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Research article

Variety of double knock out barrier option for sustainable financial management

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Abstract: Options are financial contracts that are based on an underlying security and are useful for both hedging and speculating on future market trends. New financial tools are constantly being developed for sustainable financial management. In order to define new financial instruments, the BS Hamiltonian, in conjunction with a potential function, is particularly important for modelling path-dependent options. It is demonstrated here how supersymmetry provides a natural framework for generating various options, particularly using higher order supersymmetry to find and examine numerous isospectral partners of the double knock out barrier option.

Keywords: resource management; sustainability; higher order supersymmetry; Black-Scholes hamiltonians; double knock out barrier potential; option Pricing

1. Introduction

The stock market is currently one of the fastest growing areas in the modern banking and corporate world, and it plays an important role in our daily lives. In classical physics, the location of a particle at any given time t is a deterministic function of t that can be described by Newton's law and consequently, the stock price follows a predictable pattern in this frame. As a result, it's similar to the progression of a stock price with zero volatility. However, we observe that the stocks are always exchanged at fixed prices, which represents the particle property and that the stock price fluctuates frequently in the market, reflecting the wave nature, implying that the stock contains a wave-particle dualism,this is similar to the scenario of the non-zero fluctuating stock price. It is a well-known fact that the micro-world with wave-particle dualism can be described via Schrdinger equation in quantum mechanics. To get a clearer picture of the stock market, it is reasonable to look at it through the perspective of the quantum mechanical mode of thinking.

Knowing a problem's fundamental symmetry and using it to get the answer can be helpful in a variety of circumstances. Different algebraic structures in mathematics are used to realize the idea of symmetries. A generalised closed algebraic structure called supersymmetry (SUSY) has developed in recent years. SUSY first appeared in field theories to connect fermions and bosons. SUSY was studied in quantum mechanics as a testing ground for the non-perturbative methods of seeing SUSY breaking in field theory. It quickly became apparent that this field has a self identity and can be used in various disciplines of physics, rather than only serving as a model for studying field theory techniques. One of the most essential properties of supersymmetric quantum mechanics (SUSYQM) is that it provides insight into the factorization approach, which aids in the comprehension of analytically solved potential problems.

The Black-Scholes(BS) partial differential equation is crucial for determining the price of the simplest options, such as European call options. Using suitable transformation the BS equation can be rewritten as a Schrdinger-like equation. Again, the Hamiltonian of a Schrdinger equation may have several partner Hamiltonians with almost identical to the original spectrum. The BS Hamiltonian, in conjunction with a potential function, is now extremely important for modeling path-dependent options in order to define new financial instruments. In light of this, we'd like to find BS Hamiltonian's partners and investigate their properties.

Since only one partner of the BS Hamiltonian can be identified using first order SUSY, the obvious question is whether or not there exists another partner potential. The response is affirmative. We'll try to address this gap in this article. To obtain several partners of the BS Hamiltonian, we will use the second order supersymmetry formalism. For example, we'll look into the option of a double knock out barrier, that has the potential of a square well. This is the case in this instance, as shall be seen, other than the barrier potential, it is possible to work with a number of potentials.

It is interesting to see how closely the study is tied to environmental concerns. Green finance is a fancy example that can encourage long-term economic development. If green bonds outperform other corporate bonds, they will encourage more investors to participate in green energy projects. As a result, the design of green bonds is important for the growth of the green bond market. Green bond floating rates are related to the price of carbon. Fluctuations in carbon prices can cause interest rates on green bonds to fluctuate. If we set two carbon price boundaries, one with a higher carbon price and the other with a lower carbon price, and the coupon rate is reevaluated when the carbon price hits the boundary, then this model coincides with the model of the double-barrier option, and hence the interest rate of green bonds can be calculated using double-barrier option pricing. With this concept, issuers can be tracked and motivated to contribute more to green financing. The outline of the remaining sections of this paper is as follows: in section 2 we outline briefly literature review; in section 3 the notations are defined; in section 4 we discuss Black-Scholes Hamiltonian; in section 5 we consider option for a double knockout barrier; in section 6 we outline briefly second order supersymmetry formalism; in section 7 we show some results; a managerial insight is given in section 8 and finally section 9 is devoted to a conclusion.

