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ON IDENTIFIABILITY OF 3-TENSORS OF MULTILINEAR RANK $(1,\ L_r,\ L_r)$

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Abstract. In this paper, we study a specific big data model via multilinear rank tensor decompositions. The model approximates to a given tensor by the sum of multilinear rank $(1,\ L_r,\ L_r)$ terms. And we characterize the identifiability property of this model from a geometric point of view. Our main results consists of exact identifiability and generic identifiability. The arguments of generic identifiability relies on the exact identifiability, which is in particular closely related to the well-known "trisecant lemma" in the context of algebraic geometry (see Proposition 2.6 in [1]). This connection discussed in this paper demonstrates a clear geometric picture of this model.

1. Introduction.

1.1. Content of the paper.

The importance and usefulness of tensors that are characterized by multiway arrays for big data sets, has been increasingly recognized in the last decades, as testified by a number of surveys [15, 20, 14, 5, 17] and among others. Identifiability property (see [3, 10, 2, 9]), including both exact and generic identifiability, is critical for tensor models in various applications, and widely used in many areas, such as signal processing, statistics, computer science, and so on. For instance, in signal processing, the tensor encodes data from received signals and one needs to decompose the tensor to obtain the transmitted signals. If the uniqueness

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does not hold, one may not recover the transmitted signals. Therefore, to establish the uniqueness property of appropriate tensor decomposition is not only mathematically interest but also necessary in various real applications. Extensive studies under the framework of algebraic geometry have provided various characteristics involving tensor rank and dimensions to ensure generic identifiability.

In this paper, we consider the model of low multilinear rank tensor decomposition (**LRD**). The initial idea was proposed by De Lathauwer [6, 7, 8], where the rank-1 tensors in CP decomposition is replaced by tensors in multilinear rank $(1,\ L_r,\ L_r)$ terms. Such approach allows us to model more complex phenomena and to analyze big data sets with complex structures, especially for the cases that tensor components cannot be represented as rank-1 tensors. We extend the theoretical frameworks by establishing the uniqueness conditions of **LRD**, which are critical for the applications of tensor-based approaches in handling big data sets. More specifically, if a tensor can be written in a unique manner as a sum of tensors of low multilinear rank, then this decomposition may reveal (meaningful) characteristics that are more general than the components extracted from CP decomposition. The uniqueness property of **LRD** can be theoretically guaranteed with mild conditions under our framework, and we provide the new uniqueness criterion of multilinear-rank tensor decomposition that closely relates to the applications of **LRD** in blind source separation in signal processing. The theoretical contributions of establishing the explicit uniqueness criterion of **LRD** may play significant role in the application domains of tensor-based methods for big data analysis [4].

1.2. **Definitions.**

Definition 1.1. (see Chapter III in [19]) Let $\mathbb{K}$ be a field $\mathbb{C}$ or $\mathbb{R}$ and let $A_1, \dots, A_n$ be $\mathbb{K}$-vector spaces. The *tensor product space* $A_1 \otimes \cdots \otimes A_n$ is the quotient module $\mathbb{K}(A_1, \dots, A_n)/R$ where $\mathbb{K}(A_1, \dots, A_n)$ is the free module generated by all n-tuples $(a_1, \dots, a_n) \in A_1 \times \cdots \times A_n$ and R is the submodule of $\mathbb{K}(A_1, \dots, A_n)$ generated by elements of the form

$$(a_1, \dots, \alpha a_k + \beta a_k', \dots, a_n) - \alpha(a_1, \dots, a_k, \dots, a_n) - \beta(a_1, \dots, a_k', \dots, a_n)$$

for all $a_k, a_k' \in A_k, \alpha, \beta \in \mathbb{K}$, and $k \in \{1, \dots, n\}$. We write $a_1 \otimes \cdots \otimes a_n$ for the element $(a_1, \dots, a_n) + R$ in the quotient space $\mathbb{K}/R$.

An element of $A_1 \otimes \cdots \otimes A_n$ that can be expressed in the form $a_1 \otimes \cdots \otimes a_n$ is called *decomposable*. The symbol $\otimes$ is called the *tensor product* when applied to vectors from abstract vector spaces.

