian Floer complex $CF_*(L, H) := CF_*^{(-\infty, \infty)}(L, H)$. Since the resulting homology, the *filtered Lagrangian Floer homology* of H , does not depend on $H \in \mathcal{H}_{ac}$ (due to continuation isomorphisms), one can extend this construction to all asymptotically constant Hamiltonians. Let $H \in \mathcal{H}_{ac}$ be an arbitrary Hamiltonian and let a be outside of $\text{Spec}(L, H)$. We define

$$HF_*^{<a}(L, H) = HF_*^{<a}(L, \tilde{H}),$$

where \tilde{H} is a C^2 -small perturbation of H so that $\varphi_{\tilde{H}}^1(O_M) \cap O_M$ is transverse. It is not hard to see that $HF_*^{<a}(L, \tilde{H})$ is independent of \tilde{H} provided that \tilde{H} is sufficiently close to H .

We denote by $i_*^a : HF_*^{<a}(L, H) \rightarrow HF_*(L, H)$ the induced inclusion maps. It is well known that for $H_f = \pi^* f$ where $\pi : T^*M \rightarrow M$ is the projection map and f is a Morse function on M , $HF_*(L, H_f)$ is canonically isomorphic to the singular homology $H_*(M)$, and hence $H_*(M) \cong HF(L, H)$ for all $H \in \mathcal{H}_{ac}$. Using this identification, for $\alpha \in H_*(M)$ and $H \in \mathcal{H}_{ac}$ we define

$$\ell(\alpha, H) = \inf \{a \in \mathbb{R} \setminus \text{Spec}(L, H) \mid \alpha \in \text{Im}(i_*^a)\}.$$

By convention, we have $\ell(0, H) = -\infty$.

Proposition 3. *The Lagrangian spectral invariant $\ell : H_*(M) \setminus \{0\} \times \mathcal{H}_{ac} \rightarrow \mathbb{R}$ has the following properties:*

- (a) $\ell(\alpha, H) \in \text{Spec}(L, H)$, in particular it is a finite number.
- (b) ℓ is Lipschitz in H in the C^0 -topology.
- (c) $\ell([M], H) = -\ell([pt], \bar{H})$ with $\bar{H}(t, H) = -H(-t, H)$.
- (d) $\ell([pt], H) \leq \ell(\alpha, H) \leq \ell([M], H)$ for all $\alpha \in H_*(M) \setminus \{0\}$.
- (e) $\ell(\alpha, H) = \ell(\alpha, K)$, when $\varphi_H = \varphi_K$ in the universal covering of the group of Hamiltonian diffeomorphisms, and H, K are normalized.
- (f) $\ell(\alpha \cap \beta, H \# K) \leq \ell(\alpha, H) + \ell(\beta, K)$, where $(H \# K)(t, x) = H(t, x) + K(t, (\varphi_H^t)^{-1}(x))$.
- (g) If $\varphi_H(O_M) = \varphi_K(O_M)$, then there exists $C \in \mathbb{R}$ such that $\ell(\alpha, H) = \ell(\alpha, K) + C$ for all $\alpha \in H_*(M) \setminus \{0\}$.
- (h) Let $f : M \rightarrow \mathbb{R}$ be a smooth function, and let $H_f : T^*M \rightarrow \mathbb{R}$ denote a compactly supported autonomous Hamiltonian so that $H_f = f \circ \pi$ on a ball bundle $T_R^*M := \{(q, p) \in T^*M \mid |p| \leq R\}$ containing $L^f := \{(q, \partial_q f(q)) \in T^*M \mid q \in M\}$, and $H_f = 0$ outside T_{R+1}^*M in M , where $|\cdot|$ is the norm induced by a metric ρ on M , and $\pi : T^*M \rightarrow M$ is the natural projection map. Then $\ell(\alpha, H_f) = c_{LS}(\alpha, f)$ for all $\alpha \in H_*(M) \setminus \{0\}$.

3. The proof of the main theorem

Our main theorem follows immediately from the following lemma.

Lemma 4. *Let $H \in \mathcal{H}_{ac}$ and $\alpha, \beta \in H_*(M) \setminus \{0\}$ with $\deg(\alpha) < n$. If the intersections of O_M and $\varphi_H(O_M)$ are isolated, then*

$$\ell(\alpha \cap \beta, H) < \ell(\beta, H).$$

Proof. Since the intersections of O_M and $\varphi_H(O_M)$ are isolated, we can pick a small open neighborhood U of $O_M \cap \varphi_H(O_M)$ in M so that $H_k(\bar{U}) = 0$ for all $k > 0$. Let $f : M \rightarrow \mathbb{R}$ be a C^2 -small function such that $f = 0$ on \bar{U} and $f < 0$ on $M \setminus \bar{U}$, and let H_f be the lift of f to the cotangent bundle T^*M as in Proposition 3(h). We claim that for any $\alpha \in H_{<n}(M)$, it holds that