2. Literature Review

In literary works, the phrases ”Sustainability” and ”Triple bottom line” (TBL) are commonly used in the same sentence. Since it is obvious that society, the economy, and the environment will determine the course of the future, we need to consider how we run our businesses. The references [1–3] highlight the connection between ”sustainability” and ”triple bottom line”. To make more contribution to green finance, the double-barrier option is used in [4]. There has already been a surge of interest to develop new framework for economic sustainability [5–7]. On the other hand, various physics tools have been applied to analyze financial issues [8–16]. Quantum field theory can be used to solve a wide range of financial problems [17–19]. Option pricing with stochastic volatility was investigated using path integral formalism and also Hamiltonians and potentials were employed in the models of pricing [20, 21]. In addition, supersymmetric(SUSY) method has been used for option pricing [22, 23]. To show the novelty of the study, the author contribution table is provided bellow.

Table 1. Contribution of the authors.

Author(s)	Financial issues	Utilized physics tools	Application of SUSY	Abundance and diversity of options
Sarkar et al. [5]	Yes	No	No	No
Hosseini et al. [8]	Yes	Yes	No	No
Ramos et al. [9]	Yes	Yes	No	No
Orrell D [10, 11]	Yes	Yes	No	No
Haven E [12, 13]	Yes	Yes	No	No
Ahn et al. [14]	Yes	Yes	No	No
Schaden M [18]	Yes	Yes	No	No
Hicks W [21]	Yes	Yes	No	No
Jana et al. [22]	Yes	Yes	Yes	No
Halperin I [23]	Yes	Yes	Yes	No
Bagrov et al. [24, 25]	No	Yes	Yes	No
Fernndez et al. [27, 28]	No	Yes	Yes	No
This Paper	Yes	Yes	Yes	Yes

3. Notation

The following notations are used in this the model.

- C represents the option price
- S represents the stock price
- σ represent the stock price volatility
- r risk-free spot interest rate
- t represent the time
- ϵ scaling parameter(≥ 0)

4. Black-Scholes Hamiltonian

The BS model is used to analyze the price of an option, and it can also be used to predict how financial instruments such as future contracts and stock shares will change in price over time. For option pricing with constant volatility, the BS equation is given by [20]

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC \tag{4.1}$$

Performing the transformations

$$C(S,t) = e^{\epsilon t}\psi(S) \quad \text{and} \quad S = e^x, \quad -\infty < x < \infty \tag{4.2}$$

to the BS equation (4.1) we obtain**[See Appendix A]**

$$\begin{aligned} H_{BS}\psi &= \epsilon\psi \\ H_{BS} &= -\frac{\sigma^2}{2}\frac{d^2}{dx^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{d}{dx} + r \end{aligned} \tag{4.3}$$

where H_{BS} is the Black-Scholes Hamiltonian. If we consider the BS equation (4.3) with a potential $V(x)$ then we have

$$[H_{BS} + V]\psi = \epsilon\psi \tag{4.4}$$

which is the more general form of the BS equation. Let us now apply the following transformation

$$\psi = \rho\phi, \quad \rho = exp[(\frac{1}{2} - \frac{r}{\sigma^2})x] \tag{4.5}$$

Then Eq. (4.4) becomes**[See Appendix B]**

$$-\frac{d^2\phi}{dx^2} + \frac{2V}{\sigma^2}\phi = \lambda\phi, \quad \lambda = \frac{2\epsilon}{\sigma^2} - (\frac{r}{\sigma^2} + \frac{1}{2})^2 \tag{4.6}$$

which looks like a Schrödinger equation with a potential $\frac{2V}{\sigma^2}$.

