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Research article

Conservation laws analysis of nonlinear partial differential equations and their linear soliton solutions and Hamiltonian structures

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Abstract: This article mainly uses two methods of solving the conservation laws of two partial differential equations and a system of equations. The first method is to construct the conservation law directly and the second method is to apply the Ibragimov method to solve the conservation laws of the target equation systems, which are constructed based on the symmetric rows of the target equation system. In this paper, we select two equations and an equation system, and we try to apply these two methods to the combined KdV-MKdV equation, the Klein-Gordon equation and the generalized coupled KdV equation, and simply verify them. The combined KdV-MKdV equation describes the wave propagation of bound particles, sound waves and thermal pulses. The Klein-Gordon equation describes the nonlinear sine-KG equation that simulates the motion of the Josephson junction, the rigid pendulum connected to the stretched wire, and the dislocations in the crystal. And the coupled KdV equation has also attracted a lot of research due to its importance in theoretical physics and many scientific applications. In the last part of the article, we try to briefly analyze the Hamiltonian structures and adjoint symmetries of the target equations, and calculate their linear soliton solutions.

Keywords: conservative law; Hamiltonian structure; linear soliton solution; direct construction method; Ibragimov method
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1. Introduction

In the study of differential equations in the past, the conservation law is undoubtedly a very important part, especially in terms of integrability and linearization. How to solve the conservation law of the equation has become a first problem to be faced [1–5]. Hence in this paper, we select three different types of differential equations and apply two different methods to solve the conservation laws of the partial differential equations.

The first method of calculating conservation laws does not require the equation to have variational conditions [6–9]. Its principle is to replace symmetry with the adjoint symmetry of a partial differential equation. Then the invariance condition on symmetry is replaced by the invariance condition on adjoint symmetry, and there is a direct explicit formula to calculate the multiplier and obtain the corresponding conservation law. In the calculation process, the adjoint invariance condition is replaced by an additional deterministic equation, and the determinant equation of the adjoint equation is expanded by writing an additional equation, thereby obtaining a linear deterministic equation system. If the adjoint symmetry of the equation is solved by this system, the multiplier that can be used to solve the conservation law is found.

The second part is to use a method to solve the conservation law of partial differential equations, which is actually a derivation of Norther's theorem [10–14]. In theory, Any Lie point, Lie-Bäcklund, and nonlocal symmetry can derive the corresponding conservation laws. In the last part of this paper, we briefly analyze the Hamiltonian structures and adjoint symmetries [15–19] of the selected equations, and calculate their linear soliton solutions by traveling wave transformation [20, 21].

In this paper, we apply these methods to three nonlinear partial differential equations: the combined KdV-MKdV equation [22–26], the Klein-Gordon equation [27] and the generalized coupled KdV equations [28, 29].

2. Brief introduction to the method of directly constructing conservation law

First we consider a system of partial differential equations with N independent variables $\boldsymbol{u} = (u^1, \ldots, u^N)$ and $n+1$ independent variables $(t, \boldsymbol{x})$, the forms are as follows:

$$G^i = \frac{\partial u^j}{\partial t} + g^j(t, \boldsymbol{x}, \boldsymbol{u}, \partial_{\boldsymbol{x}}\boldsymbol{u}, \ldots, \partial^m_{\boldsymbol{x}}\boldsymbol{u}) = 0, j = 1, \ldots, N \tag{2.1}$$

with $\boldsymbol{x}$ derivatives of $\boldsymbol{u}$ up to some order m. And we can use $\partial_{\boldsymbol{x}}\boldsymbol{u}, \partial^2_{\boldsymbol{x}}\boldsymbol{u}$, etc. to represent all the derivatives of u^j with respect to x^i. We denote partial derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i}$ with $\boldsymbol{x}$ derivatives of $\boldsymbol{u}$ up to some order m. We denote partial derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i}$ by subscripts t and i respectively. And likewise, D_t and D_i represent the total derivatives with respect to x^i and t. We set

$$(\mathcal{L}_g)^j_\alpha V^\alpha = \frac{\partial g^j}{\partial u^\alpha}V^\alpha + \frac{\partial g^j}{\partial u^\alpha_i}D_i V^\alpha + \cdots + \frac{\partial g^j}{\partial u^\alpha_{i_1\cdots i_m}}D_{i_1\cdots i_m}V^\alpha \tag{2.2}$$

and we can let $(\mathcal{L}^*_g)$ denote the adjoint operator defined by

$$(\mathcal{L}^*_g)^j_\alpha W^j = \frac{\partial g^j}{\partial u^\alpha}V^\alpha - D_i(\frac{\partial g^j}{\partial u^\alpha_i}W^j) + \cdots + (-1)^m D_{i_1\cdots i_m}(\frac{\partial g^j}{\partial u^\alpha_{i_1\cdots i_m}}W^j) \tag{2.3}$$

acting on arbitrary functions V^α, W^j respectively. It is well known that the determining equation of the system (2.1) with respect to symmetric $X = \eta^j\frac{\partial}{\partial u^j}$ is

$$D_t\eta^j + (\mathcal{L}_g)^j_\alpha\eta^\alpha = 0, j = 1, \cdots, N \tag{2.4}$$

for all solutions $\boldsymbol{u}(t, \boldsymbol{x})$ of Eqs. (2.1). The above decision equation can also be used to solve the higher order symmetry of the Eqs. (2.1), such as $\eta^j(t, \boldsymbol{x}, \boldsymbol{u}, \partial\boldsymbol{u}, \ldots, \partial^p\boldsymbol{u})$, where $\partial^k\boldsymbol{u}$ denotes all kth order derivatives of $\boldsymbol{u}$ with respect to all independent variables $t, \boldsymbol{x}$. And the adjoint of Eq. (2.4) is given by

$$-D_t\omega_j + (\mathcal{L}^*_g)^\alpha_j\omega_\alpha = 0, \; j = 1, \cdots, N \tag{2.5}$$

which is the determining equation for the adjoint symmetries $\omega_j(t,\boldsymbol{x},\boldsymbol{u},\partial\boldsymbol{u},\ldots,\partial^p\boldsymbol{u})$ of the Eqs. (2.1). In general, solutions of the adjoint symmetry Eq. (2.5) are not solutions of the symmetry Eq. (2.4), and there is no interpretation of adjoint symmetries in terms of an infinitesimal generator leaving anything invariant. Then Let

$$\mathcal{D}_t = \partial_t - ((g^j)\partial_{u^j} + (D_i g^j)\partial_{u_i^j} + \cdots) \tag{2.6}$$

which is the total derivative with respect to t on the solution space of Eqs. (2.1) (In particular, $D_t = \mathcal{D}_t$ when acting on all solutions $\boldsymbol{u}(t,\boldsymbol{x})$.) Then the determining equations explicitly become

$$\begin{aligned} 0 =&\mathcal{D}_t\eta^j + (\mathcal{L}_g)^j_\alpha\eta^\alpha \\ =&\frac{\partial\eta^j}{\partial t} - (\frac{\partial\eta^j}{\partial u^\alpha}g^j + \frac{\partial\eta^j}{\partial u_i^\alpha}D_i g^\alpha + \cdots + \frac{\partial\eta^j}{\partial u^\alpha_{i_1\cdots i_m}}D_{i_1}\cdots D_{i_m}g^\alpha) \\ &+ \frac{\partial g^\alpha}{\partial u^\alpha}\eta^\alpha + \cdots + \frac{\partial g^j}{\partial u^\alpha_{i_1\cdots i_m}}D_{i_1}\cdots D_{i_m}\eta^\alpha,\ j = 1,\ldots,N \end{aligned} \tag{2.7}$$

for $\eta^j(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^p_{\boldsymbol{x}}\boldsymbol{u})$ and

$$\begin{aligned} 0 =& -\mathcal{D}_t\omega_j + (\mathcal{L}_g^*)^\alpha_j\omega_\alpha \\ =& -\frac{\partial\omega_j}{\partial t} + (\frac{\partial\omega_j}{\partial u^\alpha}g^j + \frac{\partial\omega_j}{\partial u_i^\alpha}D_i g^\alpha + \cdots + \frac{\partial\eta_j}{\partial u^\alpha_{i_1\cdots i_m}}D_{i_1}\cdots D_{i_m}g^\alpha) \\ &+ \frac{\partial g^\alpha}{\partial u^\alpha}\omega_\alpha + \cdots + \frac{\partial g^j}{\partial u^\alpha_{i_1\cdots i_m}}D_{i_1}\cdots D_{i_m}\omega_\alpha,\ j = 1,\ldots,N \end{aligned} \tag{2.8}$$

for $\omega_j(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^p_{\boldsymbol{x}}\boldsymbol{u})$. The solutions of Eqs. (2.7) and (2.8) yield all symmetries and adjoint symmetries up to any given order p.

Definition1 A local conservation law of PDE system (2.1) is a divergence expression

$$D_t\phi^t(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^k_{\boldsymbol{x}}\boldsymbol{u}) + D_i\phi^i(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^k_{\boldsymbol{x}}\boldsymbol{u}) = 0$$

for all solutions $u(t,\boldsymbol{x})$ of Eqs. (2.1); ϕ^t and ϕ^i are called the conserved densities.

Definition2 Multipliers for PDE system (2.1) are a set of expressions

$$\lambda_1(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^q_{\boldsymbol{x}}\boldsymbol{u}),\cdots,\lambda_N(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^q_{\boldsymbol{x}}\boldsymbol{u})$$

satisfying

$$(u_t^j + g^j)\lambda_j = D_t\phi^t + D_i\phi^i \tag{2.9}$$

for some expression $\phi^t(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^k_{\boldsymbol{x}}\boldsymbol{u})$ and $\phi^i(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^k_{\boldsymbol{x}}\boldsymbol{u})$ for all functions $\boldsymbol{u}(t,\boldsymbol{x})$.

Obviously $\phi^t = D_i\theta^i$ and $\phi^i = -D_t\theta^i + D_j\psi^{ij}$ are trivial conservation laws of the system, for some erpressions $\theta^i(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^{k-1}_{\boldsymbol{x}}\boldsymbol{u})$ and $\psi^{ij}(t,\boldsymbol{x},\boldsymbol{u},\partial_{\boldsymbol{x}}\boldsymbol{u},\ldots,\partial^q_{\boldsymbol{x}}\boldsymbol{u})$ with $\psi^{ij} = -\psi^{ji}$, i.e. satisfying

$$D_t\phi^t + D_i\phi^i = D_t(D_i\theta^i) + (-D_t\theta^i + D_j\psi^{ij}) = 0. \tag{2.10}$$

The purpose of the following article is to calculate the corresponding conservation law multipliers based on differential equations. Next we consider $D_t\phi^t + D_i\phi^i$ and it is well konwn that

$$D_t\phi^t = \frac{\partial\phi^t}{\partial t} + \frac{\partial\phi^t}{\partial u^j}u_t^j + \frac{\partial\phi^t}{\partial u_i^j}u_{ti}^j + \cdots + \frac{\partial\phi^t}{\partial u^j_{i_1\cdots i_k}}u^j_{ti_1\ldots i_k} = \partial_t\phi^t + \mathcal{L}_{\phi^t}u_t^j$$

where $(\mathcal{L}_{\phi^t})_j = (\frac{\partial\phi^t}{\partial u^j}) + (\frac{\partial\phi^t}{\partial u^j_t})D_i + \cdots + (\frac{\partial\phi^t}{\partial u^j_{i_1\cdots i_k}}D_{i_1\cdots D_{i_k}})$ denotes the linearization operator of ϕ^t. Then we can obtain

$$\begin{aligned}(\mathcal{L}_{\phi^t})_j u^j_t =&(\mathcal{L}_{\phi^t})_j(u^j_t + g^j) - (\mathcal{L}_{\phi^t})_j g^j\\ =&(u^j_t + g^j)\hat{E}_{u^j} - (\mathcal{L}_{\phi^t})_j g^j + D_i\Gamma^i\end{aligned}$$

where Γ^i is given by an expression proportional to $u^j_t + g^j$ and where

$$\hat{E}_{u^j} = \partial_{u^j} - D_i\partial_{u^j_i} + D_iD_j\partial_{u^j_{ik}} + \cdots \tag{2.11}$$

is a restricted Euler operator. So we have

$$D_t\phi^t = \partial_t\phi^t - (\mathcal{L}_{\phi^t})_j g^j + D_i\Gamma + (u^j_t + g^j)\hat{E}_{u^j}(\phi^t). \tag{2.12}$$