The elements of $A_1 \otimes \cdots \otimes A_n$ are called *order-n tensors* and $I_k = \dim A_k$, $k = 1, \dots, n$ are the *dimensions* of the tensors.

If $U \cong \mathbb{K}^l, V \cong \mathbb{K}^m, W \cong \mathbb{K}^n$, we may identify

$$\mathbb{K}^l \otimes \mathbb{K}^m \otimes \mathbb{K}^n = \mathbb{K}^{l \times m \times n}$$

through the interpretation of the tensor product of vectors as a tensor via the Segre outer product,

$$[u_1, \dots, u_l]^T \otimes [v_1, \dots, v_m]^T \otimes [w_1, \dots, w_n]^T = [u_i v_j w_k]_{i,j,k=1}^{l,m,n}.$$

Definition 1.2. The *Khatria-Rao Product* is the "matching columnwise" Segre outer product. Given matrices $A = [a_1, \dots, a_K] \in \mathbb{K}^{I \times K}$ and $B = [b_1, \dots, b_K] \in \mathbb{K}^{J \times K}$, their Khatria-Rao product is denoted by $A \odot B$. The result is a matrix of size $(IJ) \times K$ defined by

$$A \odot B = [a_1 \otimes b_1 \quad a_2 \otimes b_2 \quad \cdots \quad a_K \otimes b_K].$$

If a and b are vectors, then the Khatria-Rao and Segre outer products are identical, i.e., $a \otimes b = a \odot b$.

Given standard orthnormal bases $e_1^{(k)}, \ldots, e_{I_k}^{(k)}$ for $A_k \cong \mathbb{K}^{I_k}, k = 1, \ldots, N$, any tensor $\mathcal{X}$ in $A_1 \otimes \cdots \otimes A_N \cong \mathbb{K}^{I_1 \times \cdots \times I_N}$, can be expressed as a linear combination

$$\mathcal{X} = \sum_{i_1,\ldots,i_N=1}^{I_1,\ldots,I_N} t_{i_1\cdots i_N} e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_N}^{(N)}.$$

In older literature, the $t_{i_1\cdots i_N}$'s are often called the *components* of $\mathcal{X}$. $\mathcal{X}$ has rank one or rank-1 if there exist non-zero $a^{(i)} \in A_i$, $i = 1, \ldots, N$, so that $\mathcal{X} = a^{(1)} \otimes \cdots \otimes a^{(N)}$ and $a^{(1)} \otimes \cdots \otimes a^{(N)}$ is the Segre outer product.

The *rank* of $\mathcal{X}$ is defined to be the smallest r such that it may be written as a sum of r rank-1 tensors, i.e.,

$$rank(\mathcal{X}) = \min \left\{ r : \mathcal{X} = \sum_{p=1}^{r} a_p^{(1)} \otimes \cdots \otimes a_p^{(N)} \right\}.$$

Definition 1.3. The *n-th flattening map* on any tensor $\mathcal{X} = [t_{i_1\ldots i_N}]_{i_1,\ldots,i_N=1}^{I_1,\ldots,I_N} \in \mathbb{K}^{I_1 \times \cdots \times I_N}$ is the function (see Section 2 of [11])

$$\flat_n : \mathbb{K}^{I_1 \times \cdots \times I_N} \to \mathbb{K}^{I_n \times (I_1 \ldots \hat{I}_n \ldots I_N)}$$

defined by

$$(\flat_n(\mathcal{X}))_{ij} = (\mathcal{X})_{s_n(i,j)},$$

where $s_n(i,j)$ is the j-th element in lexicographic order in the subset of $\langle I_1 \rangle \times \cdots \times \langle I_N \rangle$ consisting of elements that have n-th coordinate equal to i, and by convention a caret over any entry of a N-tuple means that the respective entry is omitted.

