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Theory article

Beam bending and Λ-fractional analysis

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Abstract: Since the global stability criteria for Λ-fractional mechanics have been established, the Λ-fractional beam bending problem is discussed within that context. The co-existence of the phase phenomenon is revealed, allowing for elastic curves with non-smooth curvatures. The variational bending problem in the Λ-fractional space is considered. Global minimization of the total energy function of beam bending is necessarily applied. The variational Euler-Lagrange equation yields an equilibrium equation of the elastic curve, with the simultaneous possible corners being expressed by Weierstrass-Erdmann corner conditions.

Keywords: Λ-fractional derivative; Λ-fractional space; initial space; local stability; global stability; coexistence of phases; elastic curve; beam bending

1. Introduction

Lately, various models in mechanics have been discussed within the context of fractional calculus, as described by Bagley et al. as either viscoelastic models or simulating experiments [1,2]. Fractional deformations of viscoelastic materials have been studied by Atanackovic [3]. Moreover, Mainardi [4] has presented the wave propagation in viscoelastic materials. Realizing the need for a global analysis in space, Lazopoulos [5] presented the deformation of a non-homogeneous bar with possible voids. It was proved in that analysis, that Noll's axiom, Truesdell [6], in mechanics of local action is no longer valid. Eringen [7] introduced the postulate of non-local action for micro and nanomaterials. In those structures, the stress at a point depends upon the strain of the points in a region located around that point.

Fractional calculus, including non-local derivatives, has recently been applied to modern engineering advances, mechanics, physics, biology, bioengineering, etc. Leibniz [8] suggested the use of fractional calculus, which was pursued by many other famous mathematicians such as Liouville [9] and Riemann [10]. Information regarding fractional calculus and the corresponding fractional differential derivatives may be found in various texts such as Samko et al. [11], Poldubny [12], Oldham et al. [13] and Miller et al. [14]. It is pointed out that fractional derivatives are not mathematical derivatives, satisfying the differential topology prerequisites Chillingworth [15]. Those prerequisites are as follows:

1. Linearity: $D\big(af(x)+bg(x)\big)=aDf(x)+bDg(x)$.

2. Leibniz rule: $D\big(f(x)g(x)\big)=Df(x)g(x)+f(x)Dg(x)$.

3. Chain rule: $D\big(g(f)\big)(x)=Dg(f(x)Df(x)$.

Hence, fractional calculus may not generate the fractional differential geometry necessary to study problems in physics, biology, etc. Nevertheless, its application was considered crucial in various areas demanding a non-local analysis. Λ-fractional analysis was introduced by Lazopoulos [16] to fill the gap of mathematical accuracy concerning fractional calculus. Lazopoulos [17–20] applied Λ-fractional analysis to mechanics, differential equations, geometry, physics, etc, in the context of a non-local analysis. Moreover, many scientists have addressed the complex body structure problem, such as Oskouie et al. [21], Liu et al. [22], Lazopoulos et al. [17,19], Zorica et al. [23], Sumelka et al. [24], Sidhardh et al. [25], Stempin et al. [26], Beda et al. [27] and Mohammadi et al. [28]. Although a Λ-fractional analysis responds to the need for global considerations of a problem, the necessity for global variational procedures has been overlooked. Indeed, the various variational procedures should be valid under the additional Weierstrass-Erdmann corner conditions. That variational procedure changes the stability character from a local to a global one. Lazopoulos [29] pointed out that only globally stable analyses are allowed in fractional calculus problems in physics, mechanics, engineering, biology, economics, etc. The co-existence of the phase phenomena, as introduced by Ericksen in [30], may arise in fractional analysis problems. Lazopoulos [30] previously presented the co-existence of phases in the extension of fractional bars. The Weierstrass-Erdmann corner conditions Gelfand and Fomin [31], should necessarily be considered.

Under the action of a transversal force, the Λ-fractional deformation of a cantilever beam acting upon the free end of the beam is studied. Working into the frame of the Λ-fractional analysis, the non-continuous curvatures of the elastic curve are allowed into the Λ-fractional space. The elastic curve is defined within the Λ-fractional space and is then transferred into the initial space. Let us point out that fractional derivatives do not exist in the initial space and strains are not allowed to be transferred.