Then in order to ensure that the Eq. (2.10) holds, $\partial_t\phi^t - (\mathcal{L}_{\phi^t})_j g^j$ which involes $u^j_t + g^j$ must cancel $D_i\phi^i$, so we have

$$D_i\phi^i = -(\partial_t\phi^t - (\mathcal{L}_{\phi^t})_j g^j). \tag{2.13}$$

Then after combining the expressions (2.12) and (2.13) we obtain

$$D_t\phi^t + D_i(\phi^i - \Gamma^i) = (u^j_t + g^j)\Lambda_j \tag{2.14}$$

with

$$\Lambda_j = \hat{E}_{u^j}(\phi^t),\ j = 1, \cdots, N.$$

Next, by solving the determination equation

$$E_{u^j}((u^j_t + g^j)\lambda_j) = E_{u^j}(D_t\phi^t + D_i\phi^i) = 0 \tag{2.15}$$

where

$$E_{u^j} = \partial_{u^j} - D_i\partial_{u^j_i} - D_t\partial_{u^j_t} + D_iD_k\partial_{u^j_{ik}} + D_tD_k\partial_{u^j_{ik}} + \cdots .$$

Then this yeilds

$$0 = E_{u^j}((u^j_t + g^j)\lambda_j) = -D_t\Lambda_j + (\mathcal{L}^*_g)^\alpha_j\Lambda_\alpha + (\mathcal{L}^*_\Lambda)_{j\alpha}(u^\alpha_t + g^\alpha),\ j = 1, \ldots, N, \tag{2.16}$$

where

$$(\mathcal{L}_\Lambda)_{j\alpha}V^\alpha = \frac{\partial\Lambda_j}{\partial u^\alpha}V^\alpha + \frac{\partial\Lambda_j}{\partial u^\alpha_i}D_iV^\alpha + \cdots + \frac{\partial\Lambda_j}{\partial u^\alpha_{i_1\cdots i_p}}D_{i_1}\cdots D_{i_p}V^\alpha \tag{2.17}$$

and

$$(\mathcal{L}^*_\Lambda)_{j\alpha}W^j = \frac{\partial\Lambda_j}{\partial u^\alpha}W^j - D_i(\frac{\partial\Lambda_j}{\partial u^\alpha_i}W^j) + \cdots + (-1)^pD_{i_1}\cdots D_{i_p}(\frac{\partial\Lambda_j}{\partial u^\alpha_{i_1\cdots i_p}}W^j) \tag{2.18}$$

acting on arbitrary functions V^α, W^j. And we know

$$D_t = \mathcal{D}_t + (u^\alpha_t + g^\alpha)\partial_{u^j} + (u^\alpha_{ti} + D_ig^\alpha)\partial_{u^p_i} + \cdots$$

which yeilds $D_t\Lambda_j = \mathcal{D}_t\Lambda_j + (\mathcal{L}_\Lambda)_{j\alpha}G^\alpha$. So by expansion (2.16), we can get

$$\begin{aligned}
0 =& -\mathcal{D}_t\Lambda_j + (\mathcal{L}_g^*)_j^\alpha\Lambda_\alpha \\
=& -\frac{\partial\Lambda_j}{\partial t} + (\frac{\partial\Lambda_j}{\partial u^\alpha}g^j + \frac{\partial\Lambda_j}{\partial u_i^\alpha}D_i g^\alpha + \cdots + \frac{\partial\Lambda_j}{\partial u_{i_1\cdots i_m}^\alpha}D_{i_1}\cdots D_{i_p}g^\alpha) \\
&+ \frac{\partial g^\alpha}{\partial u^\alpha}\Lambda_\alpha + \cdots + \frac{\partial g^j}{\partial u_{i_1\cdots i_m}^\alpha}D_{i_1}\cdots D_{i_m}\Lambda_\alpha, j = 1,\ldots,N.
\end{aligned} \tag{2.19}$$

Then by comparing the coefficients of $G^j, D_iG^j, \ldots, D_{i_1}\cdots D_{i_p}G^j$, we can get the corresponding determining equations:

$$0 =(-1)^{p+1}\frac{\partial\Lambda_j}{\partial u_{i_1\cdots i_p}^j} + \frac{\partial\Lambda_\alpha}{\partial u_{i_1\cdots i_p}^j}, \tag{2.20}$$

$$\begin{aligned}
0 =&(-1)^{q+1}\frac{\partial\Lambda_j}{\partial u_{i_1\cdots i_q}^j} + \frac{\partial\Lambda_\alpha}{\partial u_{i_1\cdots i_q}^j} - C_{q+1}^q D_{i_{q+1}}\frac{\partial\Lambda_\alpha}{\partial u_{i_1\cdots i_{q+1}}^j} + \cdots \\
&+ (-1)^{p-q}C_p^q D_{i_{q+1}}\cdots D_{i_p}\frac{\partial\Lambda_\alpha}{\partial u_{i_1\cdots i_p}^j}, q = 1,\cdots,p-1,
\end{aligned} \tag{2.21}$$

$$0 = -\frac{\partial\Lambda_j}{\partial u^j} + \frac{\partial\Lambda_\alpha}{\partial u^j} - D_i\frac{\partial\Lambda_\alpha}{\partial u_i^j} + \cdots + (-1)^p D_{i_1\cdots D_{i_p}}\frac{\partial\Lambda_\alpha}{\partial u_{i_1\cdots i_p}^j}, \tag{2.22}$$

$$j = 1,\cdots,N; \alpha = 1,\cdots,N; q = 1,\cdots,p-1$$

where $C_r^q = \frac{r!}{q!(r-q)!}$. Then the expression of the corresponding multiplier Λ_j is obtained by solving the above decision equations. After obtaining the multiplier, we can solve the corresponding conservation law. Next, we introduce the theorem:

Theorem 2: For the differential equation system (2.1), the conserved densities of any nontrivial conservation law in normal form are given in terms of the multipliers by

$$\phi^t = \int_0^1 d\lambda(\boldsymbol{u}^j - \widetilde{\boldsymbol{u}}^j)\Lambda_j[\boldsymbol{u}_\lambda] + t\int_0^1 d\lambda K(\lambda t, \lambda\boldsymbol{x}), \tag{2.23}$$

$$\begin{aligned}
\phi^i =& x^i\int_0^1 d\lambda\lambda^n K(\lambda t, \lambda\boldsymbol{x}) + \int_0^1 d\lambda(S_i[\boldsymbol{u} - \widetilde{\boldsymbol{u}}, \Lambda[\boldsymbol{u}_\lambda]; g[\boldsymbol{u}_\lambda]] + S^i[\boldsymbol{u} - \widetilde{\boldsymbol{u}}, g[\boldsymbol{u}_\lambda - \lambda g[\boldsymbol{u}] \\
&+ (1-\lambda)\widetilde{\boldsymbol{u}}_t; \Lambda[\boldsymbol{u}_\lambda]]]])
\end{aligned} \tag{2.24}$$

where

$$\boldsymbol{u}_\lambda^j = \lambda u + (1-\lambda)\widetilde{u}, K(t,\boldsymbol{x}) = (\widetilde{u}_t^j + g^j[\widetilde{\boldsymbol{u}}])\Lambda_j[\widetilde{\boldsymbol{u}}],$$

$$S^i[V,W;g] = \sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}(-1)^k(D_{i_1}\cdots D_{i_l}V^\rho)D_{j_1}\cdots D_{j_k}(W_j\frac{\partial g^j}{\partial u_{ii_1\cdots i_kj_1\cdots j_l}^j}),$$

$$S^i[V,W;\Lambda] = \sum_{l=0}^{p-1}\sum_{k=0}^{p-l-1}(-1)^k(D_{i_1}\cdots D_{i_l}V^\rho)D_{j_1}\cdots D_{j_k}(W_j\frac{\partial\Lambda_j}{\partial u_{ii_1\cdots i_kj_1\cdots j_l}^j}).$$

3. Application of directly constructing conservation laws

3.1. Combined KdV-MKdV equation

Firstly, we choose an equation named the combined KdV-MKdV equation and its form is as follows:

$$G = u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx} = 0 \tag{3.1}$$

where α and β are arbitrary constants. And its symmetries with infinitesimal genertor $Xu = \eta$ satisfies the determining equation

$$D_t\eta + \alpha u_x\eta + \alpha u D_x\eta + \beta u^2 D_x\eta + 2\beta u u_x\eta + D_x^3\eta = 0 \tag{3.2}$$

where $D_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t} + \cdots$ and $D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t} + \cdots$ are total derivative operators with respect to t and x. And the adjoint of Eq. (3.1) is given by

$$-D_t\omega - \alpha u D_x\omega - \beta u^2 D_x\omega - D_x^3\omega = 0 \tag{3.3}$$

when $G = 0$, which is the determining equation for the adjoint symmetries ω of the cKMK(the combined KdV-MKdV equation) equation. Next, we calculate the conservation law multiplier of the equation, namely finding Λ, which satisfies

$$D_t\phi^t + D_x\phi^x = (u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx})\Lambda_0 + D_x(u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx})\Lambda_1 + \cdots \tag{3.4}$$

with no dependence on u_t and its differential consequences. This yields the multiplier

$$D_t\phi^t + D_x(\phi^x - \Gamma) = (u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx})\Lambda, \Lambda = \Lambda_0 - D_x\Lambda_1 + \cdots \tag{3.5}$$

where $\Gamma = 0$ when u is restricted to be a cKMK solution. Next we set Λ to be related to x, t, u, u_x, u_{xx}, the determing equation becomes

$$E_u(G\Lambda) = -D_t\Lambda - \alpha u D_x\Lambda - \beta u^2 D_x\Lambda - D_x^3\Lambda + \Lambda_u G - D_x(\Lambda_{u_x}G) + D_x^2(\Lambda_{u_{xx}}G) = 0. \tag{3.6}$$

Then by comparing the coefficients of G and D_iG, we can get the equation

$$0 = -D_x\Lambda_{u_x} + D_x^2\Lambda_{u_x}, \tag{3.7}$$

$$0 = \Lambda_{u_x} - D_x\Lambda_{u_{xx}} \tag{3.8}$$

and we can notice that (3.7) is a differential consequence of (3.8). The highest coefficient in formula (3.8) is $\Lambda_{u_{xx}u_{xx}}u_{xxx}$, and we know that Λ is not related to u_{xx}. It yeilds $\Lambda_{u_{xx}u_{xx}} = 0$, then Λ has the following form:

$$\Lambda = a(t, x, u, u_x)u_{xx} + b(t, x, u, u_x). \tag{3.9}$$

Then the remaining terms in formular (3.8), after some cancellations, are of first order

$$0 = b_{u_x} - a_u u_x - a_x. \tag{3.10}$$

Next we extract the coefficient of u_{xxxx} in (3.6) and these yeild

$$D_x a = a_x + a_u u_x + a_{u_x}u_{xx} = 0 \tag{3.11}$$

and a is not related with u_{xx}, so we can obtain $a_{u_x} = 0$, namely $a(t, x, u)$. Similarly, we can easily get

$$a_u = a_x = 0. \tag{3.12}$$

According to (3.10), we can deduce that $b_{u_x} = 0$. We replace (3.9) into (3.6), then we can find the coefficient of u_{xxx}, i.e.