$$\ell(\alpha, H_f) < 0. \tag{3.1}$$

For this end, we first prove that $c_{LS}(\alpha, f) < 0$ for all $\alpha \in H_{<n}(M)$. In fact, if there exists a homology class $\alpha_l \in H_{<n}(M)$ such that $c_{LS}(\alpha_l, f) = 0$, then we have $c_{LS}(\alpha_l \cap [M], f) = c_{LS}(\alpha_l, f) = 0$. It follows from Proposition 2(3) that $c_{LS}([M], f) = \max_M f = 0$. So we have $c_{LS}(\alpha_l \cap [M], f) = c_{LS}([M], f)$ with $\alpha_l \in H_{<n}(M) \setminus \{0\}$. Then by Proposition 2(5) the zero level set \bar{U} of f is homologically non-trivial – a contradiction. Therefore, for any $\alpha \in H_{<n}(M)$, we have $c_{LS}(\alpha, f) < 0$. This, together with Proposition 3(h), yields $\ell(\alpha, H_f) < 0$.

Next we show that for sufficiently small $\varepsilon > 0$

$$\ell(\alpha \cap \beta, H) = \ell(\alpha \cap \beta, \varepsilon H_f \sharp H). \quad (3.2)$$

Observe that $\varphi_{H_f}^t(q, p) = (q, p + t\partial_q f(a)) \in T^*M$ for $t \in [0, 1]$ and $(q, p) \in T_R^*M$. Set $L_R^H = \varphi_H^1(O_M) \cap T_R^*M$. Then we have

$$\varphi_{\varepsilon H_f}(L_R^H) = \{(q, p + \varepsilon df(q)) \mid (q, p) \in L_R^H\}.$$

Since $L_R^H \cap \pi^{-1}(O_M \setminus U)$ is compact and has no intersections with O_M , we deduce that for small enough $\varepsilon > 0$, $\varphi_{\varepsilon H_f}(L_R^H) \cap \pi^{-1}(O_M \setminus U)$ has no intersections with O_M as well. For $(q, p) \in T^*M$ with $R \leq \|(q, p)\| \leq R + 1$ we have

$$d_g(\varphi_{\varepsilon H_f}(q, p), (q, p)) \leq \left\| \int_0^1 \frac{d}{dt} \varphi_{\varepsilon H_f}^t(q, p) dt \right\| \leq \varepsilon \sup_{R \leq \|(q, p)\| \leq R+1} \|X_{H_f}\|,$$

where d_g is the distance function induced by some Riemannian metric g on M . Therefore, for sufficiently small $\varepsilon > 0$, $\varphi_{\varepsilon H_f}(T_{R+1}^*M \setminus T_R^*M)$ does not intersect O_M . Note that the Hamiltonian diffeomorphism $\varphi_{H_f}^\varepsilon$ is supported in T_{R+1}^*M , we conclude that $\varphi_{\varepsilon H_f} \varphi_H(O_M) \cap \pi^{-1}(O_M \setminus U)$ does not intersect O_M provided that $\varepsilon > 0$ is sufficiently small. On the other hand, we have that $\varphi_{\varepsilon H_f} \varphi_H(O_M) \cap \pi^{-1}(U) = \varphi_H(O_M) \cap \pi^{-1}(U)$ because $f = 0$ on U . So if $\varepsilon > 0$ is sufficiently small then the Lagrangians $\varphi_{\varepsilon H_f} \varphi_H(O_M)$ and $\varphi_H(O_M)$ have the same intersections with O_M . A direct calculation shows that for every such intersection point, the two action values corresponding to $\varepsilon H_f \sharp H$ and H are the same. Indeed, there is a one-to-one correspondence between the set $\varphi_H(O_M) \cap O_M$ and the set $\mathcal{P}(H) := \{x \in \mathcal{P} \mid \dot{x} = X_H(x(t))\}$ of Hamiltonian chords by sending $q \in \varphi_H(O_M) \cap O_M$ to $x = \varphi_H^t(\varphi_H^{-1}(q))$. So we get a bijective map defined by

$$\Upsilon : \text{Crit}(\mathcal{A}_H) \longrightarrow \text{Crit}(\mathcal{A}_{\varepsilon H_f \sharp H}), \quad x(t) \longmapsto \varphi_{\varepsilon H_f}^t(x(t)).$$