Since the Λ-fractional analysis is quite new, it is outlined and applied upon the deformation of a cantilever beam under a transversely applied load upon the free end of the beam. The two spaces (i.e., the initial and the corresponding Λ-fractional space) are defined. The deformation of the beam is studied in the Λ-fractional space by adopting the conventional beam theory. Nevertheless, only globally stable deflections of the beam are allowed. It means that non-continuous curvatures of the elastic line are allowed. The deformed elastic curve of the beam in the initial space is defined, transferring the elastic curve from the Λ-fractional space into the initial space.

2. Basics of fractional calculus

By first establishing the differential calculus with the introduction of local derivatives, Leibnitz [8] suggested the idea of fractional derivatives. He had foreseen the importance of the fractional derivative, which was later recognized as an indispensable tool for a recently important branch of applied mathematics (i.e., fractional calculus); its main characteristic is global analysis. Fractional calculus has been employed to improve the description of various globally dependent phenomena in various branches of applied sciences, such as physics, mechanics, biology, economy, etc. Information concerning fractional calculus may be found in many books [10–13]. In fact, fractional calculus is a global mathematical analysis.

The fractional integrals are defined by the following for a fractional dimension $0 < \gamma$:

$$ {}_{a}I_{x}^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_{a}^{x \int} \frac{f(s)}{(x-s)^{1-\gamma}} ds \tag{1} $$

$$ {}_{x}I_{b}^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_{x}^{b} \frac{f(s)}{(s-x)^{1-\gamma}} ds \tag{2} $$

The left and right fractional integrals are defined by Eqs 1 and 2, respectively. Furthermore, the order of fractional integrals is γ and the Euler's Gamma function is $\Gamma(\gamma)$. Moreover, the Riemann-Liouville left fractional derivative (R-L) is defined by the following:

$$ {}_{a}^{RL}D_{x}^{\gamma} f(x) = \frac{d}{dx}\left({}_{a}I_{x}^{1-\gamma}\big(f(x)\big)\right) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_{a}^{x} \frac{f(s)}{(x-s)^{\gamma}} ds \tag{3} $$

whereas the right Riemann-Liouville's fractional derivative (R-L) is defined by the following:

$$ {}_{x}^{RL}D_{b}^{\gamma} f(x) = \frac{d}{dx}\left({}_{x}I_{b}^{1-\gamma}(f(x))\right) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_{x}^{b} \frac{f(s)}{(s-x)^{\gamma}} ds \tag{4} $$

Fractional derivatives and integrals are related by the following expression:

$$ {}_{a}^{RL}D_{x}^{\gamma}\big({}_{a}I_{x}^{\gamma} f(x)\big) = f(x) \tag{5} $$

Lazopoulos [16] proposed the Λ-Fractional analysis to bridge the inability of fractional derivatives to generate differentials. That analysis fills up the gap existing in fractional calculus, since fractional derivatives are not sufficient to generate differentials. Consequently, basic mathematical tools such as calculus of variations, differential geometry, field theorems, existence and uniqueness theorems of differential equations, etc, may not work out in fractional calculus based upon the well-known fractional derivatives. However, those are indispensable mathematical tools in science and scientific applications. Nevertheless, fractional variational procedures, fractional differentials, fractional field theories, fractional differential geometries, etc, have already been proposed; however, their accuracy is questionable. The Λ-fractional analysis proposes a dual space the Λ-fractional space, where everything behaves conventionally, similar to what occurs in a local differential analysis. Since the derivatives in the proposed Λ-space are local, differentials may be generated and mathematically established. Hence, fractional differential geometry is correctly generated, with many mathematical tools such as field theorems and variational procedures.