$$-b_u - \alpha ua - \beta u^2 a + b_u + \alpha ua - \beta u^2 a = 0 \tag{3.13}$$

and the coefficient of u_{xx} is

$$-3u_x b_{uu} - 3b_{xu} + 3(\alpha a + 2\beta ua - b_{uu})u_x = 0. \tag{3.14}$$

It yeilds that $b_{uu} = \alpha a + 2\beta ua$, $a_t = -3b_{xu}$. Hence we have

$$b = \frac{\alpha a(t)}{2}u^2 + \frac{1}{3}\beta a(t)u^3 + c(x, t)u + b_2(x, t) \tag{3.15}$$

from (3.14) and

$$b_{xu} = c' = -\frac{1}{3}a_t. \tag{3.16}$$

We can obtain the form of $b(x, t, u)$:

$$b = \frac{\alpha}{2}au^2 + \frac{1}{3}\beta au^3 + (b_1(t) - \frac{1}{3}a_t x)u + b2(x, t). \tag{3.17}$$

Then Λ has the form:

$$\Lambda = a(t)u_{xxx} + b(t, x, u) = a(t)u_{xx} + \frac{\alpha}{2}au^2 + \frac{1}{3}\beta au^3 + (b_1(t) - \frac{1}{3}a_t x)u + b_2(x, t). \tag{3.18}$$

Then after taking (3.18) back into the decision Eq. (3.6), we can get

$$-b_{2t} - \beta u^2 b_{2x} - ub_{1t} - b_{2xxx} + \frac{u_x}{3}a_{tt} - \frac{\alpha u^2}{6}a_t - \alpha ub_{2x} = 0. \tag{3.19}$$

Then by comparing the coefficients of u and u^2, we can obtain

$$b_{2t} = -b_{2xxx}, b_{1t} - \frac{1}{3}xa_{tt} = \alpha b_{2x}, \beta b_{2x} = -\frac{\alpha}{6}a_t, \tag{3.20}$$

then solve them. We get the general forms of $a(t), b_1(t)$ and $b_2(x, t)$

$$a = -\frac{6\beta c_1}{\alpha}t + c_3, b_1 = -\alpha c_1 t + c_4, b_2 = c_1 x + c_2. \tag{3.21}$$

So the general form of Λ is

$$\Lambda = (-\frac{6\beta c_1}{\alpha}t + c_3)u_{xx} + \frac{\alpha}{2}(-\frac{6\beta c_1}{\alpha}t + c_3) + \frac{1}{3}\beta u^3(-\frac{6\beta c_1}{\alpha}t + c_3) + (-\alpha c_1 t + c_4 + \frac{2\beta c_1}{\alpha}x)u + c_1 x + c_2, \tag{3.22}$$

where $c_i (i = 1, 2, 3, 4)$ are arbitrary constants. It yields that

$$\begin{aligned} \Lambda_1 &= -\frac{6\beta}{\alpha}tu_{xx} - 3\beta tu^2 - \frac{2\beta^2 tu^3}{\alpha} - \alpha tu + \frac{2x^2\beta u}{\alpha}, \\ \Lambda_2 &= 1, \Lambda_3 = u_{xx} + \frac{\alpha}{2}u^2 + \frac{\beta}{3}u^3, \Lambda_4 = u. \end{aligned} \tag{3.23}$$

Next, we find the conservation law ϕ_i^x, ϕ_i^t according to Λ_i.

According to the **Theorem.1**, we can take $\widetilde{u} = 0$ so that $K = 0$. So we have

$$\begin{aligned} \phi^t &= \int_0^1 d\lambda (u^j)\Lambda_j[\lambda u], \\ \phi^i &= \int_0^1 d\lambda (S^i[u, \Lambda[\lambda u]; g[\lambda u]] + S^i[u, g[\lambda u] - \lambda g[u]; \Lambda[\lambda u]]). \end{aligned} \tag{3.24}$$

Firstly, for Λ_1,

$$\begin{aligned} \phi_1^t &= \int_0^1 d\lambda u \Lambda(t, x, \lambda x, \lambda \partial_x u, \lambda \partial_x^2 x, \dots) \\ &= \int_0^1 -\frac{6\beta}{\alpha} tuu_{xx}\lambda - 3\beta tu^3\lambda^2 - \frac{2\beta^2 tu^4}{\alpha}\lambda^3 - \alpha tu + \frac{2x^2\beta u^2}{\alpha}\lambda \, d\lambda \\ &= -\frac{\beta^2 u^4 t}{2\alpha} - \beta u^3 t + \frac{1}{2}\left(-\frac{6\beta tuu_{xx}}{\alpha} - \alpha tu^2 + \frac{2x\beta u^2}{\alpha}\right) + xu \end{aligned} \tag{3.25}$$

and similarly, the ϕ_1^x has the following form:

$$\begin{aligned} \phi_1^x &= \int_0^1 d\lambda (S^x[u, \Lambda[\lambda u]; g[\lambda u]] + S^x[u; g[\lambda u] - \lambda g[u]; \Lambda[\lambda u]]) \\ &= \int_0^1 u\Lambda[\lambda u](\alpha\lambda u + \beta\lambda^2 u^2) + uD_x^2\Lambda[\lambda u] - u_x D_x \Lambda[\lambda u] + u_{xx}\Lambda[\lambda u] + u_x(g[\lambda u] - \lambda g[u])(-\frac{6\beta t}{\alpha}) \\ &\quad - uD_x((g[\lambda u] - \lambda g[u])(-\frac{6\beta t}{\alpha})) \\ &= -\frac{\beta^3 tu^6}{3\alpha} - \beta^2 tu^5 + \frac{x\beta^2 u^4}{2\alpha} - \beta tu^4\alpha - \frac{5u^3\beta^2 tu^3}{3} + x\beta u^3 - 6u^2\beta tu_{xx} + \frac{u\beta u_x}{\alpha} - \frac{x\beta u_x^2}{\alpha} - \frac{3u\beta tu_{xxxx}}{\alpha} \\ &\quad + \frac{3u_x\beta tu_{xxx}}{\alpha} - \frac{3\beta^2 tu_x^2 u^2}{\alpha} - \alpha utu_{xx} - \frac{3\beta tu_{xx}^2}{\alpha} + \frac{\alpha xu^2}{2} + \frac{\alpha tu_x^2}{2} + \frac{2ux\beta u_{xx}}{\alpha} + xu_{xx} - u_x. \end{aligned} \tag{3.26}$$

We can simply verify that

$$D_x\phi_1^x + D_t\phi_1^t = (-\frac{2\beta^2 u^3 t}{\alpha}t - \alpha tu + \frac{2x\beta u}{\alpha} - 3u^2\beta t + x - \alpha tu - \frac{3\beta tu_{xx}}{\alpha} + \alpha u^2)G - \frac{3u\beta t}{\alpha}D_x^2 G = 0 \tag{3.27}$$

when u is a solution of G.

And for $\Lambda_2 = 1$, the we can easily obtain

$$\begin{aligned} \phi_2^t &= \int_0^1 d\lambda u\Lambda(t, x, \lambda x, \lambda\partial_x u, \lambda\partial_x^2 x, \dots) = \int_0^1 u d\lambda = u, \\ \phi_2^x &= \int_0^1 d\lambda (S^x[u, \Lambda[\lambda u]; g[\lambda u]] + S^x[u; g[\lambda u] - \lambda g[u]; \Lambda[\lambda u]]) \\ &= \int_0^1 \lambda\alpha u^2 + \beta\lambda^2 u^3 + u_{xx} \, d\lambda \\ &= \frac{\alpha}{2}u^2 + \frac{\beta}{3}u^3 + u_{xx}. \end{aligned} \tag{3.28}$$

We can also verify that

$$D_x\phi_2^x + D_t\phi_2^t = G = 0 \tag{3.29}$$

when u is a solution of G. Identically

$$\begin{aligned}
\phi_3^t =& \int_0^1 \lambda u u_{xx} + \frac{\alpha}{2}u^3\lambda^2 + \frac{\beta}{3}u^4\lambda^3\, d\lambda \\
=& \frac{1}{2}uu_{xx} + \frac{\alpha}{6}u^3 + \frac{\beta}{12}u^4, \\
\phi_3^x =& \int_0^1 d\lambda(S^x[u, \Lambda[\lambda u]; g[\lambda u]] + S^x[u; g[\lambda u] - \lambda g[u]; \Lambda[\lambda u]]) \\
=& \frac{\beta^2}{18}u^6 + \frac{\alpha\beta}{6}u^5 + \frac{5\beta}{6}u^3u_{xx} + \frac{\alpha^2u^4}{8} + u^2u_{xx}\alpha + \frac{u^2u_x^2\beta}{2} + \frac{u_{xx}^2}{2} - \frac{u_xu_{xxx}}{2} + \frac{uu_{xxxx}}{2}
\end{aligned} \tag{3.30}$$

for $\Lambda_3 = u_{xx} + \frac{\alpha}{2}u^2 + \frac{\beta}{3}u^3$. We can verify it by

$$D_x\phi_3^x + D_t\phi_3^t = (\frac{u}{2})D_x^2G + (\frac{\beta}{3}u^3 + \frac{u_{xx}}{2} + \frac{\alpha}{2}u^2)G = 0 \tag{3.31}$$

when u is a solution of G.

The last one is $\Lambda_4 = u$,

$$\begin{aligned}
\phi_4^t =& \int_0^1 \lambda u^2 d\lambda = \frac{u^2}{2}, \\
\phi_4^x =& \int_0^1 \lambda u^2(\alpha\lambda u + \beta\lambda^2u^2) + u(\lambda u_{xx}) - u_x\lambda u_x + u_{xx}\lambda u\, d\lambda \\
=& \frac{\alpha}{3}u^3 + \frac{\beta}{4}u^4 - \frac{u_x^2}{2} + uu_{xx}.
\end{aligned} \tag{3.32}$$

Then we can easily obtain

$$D_x\phi_4^x + D_t\phi_4^t = uG = 0$$

when u is a solution of G.