For a tensor $\mathcal{X} = [t_{ijk}] \in \mathbb{K}^{l \times m \times n}$,

$$\begin{aligned} r_1 &= \dim span_{\mathbb{K}}\{\mathcal{X}_{1\bullet\bullet}, \ldots, \mathcal{X}_{l\bullet\bullet}\}, \\ r_2 &= \dim span_{\mathbb{K}}\{\mathcal{X}_{\bullet 1\bullet}, \ldots, \mathcal{X}_{\bullet m\bullet}\}, \\ r_3 &= \dim span_{\mathbb{K}}\{\mathcal{X}_{\bullet\bullet 1}, \ldots, \mathcal{X}_{\bullet\bullet n}\}. \end{aligned}$$

Here

$$\begin{aligned} \mathcal{X}_{i\bullet\bullet} &= [t_{ijk}]_{j,k=1}^{m,n} \in \mathbb{K}^{m \times n}, \\ \mathcal{X}_{\bullet j\bullet} &= [t_{ijk}]_{i,k=1}^{l,n} \in \mathbb{K}^{l \times n}, \\ \mathcal{X}_{\bullet\bullet k} &= [t_{ijk}]_{i,j=1}^{l,m} \in \mathbb{K}^{l \times m}. \end{aligned}$$

The multilinear rank of $\mathcal{X} \in \mathbb{K}^{l \times m \times n}$ is (r_1, r_2, r_3), with r_1, r_2, r_3 defined above.

Definition 1.4. (see Definition 11 in [4]) A *decomposition of a tensor* $\mathcal{X} \in \mathbb{K}^{I \times J \times K}$ in a sum of rank-$(1,\ L_r,\ L_r)$ terms, $1 \le r \le R$, is a decomposition of $\mathcal{X}$ of the form

$$\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r,$$

in which the $(J \times K)$ matrix X_r is rank-L_r, $1 \le r \le R$, and no two of X_r's are collinear.

It is clear that in $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$ one can arbitrarily permute the different rank-$(1,\ L_r,\ L_r)$ terms $a_r \otimes X_r$. Also, one can scale X_r, provided that a_r is counter scaled. We call this decomposition to be *essentially unique* when it is only subject to these trivial indeterminacies.

Definition 1.5. Let $\mu_{\mathbb{K}}$ be the Lebegue measure on $\mathbb{K}^{I\times R} \times \mathbb{K}^{J\times K\times R}$. Then $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$ in Definition 1.4 is *generically unique* if $\mu_{\mathbb{K}} = 0$, where $\mu_{\mathbb{K}}$ is defined by

$$\mu_{\mathbb{K}}\{\left(\mathbb{K}^{I\times R} \times \mathbb{K}^{J\times K\times R}\right) : \mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r \text{ is not unique for } a_r \in \mathbb{K}^I, X_r \in \mathbb{K}^{J\times K}\}.$$

Note that in Definition 1.4, we could require $\{a_i,\ 1 \le i \le R\}$ to be an orthogonal frame:

Definition 1.6. A decomposition of a tensor $\mathcal{X} \in \mathbb{K}^{I\times J\times K}$ in an *orthogonal frame* in a sum of rank-$(1,\ L_r,\ L_r)$ terms, $1 \le r \le R$, is a decomposition of $\mathcal{X}$ of the form

$$\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r.$$

As in Definition 1.4, X_r has rank-L_r, $1 \le r \le R$, but we need $\{a_i,\ 1 \le i \le R\}$ to be an orthogonal frame.

1.3. **Main results.** The main results of the paper are the following, and their proofs will be given in the following sections:

Theorem 1.7. *Assume* $I \ge R$, $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$ *in Definition 1.4 is essentially unique if and only if*

$$span_{\mathbb{K}}\{X_{j_1},\ \ldots,\ X_{j_s}\} \cap \Sigma_{\le L_{j_t}}(\mathbb{K}^{J\times K}) \subset \{X_{j_1},\ \ldots,\ X_{j_s}\},\ 1 \le t \le s,$$

where $\Sigma_{\le L}(\mathbb{K}^{J\times K}) = \{M \in \mathbb{K}^{J\times K} | \text{rank } M \le L\}$.

Remark 1. In reasonably small cases, one can use tools from numerical algebraic geometry such as those described in [18, 12, 13].