The Λ-fractional derivative (Λ-FD) is defined by the following:

$$ {}_{a}^{\Lambda}D_x^{\gamma} f(x) = \frac{{}_{a}^{RL}D_x^{\gamma} f(x)}{{}_{a}^{RL}D_x^{\gamma} x} \tag{6} $$

Recalling Eq 3, the Λ-FD is defined by the following:

$$ {}_{a}^{\Lambda}D_x^{\gamma} f(x) = \frac{\frac{d\, {}_{a}I_x{}^{1-\gamma} f(x)}{dx}}{\frac{d\, {}_{a}I_x{}^{1-\gamma} x}{dx}} = \frac{d_a I_x{}^{1-\gamma} f(x)}{d_a I_x{}^{1-\gamma} x} \tag{7} $$

One can define the dual, Gao [22], Λ-fractional space by *(X, F(X))* with the following:

$$ X = {}_{a} I_x{}^{1-\gamma} x, \quad F(X) = {}_{a} I_x{}^{1-\gamma} f(x(X)) \tag{8} $$

where Λ-FD is a local derivative in the dual, Λ-fractional space *(X, F(X))*. Hence, differential geometry exists in that dual space. Furthermore, various other important mathematical procedures followed in applications, such as variational analyses and field theorems, existence and uniqueness theorems of differential equations, are equipped with the demanded mathematical accuracy. Moreover, the dual to the dual rule, Gao [32], yields the initial space and transfers the results into the initial space, following the given relation:

$$ f(x) = {}_{a}^{RL}D_x^{1-\gamma} \; F\big(X(x)\big) = {}_{a}^{RL}D_x^{1-\gamma} \; {}_{a}I_x^{1-\gamma} f(x) \tag{9} $$

3. The globally stable Λ-fractional bending of a cantilever beam

As seen in Figure 1, a cantilever beam in the initial space is defined in its undeformed placement (placement at ease, Truesdell [6]) by the following:

$$ x \, \varepsilon \, L, \, 0 < L < \infty \tag{10} $$

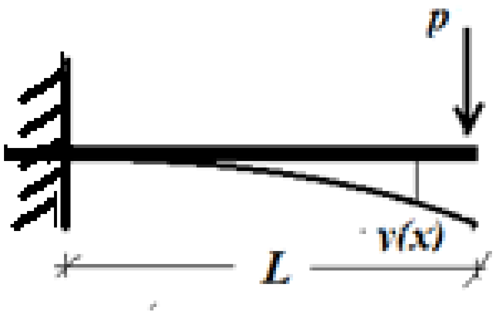

Figure 1. The cantilever beam in the initial space.

The deformed (current) placement in the initial space after the application of the transverse force p at the end $x = L$ of the bar is defined by *y(x)*.

Transferring the problem into the (dual) Λ-fractional space, the corresponding parameters of the problem *(X, U(X), Y(X), P)* are defined through the following:

$$X = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{s}{(x-s)^\gamma} ds = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \tag{11}$$

and,

$$x = (X\Gamma(3-\gamma))^{1/(2-\gamma)} \tag{12}$$

Furthermore,

$$Y(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{y(s)}{(x-s)^\gamma} ds \tag{13}$$

Moreover, by introducing the value of *x* into Eq 13, the elastic curve *Y(X)* is defined. Furthermore, the tangent of the elastic curve in the Λ-fractional space is defined by the following:

$$\tan(\Theta(X)) = \frac{dY(X)}{dX}$$

The total energy of the bar in the Λ-space is expressed by the following:

$$V = \int_0^L (W(\Theta'(X)) - PSin(\Theta(X)))dX = \int_0^L F(X)dX \tag{14a}$$

where $W\left(\Theta'(X)\right)$ is the strain energy function. In case of the general bending problem with various applied perpendicular loads, P(X) is defined as the following:

$$F(X) = W(\Theta'(X)) - P(X) \int_X^L Sin(\Theta(Z))dZ \tag{14b}$$

However, for a better understanding of the present procedure, the simplified Eq 14a will be adopted, where $W\left(\Theta'(X)\right)$ is the strain energy function. The equilibrium equation is defined by the minimum of the total energy V, as expressed by the following:

$$\frac{d}{dX} \frac{\partial W(\Theta'(X))}{\partial \Theta'(X)} - P\, Cos(\Theta(X)) = 0 \tag{15}$$

with the boundary conditions,

$$\Theta(0) = \Theta'(L^\Lambda) = 0 \tag{16}$$

That equilibrium problem should be solved to complete the solution in conventional equilibrium problems. That is valid because the local criterion of stability is adopted. However, when the global stability criteria are considered, the coexistence of the phase phenomenon appears, due to the non-local criterion of stability.