3.2. Klein-Gordon equation

We choose the Klein-Gordon equation as the second equation to study and it has the following form:

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0. \tag{3.33}$$

It is obvious that its self-adjoint and the determing equation for its symmetries with infinitesimal generator $Xu = \eta$ and the adjoint of it is all

$$D_t^2\eta - D_x^2\eta + \alpha\eta + 3\beta u^2\eta = 0. \tag{3.34}$$

We set the Λ has the expression $\Lambda(t, x, u, u_x, u_t)$. Then the determining equation for the conservation law multiplier is

$$\begin{aligned}
0 =& E_u((u_{tt} - u_{xx} + \alpha u + \beta u^3)\Lambda) \\
=& D_t^2\Lambda - D_x^2\Lambda + \alpha\Lambda + 3\beta u^2\Lambda + \Lambda_u(u_{tt} - u_{xx} + \alpha u + \beta u^3) - D_x(\Lambda_{u_x}(u_{tt} - u_{xx} + \alpha u \\
&+ \beta u^3)) - D_t(\Lambda_{u_t}(u_{tt} - u_{xx} + \alpha u + \beta u^3)).
\end{aligned} \tag{3.35}$$

By comparing the coefficient of G, we can sort out two determining equations:

$$\mathcal{D}_t^2\Lambda - D_x^2\Lambda + \alpha\Lambda + 3\beta u^2\Lambda = 0, \tag{3.36}$$

$$2\Lambda_u + \mathcal{D}_t\Lambda_{u_t} - D_x\Lambda_{u_x} = 0 \tag{3.37}$$

where $\mathcal{D}_t = \Lambda_t + u_t\Lambda_u + u_{xt}\Lambda_{u_x} + (u_{xx} - \alpha u - \beta u^3)\Lambda_{u_t}$. We start from the Eq. (3.37) and it yeilds

$$2\Lambda_u + \Lambda_{tu_t} + u_t\Lambda_{uu_t} + (u_{xx} - \alpha u - \beta u^3)\Lambda_{u_tu_t} - \Lambda_{xu_x} - u_x\Lambda_{uu_x} - u_{xx}\Lambda_{u_xu_x} = 0 \tag{3.38}$$

and since Λ does not contain u_{xx}, we can seperat it from Eq.(3.38)

$$\Lambda_{u_tu_t} = \Lambda_{u_xu_x}, \tag{3.39}$$

$$2\Lambda_u + \Lambda_{tu_t} + u_t\Lambda_{uu_t} + (-\alpha u - \beta u^3)\Lambda_{u_tu_t} - \Lambda_{xu_x} - u_x\Lambda_{uu_x} = 0. \tag{3.40}$$

Then we deal with (3.36). It is easy to verify that the coefficients of u_{xxx} and u_{xxt} are 0. So we consider the coefficient of the second order u_{xx} and u_{xt}, i.e.

$$-2u_x\Lambda_{uu_x} - 2\Lambda_{xu_x} + 2\Lambda_{tu_t} + 2u_t\Lambda_{uu_t} - 2\alpha u\Lambda_{u_tu_t} - 2\beta u^3\Lambda_{u_tu_t} = 0 \tag{3.41}$$

and from (3.36) we can get

$$-4\Lambda_u = 0 \tag{3.42}$$

and the coefficient of u_{xt} is

$$-2\Lambda_{xu_t} + 2\Lambda_{tu_t} + 2(-\alpha - \beta u^3)\Lambda_{u_tu_x} = 0. \tag{3.43}$$

Then by solving the four equations (3.39), (3.40), (3.42) and (3.43) simultaneously, we can obtain the general form of the multiplier

$$\Lambda = (-u_t + u_x)f_1(t - x) + (u_t + u_x)f_2(t + x) + c_1u_t + d(x, t) \tag{3.44}$$

where $f_1(t - x)$ is any function related to $t - x$, $f_2(t + x)$ is any function related to $t + x$, c_1 is an arbitrary constant and $d(x, t)$ is any function related to x, t. Next, we substitute the multiplier into (3.38), and compare the coefficients of u_{xx}, u_{xt}. We can get

$$2f_1'\alpha u + 2f_1'\beta u^3 + 3u^2 d\beta - d_{xx} + \alpha d - 2u^3 f_2'\beta - 2uf_2'\alpha + d_{tt} = 0. \tag{3.45}$$

We can obtain

$$f_1' = f_2', d = 0.$$

We can take their simplest form for f_1 and f_2,

$$f_1 = c_2(t - x) + c_3, f_2 = c_2(t + x) + c_4 \tag{3.46}$$

where $c_i (i = 1, 2, 3, 4)$ are arbitary constants. Therefore, the general form of multiplier is

$$\Lambda = (-u_t + u_x)(c_2(t - x) + c_3) + (u_t + u_x)(c_2(t + x) + c_4) + c_1u_t. \tag{3.47}$$

It yeilds that

$$\Lambda_1 = u_t, \Lambda_2 = 2tu_x + 2xu_t,\\ \Lambda_3 = -u_t + u_x, \Lambda_4 = u_t + u_x. \tag{3.48}$$

Next we solve the conservation law with the formula

$$\phi^t = \int_0^1 d\lambda(u^j - u_0)\Lambda_j[u_\lambda],\\ \phi^i = \int_0^1 d\lambda(S^i[u-u_0, \Lambda[u_\lambda]; g[u_\lambda]] + S^i[u, g[u_\lambda] - \lambda g[u]; \Lambda[u_\lambda]]) \tag{3.49}$$

and we choose $\tilde{u} = u_0$ which is a constant and $K = 0$. Then we can obtain that

$$\begin{aligned} \phi_1^t =& \int_0^1 u_t\Lambda[\lambda u + (1-\lambda)u_0] + (u_0 - u)\mathcal{D}_t\Lambda[\lambda u + (1 - \lambda u_0)]d\lambda\\ =& \int_0^1 u_t\lambda u_t + (u_0 - u)\mathcal{D}_t\lambda u_t \, d\lambda\\ =& \frac{1}{2}u_t^2 - \frac{1}{2}(u-u_0)(u_{xx} - \alpha u - \beta u^3),\\ \phi_1^x =& \frac{u-u_0}{2}u_{xt} - \frac{u_x u_t}{2} \end{aligned} \tag{3.50}$$

for multiplier Λ_1.

$$\begin{aligned} \phi_2^t =& \int_0^1 u_t(2t\lambda u_x + 2x\lambda u_t) - (u-u_0)\mathcal{D}_t(2\lambda tu_x + 2x\lambda u_t)\, d\lambda\\ =& tu_xu_t + xu_t^2 - (u-u_0)(tu_{xt} + u_x + xu_{xx} - \alpha xu - \beta xu^3),\\ \phi_2^x =& u_0u^3\beta t + tu\alpha u_0 - \alpha tu_0^2 - \frac{t}{2}\beta u_0^4 - \frac{t\beta}{2}u^4 + tu_{xx}(u-u_0) + xu_{xt}(u-u_0) - u_xxu_t + uu_t - u_0u_t - tu_x^2 \end{aligned} \tag{3.51}$$

for multiplier Λ_2.

$$\begin{aligned} \phi_3^t =& \int_0^1 u_t(-\lambda u_t + \lambda u_x) - (u-u_0)\mathcal{D}_t(-\lambda u_t + \lambda u_x)\, d\lambda\\ =& -\frac{1}{2}u_t^2 + \frac{1}{2}u_xu_t - (u-u_0)(-\frac{1}{2}u_{xx} + \frac{1}{2}\alpha xu + \frac{\beta}{2}xu^3 + \frac{1}{2}u_{xt}),\\ \phi_3^x =& \frac{u^3}{2}\beta u_0 + \frac{u}{2}\alpha u_0 + \frac{u_{xx}}{2}(u-u_0) - \frac{u_{xt}}{2}(u-u_0) - \frac{u_x^2}{2} + \frac{u_xu_t}{2} - \frac{\beta}{6}(u^4 + u_0^4) - \frac{\alpha u_0^2}{2} \end{aligned} \tag{3.52}$$

for multiplier Λ_3.

$$\begin{aligned} \phi_4^t =& \int_0^1 u_t(\lambda u_t + \lambda u_x) - (u-u_0)\mathcal{D}_t(\lambda u_t + \lambda u_x)\, d\lambda\\ =& \frac{1}{2}u_t^2 + \frac{1}{2}u_xu_t - (u-u_0)(\frac{1}{2}u_{xx} - \frac{1}{2}\alpha xu - \frac{\beta}{2}xu^3 + \frac{1}{2}u_{xt}),\\ \phi_4^x =& \frac{u^3}{2}\beta u_0 + \frac{u}{2}\alpha u_0 + \frac{u_{xx}}{2}(u-u_0) + \frac{u_{xt}}{2}(u-u_0) - \frac{u_x^2}{2} - \frac{u_xu_t}{2} - \frac{\beta}{6}(u^4 + u_0^4) - \frac{\alpha u_0^2}{2} \end{aligned} \tag{3.53}$$

for multiplier Λ_4.

3.3. The generalized coupled KdV equation

In the third example we try to apply to a multi-potential differential equation system and we choose the generalized coupled KdV equation. It has the form as follows:

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} - 3uu_x + 6vv_x - 3\omega_x = 0, \\ v_t + 3uv_x + \frac{1}{2}v_{xxx} = 0, \\ \omega_t + 3u\omega_x + \frac{1}{2}\omega_{xxx} = 0. \end{cases} \tag{3.54}$$

The famous KdV equation is considered to be one of the most important equations in the theory of integrable systems. It gives multiple soliton solutions with infinite number of conservation laws, double Hamiltonian structures, Lax pairs and many other physical properties. The coupled KdV equation has attracted a lot of research due to its importance in theoretical physics and many scientific applications. According to the formula (2.2) and (2.3), we can obtain by calculation

$$\begin{aligned} &(\mathcal{L})_1^1 v^1 = -3u_x v^1 - 3uD_x v^1 - \frac{1}{4}D_x^3 v^1, (\mathcal{L})_2^1 v^2 = 6vv_x v^2 + 6v_x v^2, \\ &(\mathcal{L})_3^1 v^3 = -3D_x v^3, (\mathcal{L})_1^2 v^1 = 3v_x v^1, \\ &(\mathcal{L})_2^2 v^2 = 3uD_x v^2 + \frac{1}{2}D_x^3 v^2, \\ &(\mathcal{L})_3^2 v^3 = 0, (\mathcal{L})_1^3 v^1 = 3\omega_x v^1, \\ &(\mathcal{L})_2^3 v^2 = 0, (\mathcal{L})_3^3 v^3 = 3uD_x v^3 + \frac{1}{2}D_x^3 v^3 \end{aligned} \tag{3.55}$$

and we can also obtain the adjoint form of them

$$\begin{aligned} &(\mathcal{L}^*)_1^1 \omega^1 = \frac{1}{4}D_x^3 \omega^1 + 3uD_x \omega^1, (\mathcal{L}^*)_2^1 \omega^1 = -6vD_x \omega^1, \\ &(\mathcal{L}^*)_3^1 \omega^1 = 3D_x \omega^1, (\mathcal{L}^*)_1^2 \omega^2 = 3v_x \omega^2, \\ &(\mathcal{L}^*)_2^2 \omega^2 = -\frac{1}{2}D_x^3 \omega^2 - 3uD_x \omega^2 - 3u_x \omega^2, (\mathcal{L}^*)_1^3 \omega^3 = 3\omega_x \omega^3, \\ &(\mathcal{L}^*)_2^3 \omega^1 = 0, (\mathcal{L}^*)_3^3 \omega^3 = -\frac{1}{2}D_x^3 \omega^3 - 3uD_x \omega^3 - 3u_x \omega^3. \end{aligned} \tag{3.56}$$

We set $\mathcal{D}_t = \partial_t - (g^1\partial_u + g^2\partial_v + g^3\partial_\omega) + \cdots$, where $g^1 = -\frac{1}{4}u_{xxx} - 3uu_x + 6vv_x - 3\omega_x$, $g^2 = 3uv_x + \frac{1}{2}v_{xxx}$, $g^3 = 3u\omega_x + \frac{1}{2}\omega_{xxx}$. Then we will try to calculate the form $\Lambda(t, x, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx})$, and $\boldsymbol{u}$ means u, v, ω. The determing equations according to (2.16) become

$$\begin{aligned} 0 =& E_u(u_t\Lambda_1 + g^1\Lambda_1 + v_t\Lambda_2 + g^2\Lambda_2 + \omega_t\Lambda_3 + g^3\Lambda_3) \\ =& - D_t\Lambda_1 + (\mathcal{L}_g^*)_1^\rho \Lambda_\rho + (\mathcal{L}_\Lambda^*)_{1\rho}(u_t^\rho + g^\rho). \end{aligned} \tag{3.57}$$