5. Option for a double knockout barrier

The option for a double barrier is one in which the payoff of an exotic option is decided by setting two price barriers, one upper and one lower. A double barrier option is a path dependent option that limits the stock's value to two boundaries., e^{l_1} and e^{l_2}. If the stock price falls outside the barrier, the option's value is judged to be zero.

The requirements as discussed above, can be adequately fulfilled if we choose the potential $V(x)$ as

$$V(x) = \left\{ \begin{array}{ll} \infty, & x \le l_1 \\ 0, & l_1 < x < l_2 \\ \infty, & x \ge l_2 \end{array} \right\} \tag{5.1}$$

Then the double knock out barrier's Hamiltonian is

$$H_{DB} = H_{BS} + V(x) \tag{5.2}$$

Now from Eq. (4.6) we get

$$-\frac{d^2\phi}{dx^2} = \lambda\phi \tag{5.3}$$

which is a very well-known problem in quantum mechanics, entitled as : particle in an infinite potential well. Following the boundary conditions

$$\phi(l_1) = \phi(l_2) = 0 \tag{5.4}$$

the eigenenergies and eigenfunctions of Eq. (5.3) can be found to be

$$\begin{aligned} \lambda_n &= \left[\frac{\pi(n+1)}{(l_2 - l_1)}\right]^2 \\ \phi_n(x) &= \sqrt{\frac{2}{l_2 - l_1}}\, sin[\sqrt{\lambda_n}(x - l_1)], \quad n = 0, 1, 2, ... \end{aligned} \tag{5.5}$$

As a result, the spectrum of associated double barrier equation $H_{DB}\psi = \epsilon\psi$ are given by

$$\begin{aligned} \epsilon_n &= \frac{\sigma^2}{2}\lambda_n + \gamma, \quad \gamma = \frac{1}{2\sigma^2}(r + \frac{\sigma^2}{2})^2 \\ \psi_n(x) &= \rho\phi_n(x) = \sqrt{\frac{2}{l_2 - l_1}}\, exp\left[\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)x\right] sin[\sqrt{\lambda_n}(x - l_1)], \quad n = 0, 1, 2, ... \end{aligned} \tag{5.6}$$

6. Second order supersymmetry formalism

Let H_0 and H_1 be two Hamiltonians and A be an operator, if

$$AH_0 = H_1 A \tag{6.1}$$

then it can be verified that if $\phi^{(0)}$ is an eigenstate of H_0 with eigenenergy $\lambda^{(0)}$, then $A\phi^{(0)}$ is an eigenstate of H_1 with the same eigenenergy as long as $A\phi^{(0)}$ meets the appropriate boundary conditions.

The intertwining approach is analogous to first order Darboux formalism [24, 25] or first order supersymmetry when A is constructed by means of first order differential operators. Specifically for the case of first order supersymmetry we know that if V_0 is the potential associated with the Hamiltonian H_0 and $A = \frac{d}{dx} + W(x)$, then the isospectral potential associated with the Hamiltonian H_1 is $V_1 = V_0 + 2\frac{dW}{dx}$ [26].In a similar way, if we take the operator A to be higher orders, we get a higher order supersymmetry or a higher order Darboux algorithm. In this section we display second order supersymmetry formalism to construct new exactly solvable isospectral potentials.

Let's pretend that A is a differential operator containing second-order derivative with the form [27,28]

$$\begin{aligned} A &= \frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(\beta) \\ \beta(x) &= -\frac{d}{dx}\, log\; W_{i,j}(x) \\ \gamma(\beta) &= -\frac{\beta''}{2\beta} + \left(\frac{\beta'}{2\beta}\right)^2 + \frac{\beta'}{2} + \frac{\beta^2}{4} - \left(\frac{\lambda_i^{(0)} - \lambda_j^{(0)}}{2\beta}\right)^2 \end{aligned} \tag{6.2}$$