Remark 2. A generic $b \times b$ pencil is diagonalizable (as the conditions to have repeated eigenvalues or bounded rank are closed conditions) and thus of rank b. Thus for most (more precisely, a Zariski open subset of) pencils that are not diagonalizable, a perturbation by a general rank one matrix will make it diagonalizable. And there is a normal form for a general point p of $\Sigma_{\le L}(\mathbb{K}^{J\times K})$ (L is smaller than J and K), which is

$$p = b_1' \otimes c_1' + \cdots + b_L' \otimes c_L',$$

where $\{b_1', \ldots, b_L'\}, \{c_1', \ldots, c_L'\}$ are linear independent.

We now establish a simpler condition related to the uniqueness of $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$ in Definition 1.4. More precisely, there is a set of tensors $\mathcal{X}$ of measure 0 such that for any $\mathcal{X}$ outside this set, the conditions are sufficient to guarantee the uniqueness of $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$. Notice that these conditions are not truly sufficient, since it fails to provide the conclusion on a set of problems of measure 0. It, however, illustrates very well the situations in which uniqueness should hold.

Theorem 1.8. *$\mathcal{X} = a_1 \otimes X_1 + a_2 \otimes X_2$ in Definition 1.4 is generically unique if and only if*

$$I \geq 2,\ J = K \neq \text{one of } \left\{ \frac{2L_1 + L_2}{2}, \frac{2L_2 + L_1}{2}, L_1, L_2 \right\}.$$

Theorem 1.9. *$\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$ in Definition 1.4 is generically unique if*

$$I \geq R,\ K \geq \sum_{r=1}^{R} L_r,\ J \geq 2\max\{L_i\},\ \binom{J}{\max\{L_i\}} \geq R, L_i + L_j > L_k$$

for all $1 \leq i, j, k \leq R$.

For low multilinear rank decomposition in orthogonal frame, we have the following theorem.

Theorem 1.10. *A tensor decomposition of $\mathcal{X} \in \mathbb{K}^{I\times J\times K}$,*

$$\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r,$$

as in Definition 1.6 is essentially unique if and only if for any non-identity special orthogonal matrix $E = [\varepsilon_{ij}]_{1\leq i,j\leq R}$, there exists $k,\ 1 \leq k \leq R$ such that

$$\textit{rank}\ (\varepsilon_{k1}X_1 + \cdots + \varepsilon_{kR}X_R) \neq L_1, \ldots, L_R.$$

1.4. **Outline of the paper.** In this paper, we first provide some known and preliminary results related to the tensor decompositions of multilinear rank $(1,\ L_r,\ L_r)$ terms that we are considering. Then we establish simple geometric necessary and sufficient conditions which guarantee the uniqueness of tensor decompositions of multilinear rank $(1,\ L_r,\ L_r)$ terms (see Theorem 1.7). The conditions are then relaxed to obtain simpler sufficient conditions Theorem 1.8 and Theorem 1.9. Finally, we discuss the uniqueness of tensor decompositions of multilinear rank $(1,\ L_r,\ L_r)$ terms in an orthogonal frame that provides better structures.

2. Algebraic criteria of uniqueness.

Definition 2.1. For a vector space V, V^* denotes the *dual space* of linear functionals of V, which is the vector space whose elements are linear maps from V to $\mathbb{K}$: $\{\alpha : V \mapsto \mathbb{K} | \alpha \text{ is linear}\}$. If one is working in bases and represents elements of V by column vectors, then elements of V^* are naturally represented by row vectors and the map $V \mapsto \langle \alpha, v\rangle$ is just row-column matrix multiplication. Given a basis $v_1, \ldots, v_{\mathbf{v}}$ of V, it determines a basis $\alpha_1, \ldots, \alpha_{\mathbf{v}}$ of V^* by $\langle \alpha_j, v_i\rangle = \delta_{ij}$, called the *dual basis*. Now we define $V^{\perp} = \{\alpha \in V^* | \langle \alpha, v\rangle = 0,\ \forall v \in V\}$.