The specific forms are

$$- D_t\Lambda_1 + \frac{1}{4}D_x^3\Lambda_1 + 3uD_x\Lambda_1 + 3v_x\Lambda_2 + 3\omega_x\Lambda_3 + (\frac{\partial\Lambda_1}{\partial u}G^1 - D_x(\frac{\partial\Lambda_1}{\partial u_x}G^1) + D_x^2(\frac{\partial\Lambda_1}{\partial u_{xx}}G^1)) \tag{3.58}$$

$$+(\frac{\partial\Lambda_2}{\partial u}G^2 - D_x(\frac{\partial\Lambda_2}{\partial u_{xx}}G^2) + D_x^2(\frac{\partial\Lambda_2}{\partial u_{xx}}G^2)) + (\frac{\partial\Lambda_3}{\partial u}G^3 - D_x(\frac{\partial\Lambda_3}{\partial u_{xx}}G^3) + D_x^2(\frac{\partial\Lambda_3}{\partial u_{xx}}G^3)) = 0,$$

$$-D_t\Lambda_2 - 6vD_x\Lambda_1 - \frac{1}{2}D_x^3\Lambda_2 - 3u_xD_x\Lambda_2 - 3u_x\Lambda_2 + (\frac{\partial\Lambda_1}{\partial v}G^1 - D_x(\frac{\partial\Lambda_1}{\partial v_x}G^1) + D_x^2(\frac{\partial\Lambda_1}{\partial v_{xx}}G^1)) \tag{3.59}$$

$$+(\frac{\partial\Lambda_2}{\partial v}G^2 - D_x(\frac{\partial\Lambda_2}{\partial v_{xx}}G^2) + D_x^2(\frac{\partial\Lambda_2}{\partial v_{xx}}G^2)) + (\frac{\partial\Lambda_3}{\partial v}G^3 - D_x(\frac{\partial\Lambda_3}{\partial v_{xx}}G^3) + D_x^2(\frac{\partial\Lambda_3}{\partial v_{xx}}G^3)) = 0,$$

$$-D_t\Lambda_3 + 3D_x\Lambda_1 - \frac{1}{2}D_x^3\Lambda_3 - 3u_xD_x\Lambda_3 - 3u_x\Lambda_3 + (\frac{\partial\Lambda_1}{\partial\omega}G^1 - D_x(\frac{\partial\Lambda_1}{\partial\omega_x}G^1) + D_x^2(\frac{\partial\Lambda_1}{\partial\omega_{xx}}G^1)) \tag{3.60}$$

$$+(\frac{\partial\Lambda_2}{\partial\omega}G^2 - D_x(\frac{\partial\Lambda_2}{\partial\omega_{xx}}G^2) + D_x^2(\frac{\partial\Lambda_2}{\partial\omega_{xx}}G^2)) + (\frac{\partial\Lambda_3}{\partial\omega}G^3 - D_x(\frac{\partial\Lambda_3}{\partial\omega_{xx}}G^3) + D_x^2(\frac{\partial\Lambda_3}{\partial\omega_{xx}}G^3)) = 0.$$

Then by comparing the coefficients of the derivative of G, we can separate the following determining equations:

$$-\mathcal{D}_t\Lambda_1 + \frac{1}{4}D_x^3\Lambda_1 + 3uD_x\Lambda_1 + 3v_x\Lambda_2 + 3\omega_x\Lambda_3 = 0, \tag{3.61}$$

$$-\mathcal{D}_t\Lambda_2 - 6vD_x\Lambda_1 - \frac{1}{2}D_x^3\Lambda_2 - 3uD_x\Lambda_2 - 3u_x\Lambda_2 = 0, \tag{3.62}$$

$$-\mathcal{D}_t\Lambda_3 + 3v_xD_x\Lambda_1 - \frac{1}{2}D_x^3\Lambda_3 - 3uD_x\Lambda_3 - 3u_x\Lambda_3 = 0, \tag{3.63}$$

$$-\frac{\partial\Lambda_1}{\partial v_{xx}} + \frac{\partial\Lambda_2}{\partial u_{xx}} = 0, -\frac{\partial\Lambda_1}{\partial\omega_{xx}} + \frac{\partial\Lambda_3}{\partial u_{xx}} = 0, -\frac{\partial\Lambda_3}{\partial v_{xx}} + \frac{\partial\Lambda_2}{\partial\omega_{xx}} = 0, \tag{3.64}$$

$$2\frac{\partial\Lambda_1}{\partial u_x} - 2D_x\frac{\partial\Lambda_1}{\partial u_{xx}} = 0, 2\frac{\partial\Lambda_2}{\partial v_x} - 2D_x\frac{\partial\Lambda_2}{\partial v_{xx}} = 0, 2\frac{\partial\Lambda_3}{\partial\omega_x} - 2D_x\frac{\partial\Lambda_3}{\partial\omega_{xx}} = 0, \tag{3.65}$$

$$\frac{\partial\Lambda_1}{\partial v_x} + \frac{\partial\Lambda_2}{\partial u_x} - 2D_x\frac{\partial\Lambda_2}{\partial u_{xx}}, \frac{\partial\Lambda_1}{\partial\omega_x} + \frac{\partial\Lambda_3}{\partial u_x} - 2D_x\frac{\partial\Lambda_3}{\partial u_{xx}} = 0, \frac{\partial\Lambda_2}{\partial\omega_x} + \frac{\partial\Lambda_3}{\partial v_x} - 2D_x\frac{\partial\Lambda_3}{\partial v_{xx}} = 0, \tag{3.66}$$

$$\frac{\partial\Lambda_2}{\partial u_x} + \frac{\partial\Lambda_1}{\partial v_x} - 2D_x\frac{\partial\Lambda_1}{\partial v_{xx}} = 0, \frac{\partial\Lambda_3}{\partial u_x} + \frac{\partial\Lambda_1}{\partial\omega_x} - 2D_x\frac{\partial\Lambda_1}{\partial\omega_{xx}} = 0, \frac{\partial\Lambda_3}{\partial v_x} + \frac{\partial\Lambda_2}{\partial\omega_x} - 2D_x\frac{\partial\Lambda_2}{\partial\omega_{xx}} = 0, \tag{3.67}$$

$$-D_x\frac{\partial\Lambda_1}{\partial u_x} + D_x^2\frac{\partial\Lambda_1}{\partial u_{xx}} = 0, -D_x\frac{\partial\Lambda_2}{\partial v_x} + D_x^2\frac{\partial\Lambda_2}{\partial v_{xx}} = 0, -D_x\frac{\partial\Lambda_3}{\partial\omega_x} + D_x^2\frac{\partial\Lambda_3}{\partial\omega_{xx}} = 0, \tag{3.68}$$

$$-\frac{\partial\Lambda_1}{\partial v} + \frac{\Lambda_2}{\partial u} - D_x\frac{\Lambda_2}{\partial u_x} + D_x^2\frac{\partial\Lambda_2}{\partial u_{xx}} = 0, -\frac{\partial\Lambda_1}{\partial\omega} + \frac{\Lambda_3}{\partial u} - D_x\frac{\Lambda_3}{\partial u_x} + D_x^2\frac{\partial\Lambda_3}{\partial u_{xx}} = 0, \tag{3.69}$$

$$-\frac{\partial\Lambda_2}{\partial u} + \frac{\Lambda_1}{\partial v} - D_x\frac{\Lambda_1}{\partial v_x} + D_x^2\frac{\partial\Lambda_1}{\partial v_{xx}} = 0, -\frac{\partial\Lambda_3}{\partial u} + \frac{\Lambda_1}{\partial\omega} - D_x\frac{\Lambda_1}{\partial\omega_x} + D_x^2\frac{\partial\Lambda_1}{\partial\omega_{xx}} = 0, \tag{3.70}$$

$$-\frac{\partial\Lambda_2}{\partial\omega} + \frac{\Lambda_3}{\partial v} - D_x\frac{\Lambda_3}{\partial v_x} + D_x^2\frac{\partial\Lambda_3}{\partial v_{xx}} = 0, -\frac{\partial\Lambda_3}{\partial v} + \frac{\Lambda_2}{\partial\omega} - D_x\frac{\Lambda_2}{\partial\omega_x} + D_x^2\frac{\partial\Lambda_2}{\partial\omega_{xx}} = 0. \tag{3.71}$$

By the formula (3.65) we can know that the hignest coefficient of it is $\Lambda_{1u_{xx}u_{xx}}u_{xxx}$, and Λ_1 is not related with u_{xxx}, so we obtain $\Lambda_{1u_{xx}u_{xx}} = 0$,

$$\Lambda_1 = k_1(t, x, \boldsymbol{u}, \boldsymbol{u}_x, v_{xx}, \omega_{xx})u_{xx} + b_1(t, x, \boldsymbol{u}, \boldsymbol{u}_x, v_{xx}, \omega_{xx}). \tag{3.72}$$

Similarly, we can get

$$\begin{aligned} \Lambda_2 &= k_2(t, x, \boldsymbol{u}, \boldsymbol{u}_x, u_{xx}, \omega_{xx})v_{xx} + b_2(t, x, \boldsymbol{u}, \boldsymbol{u}_x, u_{xx}, \omega_{xx}), \\ \Lambda_3 &= k_3(t, x, \boldsymbol{u}, \boldsymbol{u}_x, u_{xx}, v_{xx})\omega_{xx} + b_3(t, x, \boldsymbol{u}, \boldsymbol{u}_x, u_{xx}, v_{xx}) \end{aligned} \tag{3.73}$$

and accoring to the formula (3.66), we can obtain

$$\Lambda_{2u_{xx}u_{xx}} = 0, \Lambda_{3u_{xx}u_{xx}} = 0, \Lambda_{1v_{xx}v_{xx}} = 0, \Lambda_{1\omega_{xx}\omega_{xx}} = 0, \cdots$$

and since the formula (3.64), we have $\frac{\partial \Lambda_1}{\partial v_{xx}} = \frac{\partial \Lambda_2}{\partial u_{xx}}$.