where $W_{i,j} = (\phi_i^{(0)}\phi_j'^{(0)} - \phi_i'^{(0)}\phi_j^{(0)})$ is the corresponding Wronskian and $\phi_i^{(0)}$ and $\phi_j^{(0)}$ are eigenfunctions of $H_0(= -\frac{d^2}{dx^2} + V_0(x))$ corresponding to the eigenvalues $\lambda_i^{(0)}$ and $\lambda_j^{(0)}$ and
Then from the intertwining condition (6.1) the isospectral partner Hamiltonian can be obtained as

$$H_2 = -\frac{d^2}{dx^2} + V_2(x) \tag{6.3}$$

where

$$V_2(x) = V_0(x) - 2\frac{d^2}{dx^2}\log W_{i,j}(x) \tag{6.4}$$

The wave functions $\psi_i^{(2)}(x)$ and $\phi_i^{(0)}(x)$ in accordance with $V_2(x)$ and $V_0(x)$ are related by [24,25,28]

$$\psi_k^{(2)}(x) = A\phi_k^{(0)}(x) = \frac{1}{W_{i,j}(x)}\begin{vmatrix} \phi_i^{(0)} & \phi_j^{(0)} & \phi_k^{(0)} \\ \phi_i'^{(0)} & \phi_j'^{(0)} & \phi_k'^{(0)} \\ \phi_i''^{(0)} & \phi_j''^{(0)} & \phi_k''^{(0)} \end{vmatrix}, \quad i, j \neq k \tag{6.5}$$

The eigenstates derived from $\phi_i^{(0)}$ and $\phi_j^{(0)}$ are offered by [24,25,28]

$$f(x) \propto \frac{\phi_i^{(0)}(x)}{W_{i,j}(x)} \;\; , \;\; g(x) \propto \frac{\phi_j^{(0)}(x)}{W_{i,j}(x)} \tag{6.6}$$

Here we want the bound state solution, this has been done keeping in mind the fact that singularities will not exist in the newly developed potential if $W_{i,j}(x)$ is nodeless and it will be guaranteed if we choose two consecutive eigenfunctions of H_0 [28].

7. Results

In this section we develop some alternative potentials instead of double knockout barrier potential for which the same scenario can be realized. To achieve our intention we use the concept of second order supersymmetry as discussed above.

Let us assume that the role of the potential $V_0(x)$ is same as the the potential $V(x)$ as defined in Eq. (5.1) whose spectrums are exactly known and now consider the following cases :
Case I : In this scenario, we start with two bound states, the ground stae $\phi_0^{(0)}(x)$ and the first exited

state $\phi_1^{(0)}(x)$ of the potential $V_0(x)$ corresponding to energies $\lambda_0^{(0)}$ and $\lambda_1^{(0)}$ respectively. Then from Eq. (5.5) we have

$$\begin{aligned} \lambda_0^{(0)} &= \left[\frac{\pi}{(l_2 - l_1)}\right]^2 \\ \lambda_1^{(0)} &= \left[\frac{2\pi}{(l_2 - l_1)}\right]^2 \\ \phi_0^{(0)}(x) &= \sqrt{\frac{2}{l_2 - l_1}}\ sin[\sqrt{\lambda_0^{(0)}}(x - l_1)] \\ \phi_1^{(0)}(x) &= \sqrt{\frac{2}{l_2 - l_1}}\ sin[\sqrt{\lambda_1^{(0)}}(x - l_1)] \end{aligned} \tag{7.1}$$

and consequently,

$$W_{0,1}(x) = -\frac{4\pi(sin[\pi(l_1 - x)/(l_1 - l_2)])^3}{(l_1 - l_2)^2} \tag{7.2}$$