2.1. Proof of Theorem 1.7.

Proof. $\Leftarrow$ Assume the contrary that $\mathcal{X} = \sum_{r=1}^{R} a'_r \otimes X'_r$ is different from $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$. Since $a_1, \ldots, a_R$ are independent, we claim that

$$a'_r \in span_{\mathbb{K}}\{a_1, \ldots, a_R\}.$$

Since if not, we have

$$a_r^{'*} \in span_{\mathbb{K}}^{\perp}\{a_1, \ldots, a_R\}.$$

This implies

$$\langle \mathcal{X},\ a_r^{'*}\rangle = 0 = X'_r,$$

which is a contradiction. Therefore, we have $a'_r = \sum_{j=1}^{R} \alpha_j^r a_j$, where α_j^r are not all zero. From

$$\mathcal{X} = \sum_{r=1}^{R} a'_r \otimes X'_r = \sum_{r=1}^{R} \left(\sum_{j=1}^{R} \alpha_j^r a_r \otimes X'_j \right),$$

we know that $X_r = \sum_{j=1}^{R} \alpha_j^r X'_j$. Taking the inverse of the nonsingular $R \times R$ matrix $[\alpha_j^r]$, we have $X'_r = \sum_{j=1}^{R} \tilde{\alpha}_j^r X_j$. Consequently, there exist $r, j_1, j_2 \in \{1, \ldots, R\}$ such that $j_1 \neq j_2$ and $\tilde{\alpha}_{j_1}^r \cdot \tilde{\alpha}_{j_2}^r \neq 0$. Therefore, we obtain

$$X'_r \in span_{\mathbb{K}}\{X_{j_1}, \ldots, X_{j_s}\} \cap \Sigma_{\leq L_r}\left(\mathbb{K}^{J \times K}\right).$$

But X'_r does not belong to $\{X_{j_1}, \ldots, X_{j_s}\}$, which is a contradiction.

$\Rightarrow$ If there exists $X'_{j_t} \in span_{\mathbb{K}}\{X_{j_1}, \ldots, X_{j_s}\} \cap \Sigma_{\leq L_{j_t}}\left(\mathbb{K}^{K \times J}\right)$ such that $X'_{j_t} \notin \{X_{j_1}, \ldots, X_{j_s}\}$, Without loss of generality, we assume that $X'_{j_t} = X_1 + \chi_2 X_2 + \cdots + \chi_R X_R$. Now

$$\begin{aligned} & a_1 \otimes X_1 + \cdots + a_R \otimes X_R \\ &= a_1 \otimes X'_{j_t} - \chi_2 a_1 \otimes X_2 - \cdots - \chi_R a_1 \otimes X_R + a_2 \otimes X_2 + \cdots + a_R \otimes X_R \\ &= a_1 \otimes X'_{j_t} + (a_2 - \chi_2 a_1) \otimes X_2 + \cdots + (a_R - \chi_R a_1) \otimes X_R \\ &= a_1 \otimes X'_{j_t} + a'_2 \otimes X_2 + \cdots + a'_R \otimes X_R. \end{aligned}$$

So $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$ is not unique. □

Example 1. A tensor decomposition of $\mathcal{X} \in \mathbb{K}^{I \times J \times K}$,

$$\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r,$$

as in Definition 1.4 is essentially unique if the singular vectors of $X_1, \ldots, X_R$ are linear independent.

Proof. Assume the contrary that $\mathcal{X} = \sum_{r=1}^{R} a'_r \otimes X'_r$ is different from $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$, then we have

$$X'_r = \chi_1 X_1 + \cdots + \chi_R X_R.$$

Let

$$U_r = \begin{pmatrix} | & | & & | \\ u_1^r & u_2^r & \cdots & u_J^r \\ | & | & & | \end{pmatrix}$$

$$V_r = \begin{pmatrix} | & | & & | \\ v_1^r & v_2^r & \cdots & v_K^r \\ | & | & & | \end{pmatrix}$$

and $u_j^r,\ 1 \leq r \leq R,\ 1 \leq j \leq J,\ v_k^r,\ 1 \leq r \leq R,\ 1 \leq k \leq K$, are linear independent, and let

$$X_r = \sigma_1^r u_1^r \otimes v_1^r + \cdots + \sigma_{L_r}^r u_{L_r}^r \otimes v_{L_r}^r,$$

then we can see the rank of $\chi_1 X_1 + \cdots + \chi_R X_R$ should be bigger or equal to $L_i, 1 \leq i \leq R$ and equality holds only if X'_r is one of $\{X_1, \ldots, X_R\}$. And the uniqueness follows from Theorem 1.7. □