Then the genreal forms of the multiplicators become

$$\begin{aligned}
\Lambda_1 &= a_1(x,t,\boldsymbol{u},\boldsymbol{u}_x)u_{xx} + a_2(x,t,\boldsymbol{u},\boldsymbol{u}_x)v_{xx} + a_3(x,t,\boldsymbol{u},\boldsymbol{u}_x)\omega_{xx} + b_1(x,t,\boldsymbol{u},\boldsymbol{u}_x),\\
\Lambda_2 &= a_2(x,t,\boldsymbol{u},\boldsymbol{u}_x)u_{xx} + a_4(x,t,\boldsymbol{u},\boldsymbol{u}_x)v_{xx} + a_5(x,t,\boldsymbol{u},\boldsymbol{u}_x)\omega_{xx} + b_2(x,t,\boldsymbol{u},\boldsymbol{u}_x),\\
\Lambda_3 &= a_3(x,t,\boldsymbol{u},\boldsymbol{u}_x)u_{xx} + a_5(x,t,\boldsymbol{u},\boldsymbol{u}_x)v_{xx} + a_6(x,t,\boldsymbol{u},\boldsymbol{u}_x)\omega_{xx} + b_3(x,t,\boldsymbol{u},\boldsymbol{u}_x).
\end{aligned} \tag{3.74}$$

Then we start from formula (3.65), and according to the remaining items we can get

$$a_{1u_x}u_{xx} + a_{2u_x}v_{xx} + a_{3u_x}\omega_{xx} + b_{1u_x} - (a_{1x} + a_{1u}u_x + a_{1u_x}u_{xx}) = 0. \tag{3.75}$$

According to its u_{xxxx} coefficient of the formula (3.61), we have

$$\frac{1}{4}(3u_x a_{1u} + 3a_{1x} + b_{u_x} + 4u_{xx}a_{1u_x} - \Lambda_{1u_x}) = 0, \tag{3.76}$$

then it yeilds $3D_x a_1 - a_{2u_x}v_{xx} - a_{3u_x}\omega_{xx} = 0$, so we get $a_{2u_x} = a_{3u_x} = a_{1u_x} = a_{1u} = a_{1x} = 0$. Similiarly according to the forlumas (3.62) and (3.63), finally we obtain the expressions $a_i(t), i = 1, \cdots, 6$. Then we calculate from formula (3.75), we have $b_{1u_x} = 0$, similarly, $b_{2v_x} = b_{3\omega_x} = 0$. Accoring to the formula (3.66), we get

$$\begin{aligned}
&b_{1v_x} + b_{2u_x} = 0, b_{2\omega_x} + b_{3v_x} = 0,\\
&b_{1\omega_x} + b_{3u_x} = 0, b_{3v_x} + b_{2\omega_x} = 0,\\
&b_{2u_x} + b_{1v_x} = 0, b_{3u_x} + b_{1\omega_x} = 0.
\end{aligned} \tag{3.77}$$

We can take the simplest forms, i.e. $b_1,\ b_2,\ b_3$ are not related with $\boldsymbol{u}_x$. Then let us start with the formula (3.58). The coefficient of u_{xxx} is $3ua_1 - 3ua_1 + \frac{1}{4}b_u - \frac{1}{4}b_u = 0$, and the coefficient of u_{xx} is

$$\frac{1}{4}(3b_{u\omega} + 3u_x b_{uu} + 3v_x b_{uv} + 3b_{xu}) + 3v_x a_2 + 3\omega_x a_3 - a^{'}(t) - 9a_1 u_x + 3a_2 v_x + 3a_3\omega_x = 0. \tag{3.78}$$

Since a_1, a_2, a_3 don't contain u_x, v_x, ω_x, we obtain

$$\begin{aligned}
&\frac{3}{4}b_{u\omega} + 6a_3 = 0, \frac{3}{4}b_{uv} + 6a_2 = 0,\\
&\frac{3}{4}b_{uu} - 9a_1 = 0, \frac{3}{4}b_{xu} = a^{'}(t).
\end{aligned} \tag{3.79}$$

So we have

$$b_1 = 6a_1u^2 + cu + d(x,t,v,\omega)$$

and since $b_{1uv} = -8a_2$, we get $c_v = -8a_x$. It yeilds $c = -8a_2v + d^{'}$, since $\frac{\partial d}{\partial \omega} = -8a_3$.

We can denote it by

$$b_1 = 6a_1u^2 - 8a_2uv - 8a_3u\omega + \frac{4}{3}a_{1t}x + p_1(t)u + p_2(x,t,v,\omega). \tag{3.80}$$

And according to the formula (3.62), the coefficient of v_{xx} is

$$-\frac{1}{2}(3b_{2xv}+3v_xb_{2vv}+3\omega_xb_{2v\omega}+3u_xb_{2uv})-3u_xa_4-a_4^{'}(t)+18v_xa_2+6u_xa_4. \tag{3.81}$$

It yeilds that

$$\begin{aligned}&-\frac{3}{2}b_{2vv}+18a_2=0, b_{2v\omega}=0,\\&-\frac{3}{2}b_{2uv}+3a_4=0, -\frac{3}{2}b_{2xv}=a_4^{'}(t).\end{aligned} \tag{3.82}$$

We can calculate the expression

$$b_2=-6a_2v^2+2a_4uv-\frac{2}{3}a_4^{'}xv+q_1(t)+q_2(x,t,u,\omega). \tag{3.83}$$

Similarly, in the formula (3.63), the coffecient of ω_{xx} is

$$-\frac{1}{2}(3b_{3x\omega}+3v_xb_{3\omega\omega}+3u_xb_{3u\omega})-3u_xa_6. \tag{3.84}$$

It yeilds that

$$\begin{aligned}&b_{3v\omega}=0, -\frac{3}{2}b_{3u\omega}+3a_6=0,\\&b_{3\omega\omega}=0, -\frac{3}{2}b_{x\omega}=a_6^{'}(t).\end{aligned} \tag{3.85}$$

Then we can calculate the expression

$$b_3=2a_6u\omega-\frac{2}{3}a_6^{'}x\omega+r_1(t)\omega+r_2(x,t,u,v). \tag{3.86}$$

Then the general forms become

$$\begin{aligned}\Lambda_1&=a_1(t)u_{xx}+a_2(t)v_{xx}+a_3(t)\omega_{xx}+6a_1u^2-8a_2uv-8a_3u\omega+(\frac{4}{3}a_1tx+p_1(t))u+p_2(x,t,v,\omega),\\\Lambda_2&=a_2(t)u_{xx}+a_4(t)v_{xx}+a_5(t)\omega_{xx}-6a_2v^2+2a_4uv-\frac{2}{3}a_4^{'}xv+q_1(t)v+q_2(x,t,u,\omega),\\\Lambda_3&=a_3(t)u_{xx}+a_5(t)v_{xx}+a_6(t)\omega_{xx}+2a_6u\omega-\frac{2}{3}a_6^{'}x\omega+r_1(t)\omega+r_2(x,t,u,v).\end{aligned} \tag{3.87}$$

And according to the formula (3.61), the coefficient of v_{xxxxx} is $(\frac{1}{4}-\frac{1}{2})a_2=0$, so we obtain $a_2=0$. Similarly, we also get $a_3=0$. Then we separated the coefficient of u_{xx} of the formula (3.61):

$$\frac{1}{4}(3b_{xu}+3u_xb_{uu}+3v_xb_{uv}+3\omega_xb_{u\omega})-a_1^{'}(t)-9a_1u_x=0, \tag{3.88}$$

and it yeilds that

$$b_{uv}=0, b_{u\omega}=0, \frac{3}{4}b_{xu}=a_1^{'}(t), \frac{3}{4}b_{uu}=9a_1(t).$$

According to the coefficients of ω_{xxx}:$\frac{3}{4}p_{2\omega} = 3a_1(t) = 0,\ v_{xxx}$:$34p_2v = -6a_1(t)v = 0,\ v_xv_{xx}$:$3a_4 + 12a_1 = 0,\ v_x\omega_{xx}$:$3a_5 = 0,\ \omega_xv_{xx}$:$3a_5 = 0,\ \omega_x\omega_{xx}$:$3a_6 = 0$. We obtain that $p_2 = 4a_1(t)\omega,\ p_2 = -4a_1(t)v^2,\ a_5 = a_6 = 0,\ a_4 = -4a_1(t)$. And the coefficient of vv_x is

$$16a_1^{'}x + 12ua_1 + 3q_1 + 6p_1 = 0. \tag{3.89}$$

It yeilds that $a_1(t) = 0$. The remaining coefficients of the formula (3.61) can be separated into three equations for the coefficients of v_x and ω_x:

$$\begin{aligned} &-p_{1t} + 3p_{2x} = 0,\\ &3q_2 + 12ua_1v + 3q_1v + 6p_1v = 0,\\ &3r_2 - 3p_1 - 2a_6^{'}x\omega - 12a_1u + 3r_1\omega = 0. \end{aligned} \tag{3.90}$$

They yeild that

$$\begin{aligned} &c = -p_{2t} + \frac{p_{2xxx}}{4},\\ &q_2 = -4a_1uv - q_1v - 2p_1v,\\ &r_2 = p_1 + \frac{2}{3}a_6^{'}x\omega + 4a_1u - r_1\omega. \end{aligned} \tag{3.91}$$

Let us bring back them into the formulas (3.61), (3.62), (3.63), the remaining items of them can be separated into four equations

$$\begin{cases} 3p_{2x} - p_{1t} = 0,\\ -p_{2t} + \dfrac{p_{2xxx}}{4} = 0,\\ 2p_{1t} = 6p_{2x},\\ -p_{1t} + 3p_{2x} = 0. \end{cases} \tag{3.92}$$

By solving the equations, we can get

$$p_1(t) = 3c_1t + c_3, p_2(x,t) = c_1x + c_2. \tag{3.93}$$

Then we get the genreal form of $\Lambda_i, i = 1, 2, 3$.

$$\begin{aligned} &\Lambda_1 = a_1u_{xx} + 6a_1u^2 + (3c_1t + c_3)u + 4a_1\omega - 4a_1v^2 + c_1x + c_2,\\ &\Lambda_2 = -4a_1v_{xx} - 8a_1uv - 2(3c_1t + c_3)v,\\ &\Lambda_3 = 3c_1t + c_3 + 4a_1u. \end{aligned} \tag{3.94}$$

They yeild that

$$\begin{aligned} &\Lambda_{11} = u_{xx} + 6u^2 + 4\omega - 4v^2, \Lambda_{12} = -4v_{xx} - 8uv, \Lambda_{13} = 4u.\\ &\Lambda_{21} = 3ut + x, \Lambda_{22} = -6tv, \Lambda_{23} = 3t.\\ &\Lambda_{31} = 1, \Lambda_{32} = 0, \Lambda_{33} = 0.\\ &\Lambda_{41} = u, \Lambda_{42} = -2v, \Lambda_{43} = 1. \end{aligned} \tag{3.95}$$

Then we can solve the conservation laws ϕ^t, ϕ^x by multipliers according to (2.23), (2.24)

$$\begin{aligned}
\phi_1^t =& \int_0^1 u\Lambda_{11}[\lambda \boldsymbol{u}] + v\Lambda_{12}[\lambda \boldsymbol{u}] + \omega\Lambda_{13}[\lambda \boldsymbol{u}]\, d\lambda \\
=& \int_0^1 u(\lambda u_{xx} + 6u^2\lambda^{12} + 4\omega\lambda - 4v^2\lambda^2) + v(-4\lambda v_{xx} - 8\lambda^2 uv) + \omega(4\lambda u)\, d\lambda \\
=& -\frac{1}{2}u_x^2 + 2u^3 + 2\omega u - \frac{4}{3}v^2 u - 2v_x^2 - \frac{8}{3}uv^2 + 2\omega u, \\
\phi_1^x =& \int_0^1 d\lambda (S^x[\boldsymbol{u}, \Lambda[\lambda \boldsymbol{u}]; g[\lambda \boldsymbol{u}]] + S^x[u; g[\lambda \boldsymbol{u}] - \lambda g[\boldsymbol{u}]; \Lambda[\lambda \boldsymbol{u}]]) \\
=& u(\Lambda_{11}(-3\lambda u)) + \frac{u}{4}D_x^2\Lambda_1 + v(6\lambda v\Lambda_{11} + 3\lambda u\Lambda_{22}) + \frac{v\Lambda_{12}}{2} + \omega(-3\Lambda_{11} + 3\lambda u\Lambda_{13}) + \frac{\omega\Lambda_{13}}{2} \\
&+ \frac{1}{4}u_x D_x\Lambda_{11} - \frac{1}{2}v_x D_x\Lambda_{12} - \frac{1}{2}\omega_x D_x\Lambda_{13} - \frac{1}{4}u_{xx}\Lambda_{11} + \frac{1}{2}v_{xx}D_x\Lambda_{12} + \frac{1}{2}\omega_{xx}\Lambda_{13} - uD_x(g_1[\lambda \boldsymbol{u}] \\
&- \lambda g_1[\boldsymbol{u}]) + 4vD_x(g_2[\lambda \boldsymbol{u}] - \lambda g_2[\boldsymbol{u}]) + u_x(g_1[\lambda \boldsymbol{u}] - \lambda g1[\boldsymbol{u}]) - 4v_x g_2[\lambda \boldsymbol{u}] - \lambda g_2[\boldsymbol{u}] \\
=& -6\omega^2 - 6v^4 - 7v_{xx}uv + \omega_x u_x + \omega u + v_x v_{xxx} - \frac{7u_x v v_x}{3} + \frac{3u_{xx}uu_x}{8} - \frac{3u_{xx}vv_x}{4} + \frac{3u\omega_x}{2} - v_{xx}^2 \\
&+ \frac{uu_{xxxx}}{8} - vv_{xx} - \frac{3\omega u_{xx}}{2} + \frac{u_x u_{xxx}}{8} + \frac{u_{xx}u_{xxx}}{32} + \frac{3u_{xx}\omega_x}{8} - \frac{3u_x\omega}{2} + 6u^2v^2 - \frac{u^2 u_{xx}}{2} - 6u^2\omega \\
&+ 2uu_x^2 + 2v^2 u_{xx} + 12v^2\omega - \frac{4v^2 u}{3} + \frac{11v_x^2 u}{3} - \frac{9}{2}u^4
\end{aligned} \tag{3.96}$$

for $\Lambda_{11}, \Lambda_{12}, \Lambda_{13}$.