The globally stable equilibrium states have been introduced by Ericksen [30], describing the co-existence of states with low and high-strain states. Those states are similar to the co-existing phases of melting ice (i.e., the coexistence of ice and water). James [33] also describes the co-existence of the phase phenomenon in solids following Ericken's model. In the latter equilibrium placements, jumping of the strains is accepted. However, global criteria should only be considered in the present Λ-fractional analysis, since everything concerning fractional calculus is global.

It should be pointed out that only globally stable variational procedures are allowed in fractional calculus problems. Additionally, the Weierstrass-Erdman corner conditions [24] should be satisfied with the following:

$$F_{\Theta'}[X = c - 0] - F_{\Theta'}[X = c + 0] = 0 \quad (17)$$

$$(F - \Theta' F_{\Theta'})[X = c - 0] - (F - S' F_{\Theta'})[X = c + 0] = 0 \quad (18)$$

Those conditions are necessary to be fulfilled for any fractional equilibrium problem. In order to understand the difference between the locally and globally stable criteria, the diagram of the bending moment versus the curvature of the elastic curve is defined by the rotation *Θ(X)* of the cross-section of the beam in the Λ-space, as shown in Figure 2.

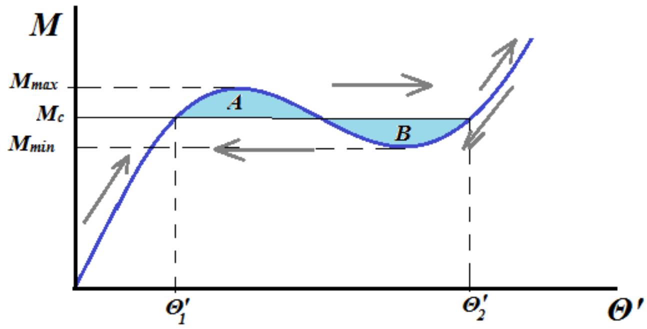

Figure 2. The bending moment-curvatures of the elastic curve diagram.

As shown in Figure 2, the bending moment M versus the curvature $\Theta'(X)$ diagram clarifies the co-existence of the phase phenomenon of the beam bending (i.e., the existence of the non-smooth curvatures satisfying the Erdmann-Weierstrass conditions, Eqs 17 and 18).

Considering the non-convex diagram of the bending moment M versus the curvature Θ' of Figure 2, the bending moment M follows the left branch from zero values. Adopting the local criterion of stability, the bending moment increases up to the local M_{max}. If the bending moment increases, the right branch also increases. Furthermore, the right branch of the diagram not only increases, but also reduces the bending moments down to the local minimum M_{min} of the diagram. Regarding the Erdmann-Weierstrass corner conditions, Eqs 18 and 19 are not valid to consider the local criterion of stability.

Nevertheless, adopting the global criterion of stability, a different trace of the mending moment curvature will be followed. Starting out from the small values of mending moments, the left part of the diagram is followed, as in the preceding case of the local criterion of stability. However, when the bending moment reaches the value M_{cr} yielding the areas $A = B$ of the diagram in Figure 2, the Weierstrass-Erdman corner conditions, Eqs 17 and 18, are valid for the coexistence of two curvatures, Θ'_1, Θ'_2. Those curvatures yield the corners in the elastic curve of the beam and are two parts of the bending beam with small and high values of the curvature. Decreasing the high curvature values will be increased down to the local minimum and then will jump to the left branch of the non-convex bending moment versus curvature diagram.

Let us consider a beam of length $l = 1$ and cross-section area α in the initial space. The left end of the bar is fixed and an axial force equal to p is transversely applied at the other. Therefore, the beam with $0 < x < 1$ in the initial space is transferred in the dual Λ-fractional space with a fractional order γ = 0.6. Hence, the bar in the Λ-space is defined by the following:

$$X = \frac{1}{\Gamma(0.4)} \int_0^x \frac{s}{(x-s)^{0.6}} ds = \frac{x^{1.4}}{\Gamma(2.4)} = 0.805x^{1.4} \tag{19}$$

Therefore, the length of the rod in the dual Λ-space is $L = 0.805$.