Notice that the Hamiltonian flow has the following property

$$(\varphi_{\varepsilon H_f})^* \theta - \theta = dF_t,$$

where the function $F : [0, 1] \times T^*M \rightarrow \mathbb{R}$ is given by $F_t = \int_0^t (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi_{\varepsilon H_f}^s ds$, see for instance [14, Proposition 9.3.1]. As a consequence, for any $x \in \mathcal{P}(H)$ we have

$$\frac{d}{dt} F_t(x(t)) = dF_t(\dot{x}(t)) + (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi_{\varepsilon H_f}^t(x(t)).$$

which implies

$$(\varphi_{\varepsilon H_f})^* \theta(\dot{x}(t)) = \theta(\dot{x}(t)) + \frac{d}{dt} F_t(x(t)) - (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi_{\varepsilon H_f}^t(x(t)).$$

Then we compute

$$\begin{aligned} \mathcal{A}_{\varepsilon H_f \sharp H}(\Upsilon(x(t))) &= \int_0^1 \varepsilon H_f(\varphi_{\varepsilon H_f}^t(x(t))) dt + \int_0^1 H_t \circ (\varphi_{\varepsilon H_f}^t)^{-1}(\varphi_{\varepsilon H_f}^t(x(t))) dt \\ &\quad - \int_0^1 \theta\left(\frac{d}{dt} \varphi_{\varepsilon H_f}^t(x(t))\right) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \varepsilon H_f(\varphi'_{\varepsilon H_f}(x(t))) dt + \int_0^1 H_f(x(t)) dt - \int_0^1 (\varphi_{\varepsilon H_f})^* \theta(\dot{x}(t)) dt \\
&\quad - \int_0^1 \theta(X_{\varepsilon H_f}(\varphi_{\varepsilon H_f}(x(t)))) dt \\
&= \mathcal{A}_H(x(t)) + \int_0^1 \varepsilon H_f(\varphi'_{\varepsilon H_f}(x(t))) - \frac{d}{dt} F_t(x(t)) dt \\
&\quad + \int_0^1 (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi'_{\varepsilon H_f}(x(t)) dt - \theta(X_{\varepsilon H_f}(\varphi_{\varepsilon H_f}(x(t)))) dt \\
&= \mathcal{A}_H(x(t)) + F_1(x(1)) - F_0(x(0)) \\
&= \mathcal{A}_H(x(t)), \tag{3.3}
\end{aligned}$$

where in the last equality we have used the fact that the value of an autonomous Hamiltonian H_f is constant along its Hamiltonian flow, and $f = 0$ on U which contains $x(1)$. Therefore, the action spectra $\text{Spec}(L, \varepsilon H_f \# H)$ and $\text{Spec}(L, H)$ are the same. Now fix a sufficiently small $\varepsilon > 0$ and consider the family of Lagrangians $\varphi_{s\varepsilon H_f} \varphi_H(O_L)$ with $s \in [0, 1]$. As before, the action spectra $\text{Spec}(L, s\varepsilon H_f \# H)$, $s \in [0, 1]$ are all the same. Since the action spectrum is a closed nowhere dense subset of \mathbb{R} , it follows from Proposition 3(b) that $\ell(\alpha \cap \beta, s\varepsilon H_f \# H)$ do not depend on s . So we have $\ell(\alpha \cap \beta, H) = \ell(\alpha \cap \beta, \varepsilon H_f \# H)$.

Combining (3.1) and (3.2), it follows from Proposition 3(f) that

$$\ell(\alpha \cap \beta, H) = \ell(\alpha \cap \beta, \varepsilon H_f \# H) \leq \ell(\alpha, \varepsilon H_f) + \ell(\beta, H) < \ell(\beta, H).$$

This completes the proof of the lemma. \square

The proof of Theorem 1. Without loss of generality we may assume that the intersections of O_M and $\varphi_H(O_M)$ are isolated, otherwise, nothing needs to prove. Set $cl(M) = k + 1$. By definition there exist $u_i \in H_{<n}(M)$, $i = 1, \dots, k$ such that $u_1 \cap \dots \cap u_k = [pt]$. We put

$$[M] = \alpha_0, \alpha_1, \dots, \alpha_k \in H_*(M), \quad \alpha_i = u_{k-i+1} \cap \alpha_{i-1}.$$

For any $\varphi \in \mathcal{H}am_c(M, \omega)$, there exists a Hamiltonian $H \in \mathcal{H}_{ac}$ such that $\varphi = \varphi_H^1$. It follows from Lemma 4 and Proposition 3(a) that there exist $k + 1$ elements $x_i \in \text{Crit}(\mathcal{A}_H)$ such that

$$\ell(\alpha_k, H) = \mathcal{A}_H(x_k) < \ell(\alpha_{k-1}, H) = \mathcal{A}_H(x_{k-1}) < \dots < \ell(\alpha_0, H) = \mathcal{A}_H(x_0).$$

Hence, all x_i , $i = 0, \dots, k$ are different. The one-to-one correspondence between the intersection points of O_M and $\varphi_H^1(O_M)$ and the critical points of \mathcal{A}_H concludes the desired result. \square

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Conflict of interest

The authors declare there is no conflict of interest.

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