2.2. Proof of Theorem 1.8.

Proof. It is sufficient to prove the case $\min\{L_1, L_2\} \le J = K < L_1 + L_2$. Let B and C denote vector spaces of dimensions J, K respectively. Split $B = B_1 \oplus B_0 \oplus B_2$ and $C = C_1 \oplus C_0 \oplus C_2$, where B_1, B_0, B_2, C_1, C_0, and C_2 are of dimensions $L_1 - l_b$, l_b, $L_2 - l_b$, $L_1 - l_c$, l_c, $L_2 - l_c$.

Consider

$$\begin{aligned} X_1 &= b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1-l_b} \otimes c_{1,L_1-l_b} + b_{0,1} \otimes c_{1,L_1-l_b+1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\ &\quad \in (B_1 \oplus B_0) \otimes (C_1 \oplus C_0) \cong \mathbb{K}^{L_1} \otimes \mathbb{K}^{L_1}, \\ X_2 &= b_{2,1} \otimes c_{2,1} + \cdots + b_{2,L_1-l_b} \otimes c_{2,L_1-l_b} + b_{0,1} \otimes c_{2,L_1-l_b+1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\ &\quad \in (B_2 \oplus B_0) \otimes (C_2 \oplus C_0) \cong \mathbb{K}^{L_2} \otimes \mathbb{K}^{L_2}, \end{aligned}$$

where

$$\begin{gathered} \{b_{0,1}, \ldots, b_{0,l_b}\}, \\ \{b_{1,1}, \ldots, b_{1,L_1-l_b}\}, \\ \{b_{2,1}, \ldots, b_{2,L_2-l_b}\}, \\ \{c_{0,1}, \ldots, c_{0,l_c}\}, \\ \{c_{1,1}, \ldots, c_{1,L_1-l_c}\}, \\ \{c_{2,1}, \ldots, c_{2,L_2-l_c}\}, \end{gathered}$$

are bases for B_0, B_1, B_2, C_0, C_1 and C_2, respectively, $J + l_b = L_1 + L_2$, and $K + l_c = L_1 + L_2$.

Suppose $\chi_1,\ \chi_2$ are both nonzero, the matrix pencil $\chi_1 X_1 + \chi_2 X_2$

$$\begin{pmatrix} \chi_1 &&&&&&&& \\ & \ddots &&&&&&& \\ && \chi_1 &&&&&& \\ &&& \chi_1+\chi_2 &&&&& \\ &&&& \ddots &&&& \\ &&&&& \chi_1+\chi_2 &&& \\ &&&&&& \chi_2 && \\ &&&&&&& \ddots & \\ &&&&&&&& \chi_2 \end{pmatrix}$$

has rank J when $\chi_1 \neq -\chi_2$, and $L_1 + L_2 - 2l_b$ when $\chi_1 = -\chi_2$. By simple computation, we know

$$\text{rank } (\chi_1 X_1 + \chi_2 X_2) \neq L_1 \textit{ or } L_2$$

if and only if

$$J, K \neq \text{one of } \left\{ \frac{2L_1 + L_2}{2}, \frac{2L_2 + L_1}{2}, L_1, L_2 \right\}.$$

Then Theorem 1.8 follows from Theorem 1.7. □

Example 2. For $\mathcal{X} \in \mathbb{K}^{2\times3\times3}$, considering the decomposition in a sum of multilinear rank $(1, 2, 2)$, we have

$$\begin{aligned} \mathcal{X} &= a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_2 \otimes c_2 + b_3 \otimes c_3) \\ &= a_1 \otimes (b_1 \otimes c_1 - b_3 \otimes c_3) + (a_1 + a_2) \otimes (b_2 \otimes c_2 + b_3 \otimes c_3), \end{aligned}$$

where $\{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}, \{a_1, a_2\}$ are bases for $\mathbb{K}^3, \mathbb{K}^3, \mathbb{K}^2$. So this is not unique.