The force p applied at the end of the bar becomes P in the Λ-space, as seen in the following:

$$P = \frac{pl^{1-\gamma}}{\Gamma(2-\gamma)} = 1.1270p \tag{20}$$

Let us consider the bending moment M-curvature $\Theta'(S)$ diagram in the dual (Λ-space). The critical bending moment M_c with the corresponding curvatures of the elastic curve Θ_1', Θ_2', satisfying the Weierstrass-Erdmann jumping conditions, Eqs 17 and 18, are shown in Figure 3. Indeed, the areas F_1 and F_2 are equal; that is, the geometrical configuration of the Weierstrass-Erdman Eqs 17 and 18.

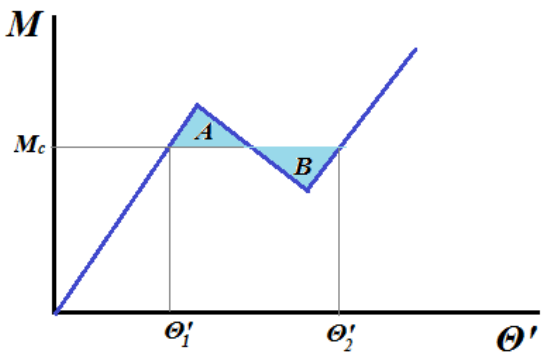

Figure 3. The non-convex stress-strain diagram in the Λ-space.

In the Λ-fractional space, the non-convex bending moment-rotation diagram is defined with the coexisting moment $M_c = 1.5$ and the corresponding coexisting curvatures $\Theta_1' = 0.03$ and $\Theta_2' = 0.12$. Furthermore, the convex branches of the diagram are parallel. Indeed,

$$M(X) = EI\Theta_1'(X) \text{ for } 0 < \Theta'(X) < \Theta_1' \tag{21}$$

$$M(X) = M_c + EI\,(\Theta'(X) - \Theta_2') \text{ for } \Theta_2' < \Theta'(X) \tag{22}$$

The fractional moments are quite low near the neighborhood of $X = L^{\Lambda} = 0.805$. Hence, starting from point $X = 0$ with high moments and curvatures, the branch BC of Figure 5 is followed. When the bending moments reach the coexisting value M_c with curvature Θ_2, two curvatures are displayed: curvature Θ_1' and curvature Θ_2'. In that case, the areas $A = B$ in the diagram of Figure 3 are equal. Reducing the stress in the left part of the diagram with line 0A is followed.

Hence, the rotation $\Theta(X)$ is defined in the branch BC of Figure 3 with the curvatures $\Theta'(X) > \Theta_2'$ by the following equation:

$$\frac{d\theta(X)}{dX} = \frac{Mc}{EI} + (\Theta'(X) - \Theta_2'), \Theta'(X) > \Theta_2' \tag{23}$$

with EI denoting the beam stiffness in the dual Λ-space. Then,

$$\Theta(X) = \int_0^X \left(\frac{Mc}{EI} + (\Theta'(X) - \Theta_2')\right) dX,\ 0<X<X_c \tag{24}$$

Furthermore, the distribution of the rotation of the cross-section is valid for the part $0 < X < X_c$ of the beam, with X_c denoting the cross-section of the beam, where jumping of the curvature takes place and the two curvatures Θ_1' and Θ_2' coexist.

However, Eq 21 is valid for the part of the beam $X_c < X < L$ with low values of the bending moment. For that region,

$$M(X)= P(L\text{-}X),\ X_c<X<L \tag{25}$$

Hence,

$$\Theta(X) = \int_{Xc} \frac{P(L-X)}{EI} dX + \Theta_c,\ X_c<X<L \tag{26}$$

Furthermore, the deflection $Y(X)$ of the beam satisfies the following equation:

$$\frac{dY(X)}{dX} = \text{Sin}\ \Theta(X) \cong \Theta(X) \tag{27}$$

Therefore, the deflection of the beam Y(X) with 0 < X < L is defined by the following:

$$Y(X) = \int \Theta(X) dX \tag{28}$$

The analysis above concerns the Λ-fractional space. The results should be transferred into the initial space. The beam elastic curve may only be transferred into the initial space. Let us point out that derivatives may not be transferred. Recalling Eq 12,