$$\begin{aligned}
\phi_t^2 =& \frac{3}{2}u^2 t + ux - 3tv^2 + 3\omega t, \\
\phi_x^2 =& -\frac{9u^3 t}{2} - \frac{3u^2 x}{2} + 3xv^2 - \frac{9\omega tu}{2} + \frac{u_{xx}u_{xx}x}{32} + \frac{3u_{xx}uu_x}{8} - \frac{3u_{xx}vv_x}{4} + \frac{3u_{xx}\omega_x}{8} + \frac{3utu_{xx}}{4} - 3tv^2 \\
&- 3x\omega + \frac{3\omega t}{2} + \frac{3tu_x^2}{4} + \frac{u_x}{4} - 3tvv_{xx} + \frac{3\omega_{xx}t}{2}
\end{aligned} \tag{3.97}$$

for $\Lambda_{21}, \Lambda_{22}, \Lambda_{23}$.

$$\phi_t^3 = u, \phi_x^3 = -\frac{3u^2}{2} + 3v^2 + \frac{u_{xx}u_{xxx}}{32} + \frac{3u_{xx}uu_x}{8} - \frac{3u_{xx}vv_x}{4} + \frac{3u_{xx}\omega_x}{8} - 3\omega \tag{3.98}$$

for $\Lambda_{31}, \Lambda_{32}, \Lambda_{33}$.

$$\begin{aligned}
\phi_t^4 =& \frac{u^2}{2} - v^2 + \omega, \\
\phi_x^3 =& -\frac{3u^3}{2} - \frac{3\omega u}{2} + \frac{u_{xx}u_{xxx}}{32} + \frac{3u_{xx}uu_x}{8} + \frac{3u_{xx}\omega_x}{8} + \frac{uu_{xx}}{4} - v^2 + \frac{\omega}{2} + \frac{u_x^2}{4} + v_x^2 - vv_{xx} + \frac{\omega_{xx}}{2} \\
&- \frac{3u_{xx}vv_x}{4}
\end{aligned} \tag{3.99}$$

for $\Lambda_{41}, \Lambda_{42}, \Lambda_{43}$.

4. Solving conservation law by Ibragimov method

In this part we try to solve the equation by Ibragimov method to obtain the conservation laws. Firstly, we introduce a theorem:

Theorem 2. Any Lie point, Lie-Bäcklund, and nonlocal symmetry

$$X = \xi^i(x, u, u_{(1)}, \dots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots)\frac{\partial}{\partial u^\alpha}$$

leads to the conservation law $D_i(C^i) = 0$, i.e.

$$\begin{aligned} C^i =&\xi^i \mathcal{L} + W^\alpha[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}) + D_jD_j(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}) - \cdots] \\ &+ D_j(W^\alpha)[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}) + \cdots] + D_jD_k(W^\alpha)[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots] + \cdots \end{aligned} \tag{4.1}$$

where

$$\mathcal{L} = \sum_{i=1}^m v^i F_i(x, u, u_{(1)}, \dots, u_{(s)}), W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \alpha = 1, ..., m.$$

By using maple, we can obtain some symmetries of the target equations. Firstly, for the combined KdV-MKdV equation (3.1), it has

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = (\frac{x}{3} - \frac{\alpha^2 t}{6\beta})\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + (\frac{-2\beta u - \alpha}{6\beta})\frac{\partial}{\partial u}. \tag{4.2}$$

We choose $X = (\frac{x}{3} - \frac{\alpha^2 t}{6\beta})\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + (\frac{-2\beta u-\alpha}{6\beta})\frac{\partial}{\partial u}$ as the symmetry used to calculate the conservation laws. According to the formula (4.1), we have

$$W = \frac{-2\beta u - \alpha}{6\beta} - (\frac{x}{3} - \frac{\alpha^2 t}{6\beta})u_x - tu_t.$$

And we note $\mathcal{L} = v(u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx})$. Then the conservation laws calculated directly become

$$\begin{aligned} C_x =&(\frac{x}{3} - \frac{\alpha^2}{6\beta}t)\mathcal{L} + W(\alpha uv + \beta u^2 v + v_{xx}) + D_xW(-v_x) + D_x^2W(v) \\ =& -\frac{\alpha^2 vtu_t}{6\beta} - \frac{uv_{xx}}{3} + \frac{2v_xu_x}{3} - vu_{xx} - \frac{\alpha u^2 v}{2} - \frac{\beta u^3 v}{3} - \frac{\alpha v_{xx}}{6\beta} - \frac{u_x x v_{xx}}{3} - tu_t v_{xx} + \frac{v_x u_{xx} x}{3} \\ &+ v_x t u_{xt} - vtu_{xxt} + \frac{vxu_t}{3} - \frac{\alpha^2 uv}{6\beta} + \frac{\alpha^2 u_x t v_{xx}}{6\beta} - \alpha t u_t uv - \beta t u_t u^2 v - \frac{\alpha^2 v_{xx} u_{xx} t}{6\beta}, \\ C_t =&t\mathcal{L} + Wv \\ =&\alpha t u_x uv + \beta u_x t u^2 v + vtu_{xxx} - \frac{uv}{3} - \frac{\alpha v}{6\beta} - \frac{vxu_x}{3} + \frac{\alpha^2 vu_x t}{6\beta}. \end{aligned} \tag{4.3}$$

We can simply verify it:

$$D_xC_x + D_tC_t = (-\frac{\alpha}{6\beta} - \frac{u}{3})(u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx}) = 0 \tag{4.4}$$

when u is a solution of Eq. (3.1) and $v = -u$.

Similarly, for the Klein-Gordon equation (3.33), it has symmetries which are as follows:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} \tag{4.5}$$

and we choose $X = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}$ as the symmetry to compute conservation law.

$$\begin{aligned} C_x =& t\mathcal{L} + W(-v_x) + D_x W v \\ =& vtu_{tt} - 2u_{xx}vt + \alpha tuv + \beta vtu^3 + tu_x v_x + xu_t v_x - vu_t - vxu_{xt}, \\ C_t =& x\mathcal{L} + W(-v_t) + vD_x W \\ =& \beta xvu^3 + \alpha xuv - xvu_{xx} - vu_{tx}t + tv_t u_x + xu_t v_t - vu_x \end{aligned} \tag{4.6}$$

where $\mathcal{L} = v(u_{tt} - u_{xx} + \alpha u + \beta u^3), W = -tu_x - xu_t$.

At last, we consider the generalized coupled KdV equation (3.54), it has the symmetries which are as follows:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial \omega}, X_4 = v\frac{\partial}{\partial \omega} + \frac{1}{2}\frac{\partial}{\partial v}, \\ X_5 &= \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{4\omega}{3}\frac{\partial}{\partial \omega} - \frac{2u}{3}\frac{\partial}{\partial u} - \frac{2v}{3}\frac{\partial}{\partial v} \end{aligned} \tag{4.7}$$

and we choose $X = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{4\omega}{3}\frac{\partial}{\partial \omega} - \frac{2u}{3}\frac{\partial}{\partial u} - \frac{2v}{3}\frac{\partial}{\partial v}$ as the symmetry to compute conservation law. We can calculate that

$$\begin{aligned} &\mathcal{L} = v_1(u_t - \frac{1}{4}u_{xxx} - 3uu_x + 6vv_x - 3\omega_x) + v_2(v_t + 3uv_x + \frac{1}{2}v_{xxx}) + v_3(\omega_t + 3u\omega_x + \frac{1}{2}\omega_{xxx}), \\ &W_1 = -\frac{2u}{3} - \frac{xu_x}{3} - tu_t, W_2 = -\frac{2v}{3} - \frac{xv_x}{3} - tv_t, W_3 = -\frac{4\omega}{3} - \frac{x\omega_x}{3} - t\omega_t. \end{aligned} \tag{4.8}$$

Then the conservation law is

$$\begin{aligned} C_x =& \frac{x}{3}\mathcal{L} + W_1(-3uv_1 - \frac{1}{4}v_{1xx}) + W_2(6vv_1 + 3uv_2 + \frac{1}{2}v_{2xx}) + W_3(-3v_1 + 3uv_3 + \frac{1}{2}v_{3xx}) + \frac{1}{4}D_x W_1 v_{1x} \\ &- \frac{1}{2}D_x W_2 v_{2x} - \frac{1}{2}D_x W_3 v_{3x} - \frac{1}{4}D_x^2 W_1 v_1 + \frac{1}{2}D_x^2 W_2 v_2 + \frac{1}{2}D_x^2 W_3 v_3 \\ =& \frac{\omega_t v_3 x}{3} - \frac{v_{3xx}x\omega_x}{6} - \frac{v_{3xx}t\omega_t}{2} - \frac{v_{1x}xu_{xx}}{12} - \frac{v_{1x}tu_{xt}}{4} + \frac{v_1 tu_{xxt}}{4} - \frac{v_3 t\omega_{xxt}}{2}\frac{v_{1xx}xu_x}{12} + \frac{v_{1xx}tu_t}{4} - 2uv_2 v \\ &- \frac{v_{2xx}xv_x}{6} - \frac{v_{2xx}tv_t}{2} + 3tv_1\omega_t - 4uv_3\omega + \frac{v_{2x}xv_{xx}}{6} + \frac{v_{2x}tv_{xt}}{2} - \frac{2v_2 v_{xx}}{3} + 3v_1 utu_t - 6vv_1 tv_t \\ &- 3uv_2 tv_t - 3uv_3 t\omega_t + \frac{u_t v_1 x}{3} + \frac{v_{3x}x\omega_{xx}}{6} + \frac{v_{3x}t\omega_{xt}}{2} - \frac{v_2 tv_{xxt}}{2} + \frac{v_t v_2 x}{3} - v_3\omega_{xx} + 2u^2 v_1 + \frac{v_{1xx}u}{6} \\ &- 4v^2 v_1 - \frac{v_{2xx}v}{3} + 4v_1\omega - \frac{2v_{3xx}\omega}{3} - \frac{v_{1xx}u_x}{4} + \frac{v_{2x}v_x}{2} + \frac{5v_{3x}\omega_x}{6} + \frac{v_1 u_{xx}}{3}, \\ C_t =& t\mathcal{L} + W_1 v_1 + W_2 v_2 + W_3 v_3 \\ =& -\frac{tv_1 u_{xxx}}{4} - 3tv_1 uu_x + 6tv_1 vv_x - 3tv_1\omega_x + 2tv_2 uv_x + \frac{tv_2 v_{xxx}}{2} + 3tv_3 u\omega_x + \frac{tv_3\omega_{xxx}}{2} - \frac{2uv_1}{3} \\ &- \frac{v_1 xu_x}{3} - \frac{2v_2 xv_x}{3} - \frac{4v_3\omega}{3} - \frac{v_3 x\omega_x}{3}. \end{aligned} \tag{4.9}$$