Example 3. For $\mathcal{X} \in \mathbb{K}^{2\times 4\times 2}$, considering the decomposition in a sum of multilinear rank $(1, 2, 2)$, we have

$$\begin{aligned}\mathcal{X} &= a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_3 \otimes c_1 + b_4 \otimes c_2)\\ &= a_1 \otimes ((b_1 + b_3) \otimes c_1 + (b_2 + b_4) \otimes c_2) + (a_2 - a_1) \otimes (b_3 \otimes c_1 + b_4 \otimes c_2),\end{aligned}$$

where $\{b_1, b_2, b_3, b_4\}, \{c_1, c_2\}, \{a_1, a_2\}$ are basis for $\mathbb{K}^4, \mathbb{K}^2, \mathbb{K}^2$. So this is not unique.

Example 4. There are explicit Weierstrass canonical forms (see Chapter 10 in [16]) of tensors in $\mathbb{K}^{2\times L\times L}$. Each of those can be decomposed in a sum of rank-$(1, L, L)$ terms as follows:

$$a_1 \otimes (b_1 \otimes c_1 + \cdots + b_L \otimes c_L) + a_2 \otimes (\lambda_1 b_1 \otimes c_1 + \cdots + \lambda_L b_L \otimes c_L),$$

but it is obviously not unique.

2.3. **Proof of Theorem 1.9.**

Proof. It is sufficient to prove the case $I = R$, $K = \sum_{r=1}^{R} L_r$. Let B and C denote vector spaces of dimensions J, K respectively. Choose the splitting of C as $C = \bigoplus_{1\le r\le R} C_r$, and fix a basis $\{b_1, \dots, b_J\}$ for B.

Without loss of generality, for $1 \le p \le R$, we can assume

$$E_{j_p} = b_{j_p,1} \otimes c_{j_p,1} + b_{j_p,2} \otimes c_{j_p,2} + \cdots + b_{j_p,L_{j_p}} \otimes c_{j_p,L_{j_p}} \in B_{j_p} \otimes C_{j_p},$$

where $\{b_{j_p,1}, \dots, b_{j_p,L_{j_p}}\} \subset \{b_1, \dots, b_J\}$ (since $J \ge 2\max\{L_i\}, \binom{J}{\max\{L_i\}} \ge R$), $\{c_{j_p,1}, \dots, c_{j_p,L_{j_p}}\}$ are bases for B_{j_p}, C_{j_p}, respectively. Further, let

$$E'_{j_t} = b'_1 \otimes c'_1 + \cdots + b'_{L_{j_t}} \otimes c'_{L_{j_t}}$$

be a general point of $\Sigma_{\le L_{j_t}}(\mathbb{K}^{J\times K})$ and set

$$E'_{j_t} = \sum_{1\le p\le s} \chi_p E_{j_p} = \sum_{1\le p\le s} \chi_p \left(b_{j_p,1} \otimes c_{j_p,1} + b_{j_p,2} \otimes c_{j_p,2} + \cdots + b_{j_p,L_{j_p}} \otimes c_{j_p,L_{j_p}}\right).$$

If there exist χ_μ, χ_ν, which are both nonzero, the pencil

$$\begin{pmatrix} x_\mu & & & & & & & & \\ & \ddots & & & & & & & \\ & & x_\mu & & & & & & \\ & & & x_\mu + x_\nu & & & & & \\ & & & & \ddots & & & & \\ & & & & & x_\mu + x_\nu & & & \\ & & & & & & x_\nu & & \\ & & & & & & & \ddots & \\ & & & & & & & & x_\nu \end{pmatrix}$$

has rank at least $L_{j_\mu} + L_{j_\nu}$, which is bigger than L_{j_t}. This implies that E'_{j_t} is not a matrix in $\Sigma_{\le L_{j_t}}(\mathbb{K}^{J\times K})$. Therefore, we prove that $E'_{j_t} \in \{E_{j_1}, \dots, E_{j_s}\}$. The uniqueness follows from Theorem 1.7. □

The following Remark can be easily obtained using elementary combinatorics.