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)}$$

Therefore,

$$Y(x) = Y\left(\frac{x^{2-\gamma}}{\Gamma(3-\gamma)}\right) \tag{29}$$

By transferring the elastic curve into the initial space, its equation is defined by the following:

$$y(x) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_0^x \frac{Y(s)}{(x-s)^\gamma} ds \tag{30}$$

4. Application

As previously mentioned, the load *p*, vertically applied upon the free end of the beam, yields the corresponding load P in the Λ-space with the following equation:

$$P = \frac{pl^{1-\gamma}}{\Gamma(2-\gamma)} \tag{31}$$

The Λ-length of the beam L is defined by the following:

$$L = \frac{l^{1-\gamma}}{\Gamma(2-\gamma)} \tag{32}$$

For beams with l = 1 in the initial space and the fractional order γ = 0.6, the length of the beam in the Λ-space is defined by the following:

$$L = 0.805 \tag{33}$$

Furthermore, the bending moment diagram in the Λ- space is defined by the following equation:

$$M_X = P(0.805 - X) \tag{34}$$

Assuming the transversal load P = 4 in the Λ-space, the bending moment diagram is shown in Figure 4.

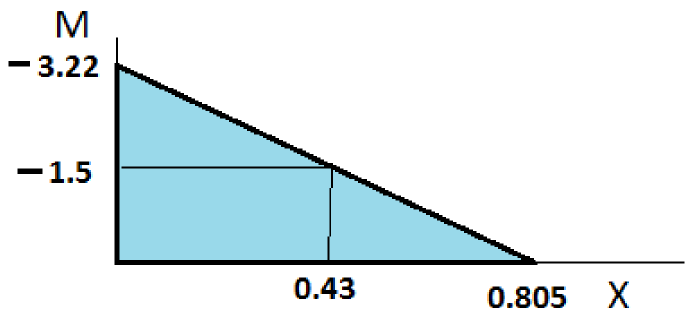

Figure 4. The beam bending moment diagram in the Λ-space.

That beam bending diagram of the bending moment M versus the curvature $\Theta'(X)$ of the elastic curve of the beam in the Λ-space is shown in Figure 5.

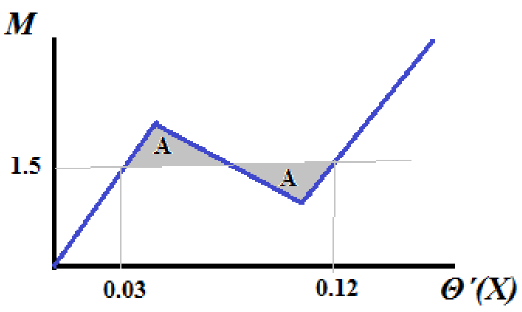

Figure 5. The non-linear bending moment-curvature diagram.

Furthermore, the diagram of the curvature Θ'(X) in the Λ-space is shown in Figure 6.

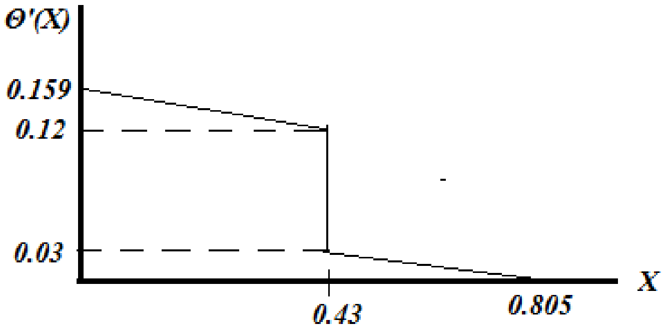

Figure 6. The diagram of the curvature Θ'(X) in the Λ-space.