5. Hamiltonian Structure and Line Soliton Solution

5.1. Hamiltonian structure

For the combined KdV and MKdV equation, We notice that it has a Hamiltonian formulation $u_t = -\mathcal{D}(\frac{\delta H}{\delta u})$, where $H = \int \frac{\alpha u^3}{6} + \frac{\beta}{12u^4} - \frac{1}{2}u_x^2 dx$ is the Hamiltonian functional, and $\mathcal{D} = D_x$ is a Hamiltonian operator. Then since D_x is a Hamiltonian operator, it can map adjoint-symmetries into symmetries, so D_x^{-1} can map symmetries into adjoint-symmetries. And we can use the above symmetry to get the adjoint symmetry of some objective equations. Applying this latter operator to the scaling symmetries, we obtain the adjoint-symmetries:

$$\begin{aligned} Q_1 =& D_x^{-1}(u_t) = v_t, Q_2 = D_x^{-1}u_x = u, \\ Q_3 =& D_x^{-1}(\frac{-2\beta u - \alpha}{6\beta} - (\frac{x}{3} - \frac{\alpha^2 t}{6\beta}))u_x - tu_t \\ =& -\frac{\alpha}{6\beta}x + \frac{\alpha^2}{6\beta}tu - \frac{xu}{3} + \alpha t\frac{u^2}{2} + \beta t\frac{u^3}{3} + tu_{xx} \end{aligned} \tag{5.1}$$

where $u = v_x$. In fact, the multiplier Λ calculated earlier in this paper is the adjoint symmetry of the variation of the objective equation.

Then for the Klein-Gordon equation, to obtain the Hamiltonian formulation, we transform the Eq. (3.33) into an equation system:

$$u_t = v, v_t = u_{xx} - \alpha u - \beta u^3. \tag{5.2}$$

The associated Hamiltonian formulation for this system is then given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = J\begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta \omega} \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.3}$$

where $H = \int \frac{v^2}{2} + \frac{u_x^2}{2} + \frac{\alpha}{2}u^2 + \frac{\beta}{4}u^4 dx$. We note that the determining equation of the objective equation and its self-adjoint determining equation is consistent:

$$D_t^2\eta - D_x^2\eta + \alpha u + \beta u^3 = 0. \tag{5.4}$$

Hence, the symmetry of the equation is consistent with the adjoint symmetry.

5.2. Line soliton solutions

A line soliton is a solitary travelling wave $u = U(x - \mu t)$ in one dimension where the parameter μ means the speed of the wave. Then we study the conservation laws of the combined KdV and MKdV equation and the Klein-Gordon equation ϕ^t, ϕ^x which doesn't contain the variables t, x. Then the conservation law is obtained by reduction

$$D_t|_{u=U(\xi)} = -\mu\frac{d}{d\xi}, D_x|_{u=U(\xi)} = \frac{d}{d\xi}, \xi = x - \mu t \tag{5.5}$$

yielding

$$\frac{d}{d\xi}((\phi^x - \mu\phi^t)) = 0. \tag{5.6}$$

So $(\phi^x-\mu\phi^t)=C$. Then we begin with the combined KdV and MKdV equation. Using the transformation $u(x,t)=U(\xi)$, we can obtain the nonlinear ordinary differential equation:

$$-\mu U^{'}+\alpha UU^{'}+\beta U^2U^{'}+U^{'''}=0 \tag{5.7}$$

for $U(\xi)$. Conservation laws (3.28), (3,30), (3.32) do not contain the variables t,x. When the first integral formula $(\phi^x-\mu\phi^t)=C$, is applied to these three conservation laws, we obtain

$$C_1=\frac{\alpha U^2}{2}+\frac{\beta U^3}{3}+U^{''}-\mu U=0, \tag{5.8}$$

$$C_2=\frac{\beta^2U^6}{18}+\frac{\alpha\beta U^5}{6}+\frac{5\beta U^3U^{''}}{6}+\frac{\alpha^2U^4}{8}+\alpha U^2U^{''}+\frac{U^2(U^{'})^2\beta}{2}+\frac{(U^{''})^2}{2} \tag{5.9}$$
$$-\frac{U^{'}U^{'''}}{2}+\frac{UU^{''''}}{2}-\mu\left(\frac{UU^{''}}{2}+\frac{\alpha U^2}{6}+\frac{\beta U^4}{12}\right),$$

$$C_3=\frac{\alpha U^2}{3}+\frac{\beta U^4}{4}-\frac{(U^{'})^2}{2}+UU^{''}-\frac{\mu U^2}{2}=0. \tag{5.10}$$

We impose the asymptotic conditions $U,U^{'},U^{''},U^{'''}\to 0$ as $|\xi|\to\infty$. Then we combine the formulas (5.8), (5.9) and (5.10), then we can calculate its general line soliton solutions:

$$U_1=\frac{144\mu e^{\xi\sqrt{\mu}}}{e^{k_1\sqrt{\mu}}\left(144\alpha^2+\frac{24\alpha e^{\xi\sqrt{\mu}}}{e^{k_1\sqrt{\mu}}}+864\beta\mu+\frac{(e^{\xi\sqrt{\mu}})^2}{(e^{k_1\sqrt{\mu}})^2}\right)}, \tag{5.11}$$

$$U_2=\frac{144\mu e^{k_1\sqrt{\mu}}}{e^{\xi\sqrt{\mu}}\left(144\alpha^2+\frac{24\alpha e^{k_1\sqrt{\mu}}}{e^{\xi\sqrt{\mu}}}+864\beta\mu+\frac{(e^{k_1\sqrt{\mu}})^2}{(e^{\xi\sqrt{\mu}})^2}\right)} \tag{5.12}$$

where k_1 is an arbitrary constant.

Figure 1 and Figure 2 display the kinds of 3D plots of $U_1(\xi)$ and $U_2(\xi)$ determined by (5.11) and (5.12), and Figure 3 and Figure 4 display the kinds of density plots of them.

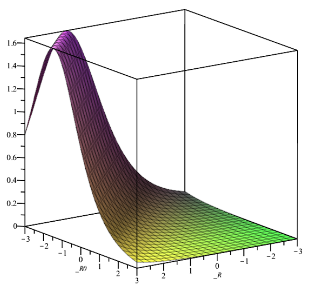

Figure 1. 3D plot of the U_1 given by Eq. (5.11) for parameters $\mu=k_1=\alpha=\beta=1$.

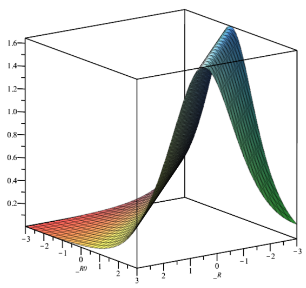

Figure 2. 3D plot of the U_2 given by Eq. (5.12) for parameters $\mu=k_1=\alpha=\beta=1$.

Then for the Klein Gordon equation, we make the transformation $u(x,t)=U(\xi)$, we obtain an ODE:

$$\mu^2U^{''}-U^{''}+\alpha U+\beta U^3=0. \tag{5.13}$$

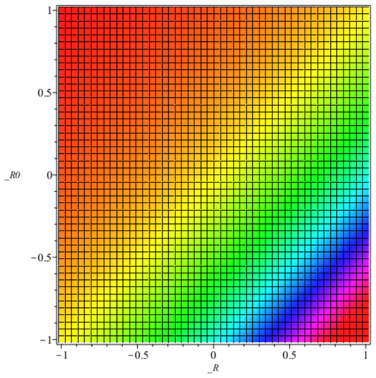

Figure 3. Density plot of the U_1 given by Eq. (5.11) for parameters $\mu = k_1 = \alpha = \beta = 1$.

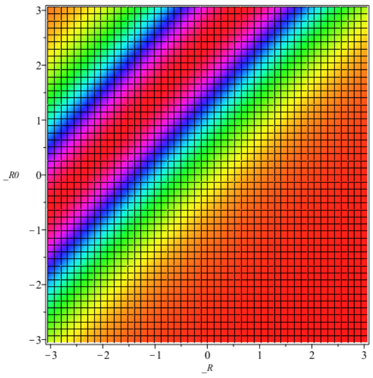

Figure 4. Density plot of the U_2 given by Eq. (5.12) for parameters $\mu = k_1 = \alpha = \beta = 1$.

And we study the related conservation laws (3.50)-(3.53), only the formula (3.50) does not contain variables x, t. We can obtain

$$-\frac{(U-u_0)\mu U''}{2} + \frac{(U')^2\mu}{2} - \mu\left(\frac{(U')^2\mu^2}{2} - \frac{(U-u_0)(U''-\alpha U-\beta U^3)}{2}\right) = 0. \tag{5.14}$$

By calculating (5.14), we can get its soliton solution

$$U_1 = u_0, U_2 = \frac{\sqrt{-\beta\alpha}}{\beta} \tag{5.15}$$

and the roots of $\xi - (\int^{U_3} \pm \frac{(\mu-1)(\mu+1)}{\sqrt{-(\mu-1)(\mu+1)a(a-u_0)(a^2\beta+\alpha)}} da) - k_2 = 0$, where k_2 is arbitrary constant.

6. Conclusion

It is well known that the study of conservation laws is very important for studying the integrability of optimal systems. In this paper, two methods are used to solve three different types of partial differential equations and systems, namely the conservation laws of the combined KdV-MKdV equation, the Klein-Gordon equation and the generalized coupled KdV equation. And these two methods are widely applicable. It can be applied not only to the case of multiple independent variables, but also to the case of multiple dependent variables and differential equation systems. In fact, the multipliers obtained in this part of the direct construction of conservation laws are actually some adjoint symmetries of the equation with variational properties. And the linear soliton solutions of the equations can be analyzed by the obtained conservation law.

In fact, the two methods used in this paper to calculate the conservation law of equations have different advantages. The adjoint equation method proposed by Ibragimov can use the symmetry of the equation to calculate the conservation law through the explicit formula. It is convenient to calculate and does not require complex analysis. It has a wide range of applications, but the results are directly affected by the symmetry of the equation. The advantage of constructing conservation laws directly is that it is not necessary to use the variational symmetry of the equation. For a partial differential equation without variational symmetry, the adjoint symmetry of the equation is used to replace the symmetry. At this time, the adjoint symmetry satisfies the linear adjoint symmetry determining equations. The symmetry invariant condition is replaced by the adjoint symmetry invariant condition, and a formula using adjoint symmetry is given. However, this method is computationally complex and does not apply to any type of equation and equation system. Both methods can be naturally applied to higher dimensional differential equations and differential equation systems. This paper mainly integrates the two methods and applies them to different types of equations and equation systems. The examples in Anco's paper [6,7] basically apply the direct construction method to the (1+1) dimensional differential equations, and this paper attempts to apply the method to the equation system.

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Conflict of interest

The authors declare that they have no known competing financial interests.

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