Remark 3. When

$$I \geq R,\ J,\ K \geq \sum_{r=1}^{R} L_r,\ L_i + L_j > L_k \quad \forall 1 \leq i, j, k \leq R,$$

a low multilinear rank tensor decomposition of $\mathcal{X}$ as in Definition 1.4 has a unique expression

$$\mathcal{X} = \sum_{r=1}^{R} a_r \otimes \left[\left(\sum_{r'=1+\sum_{u=1}^{r-1} L_u}^{\sum_{u=1}^{r} L_u} b_{r'} \otimes c_{r'} \right) \right].$$

3. Proof of Theorem 1.10.

Proof. $\Rightarrow$ Assume the contrary that $\mathcal{X} = \sum_{r=1}^{R} a'_r \otimes X'_r$ is different from $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$. Let us assume the transformation matrix between the frames $\{e'_r,\ 1 \leq r \leq R\}$ and $\{e_r,\ 1 \leq r \leq R\}$ is Q, which is a $R \times R$ special orthogonal matrix $[\varepsilon_{ir}]$. Then we have

$$[X_1 \quad \cdots \quad X_R] \odot \begin{bmatrix} e_1 \\ \vdots \\ e_R \end{bmatrix} = [X'_1 \quad \cdots \quad X'_R] \odot \begin{bmatrix} e'_1 \\ \vdots \\ e'_R \end{bmatrix} = [X'_1 \quad \cdots \quad X'_R] \odot Q \begin{bmatrix} e_1 \\ \vdots \\ e_R \end{bmatrix}.$$

Since $\{e_r,\ 1 \leq r \leq R\}$ is orthogonal, taking inner product of $\mathcal{X}$ with e_r, we have

$$X'_r = \varepsilon_{r1} X_1 + \cdots + \varepsilon_{rR} X_R,\ 1 \leq r \leq R.$$

However

$$\text{rank } X'_r = \text{rank } (\varepsilon_{r1} X_1 + \cdots + \varepsilon_{rR} X_R) \neq L_1, \ldots, L_R,$$

which is a contradiction. Therefore $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$ as in Definition 1.6 is essentially unique.

$\Leftarrow$ Assume for a special orthogonal matrix $Q = [\varepsilon_{ir}]_{R \times R}$, $\varepsilon_{i1} X_1 + \cdots + \varepsilon_{iR} X_R$ has rank L_i for any $1 \leq i \leq R$. Let

$$X'_i = \varepsilon_{i1} X_1 + \cdots + \varepsilon_{iR} X_R,\ 1 \leq i \leq R,$$

we then have $\mathcal{X} = \sum_{r=1}^{R} a_r \otimes X_r$. So it is not unique. □

Remark 4. Since the rotation matrix in the plane is

$$\begin{pmatrix} \cos\ \theta & -\sin\ \theta \\ \sin\ \theta & +\cos\ \theta \end{pmatrix},$$

a tensor decomposition of $\mathcal{X} \in \mathbb{K}^{I \times J \times K}$ in orthogonal frame, $\mathcal{X} = a_1 \otimes X_1 + a_2 \otimes X_2$ as in Definition 1.6 is essentially unique if and only if for any $\theta,\ 0 < \theta < \pi$,

$$rank\ (\cos\theta\ X_1 + \sin\theta\ X_2) \neq L_1 \text{ or } L_2,$$

and same for $rank\ (-\sin\theta\ X_1 + \cos\theta\ X_2)$.

4. Conclusion.

Different from most current approach in the analysis of big data sets, in this paper, some uniqueness characteristics of low multilinear rank tensor decomposition **LRD** are given under the framework of algebraic geometry. The proposed framework leads to a new approach for the study of identifiability properties in terms of block tensor decomposition that can be used to handle the big data sets. Several explicit uniqueness criteria for tensor decomposition of low multilinear rank terms are given.

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