Since the rotation Θ(X) of the cross-section is zero at X = 0,

$$\Theta'(X) = 0.069(0.805 - X) + 0.09, 0<X<0.43 \quad (35)$$

Hence, by integrating Eq 35, the rotation of the beam cross-section is defined by the following:

$$\Theta(X) = 0.1455X - 0.0345X^2, 0<X<0.43 \quad (36)$$

Consequently, the rotation of the cross-section at the corner cross-section is as follows:

$$\Theta(0.43) = 0.056 \quad (37)$$

Considering Eqs 34 and 37, the rotation of the cross-section of the beam is defined by the following:

$$\Theta(X) = 0.038 + 0.056X + 0.035X^2, 0.43<X<0.805 \quad (38)$$

In addition, Eqs 27 and 28 yield the following equations:

$$Y(X) = \int_0^X \Theta(X)dX = 0.073X^2 - 0.0115X^3,\ 0 < X < 0.43 \quad (39)$$

$$Y(X)=Y(0.43)+\int_{0.43}^{X} \Theta(X)dX=-0.008+0.039X+0.278X^2+0.012X^3, \text{ for } 0.43<X<0.805 \quad (40)$$

Figure 7 shows the beam deflection elastic curve Y(X) in the Λ-space.

Transferring that curve into the initial space,

$$X = \frac{1}{\Gamma(\gamma)}\int_0^x \frac{x}{(x-s)^\gamma}ds = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \quad (41)$$

where X and x are the variables in the fractional and initial spaces, respectively, and γ is the fractional order. Then, the deflection curve y(x) in the initial space is defined by the following:

$$y(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_0^x \frac{Y\left(\frac{s^{2-\gamma}}{\Gamma(3-\gamma)}\right)}{(x-s)^\gamma}ds \quad (42)$$

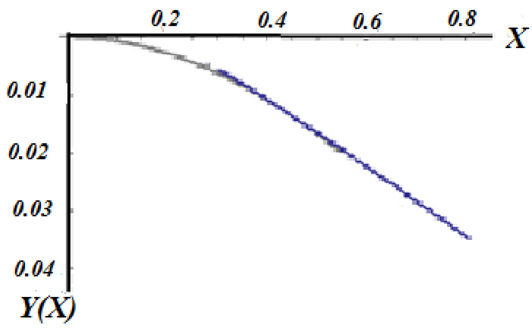

Figure 7. The deflection elastic curve in the Λ-space.

Figure 8 shows the deflection curve of the beam in the initial space.

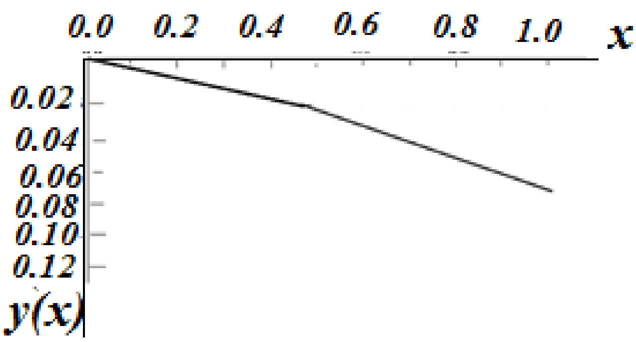

Figure 8. The deflection elastic curve in the initial space.

It is evident that only the left fractional analysis has been presented. The right fractional analysis should be performed for the complete Λ-fractional analysis, following the same ideas and procedures but for the right analysis and considering the average values in the initial space.

The present analysis might be considered as a model for discussing non-linear fractional bending structures.

5. Conclusion-further research

Bending of the Λ-fractional beam was discussed. Since fractional analysis is inherently non-local, locally stable deformations should not be considered. In the context of the Λ-fractional analysis, the stability criteria should not be local. Demanding the consideration of the additional corner Weierstrass-Erdman conditions, global variational procedures may be applied. Those conditions yield elastic deflection curves of the bending beams, with non-smooth curvatures in the Λ-fractional space, exhibiting the co-existence of phases phenomenon. The present analysis may be considered as a model for discussing fractional beam vibrations and buckling problems. The proposed theory may be applied to any problem demanding a fractional analysis. Beam problems in micro- and nanomechanics, medical engineering, bioengineering, physics, liquid crystals, signal processing, etc., should be formulated within the present context. It is pointed out that fractional variational procedures should be accepted only when they adopt global extremals and corner conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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