also fron Eq. (6.4) and (6.5),

$$\begin{aligned} &V_2(x) = -\frac{6\pi^2(cosec[\pi(l_1 - x)/(l_1 - l_2)])^2}{(l_1 - l_2)^2} \\ &\lambda_k^{(2)} = \lambda_k^{(0)} = \left[\frac{\pi(k+1)}{(l_2 - l_1)}\right]^2, k \neq 0, 1 \\ &\psi_k^{(2)} = A\phi_k^{(0)}(x) = \frac{\sqrt{2}}{(l_2-l_1)^{\frac{5}{2}}}(\pi^2(2 + cos[\tfrac{2\pi(l_1-x)}{(l_1-l_2)}])cosec^2[\tfrac{\pi(l_1-x)}{(l_1-l_2)}]sin[(-l_1 + x)\sqrt{\lambda_k^{(0)}}] \\ &+3(l_1 - l_2)\pi cos[(-l_1 + x)\sqrt{\lambda_k^{(0)}}]cot[\tfrac{\pi(l_1-x)}{(l_1-l_2)}]\sqrt{\lambda_k^{(0)}} - (l_1 - l_2)^2 sin[(-l_1 + x)\sqrt{\lambda_k^{(0)}}]\lambda_k^{(0)}), k \neq 0, 1 \end{aligned} \tag{7.3}$$

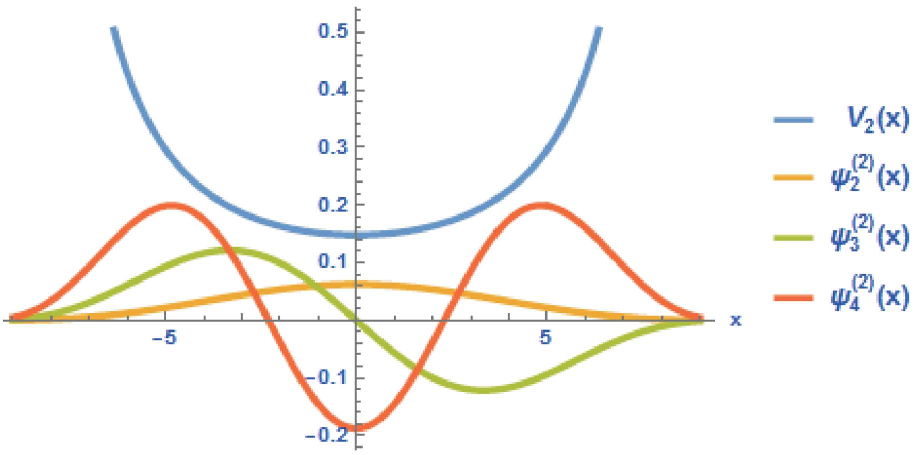

Figure 1. graphical representation of the alternative potential $V_2(x)$ and the wave functions $\psi_2^{(2)}, \psi_3^{(2)}, \psi_4^{(2)}$.

Case II : Here we start with two bound states, the first exited state $\phi_1^{(0)}(x)$ and the second exited state

$\phi_2^{(0)}(x)$ of the potential $V_0(x)$ corresponding to energies $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ respectively. Then from Eq. (5.5) we have

$$\begin{aligned} \lambda_1^{(0)} &= \left[\frac{2\pi}{(l_2 - l_1)}\right]^2 \\ \lambda_2^{(0)} &= \left[\frac{3\pi}{(l_2 - l_1)}\right]^2 \\ \phi_1^{(0)}(x) &= \sqrt{\frac{2}{l_2 - l_1}}\, sin[\sqrt{\lambda_1^{(0)}}(x - l_1)] \\ \phi_2^{(0)}(x) &= \sqrt{\frac{2}{l_2 - l_1}}\, sin[\sqrt{\lambda_2^{(0)}}(x - l_1)] \end{aligned} \tag{7.4}$$

and consequently,

$$W_{1,2}(x) = -\frac{\pi(-5sin[\pi(l_1 - x)/(l_1 - l_2)] + sin[5\pi(l_1 - x)/(l_1 - l_2)]}{(l_1 - l_2)^2} \tag{7.5}$$

fron Eq. (6.4) and (6.5),

$$\begin{aligned} &V_2(x) = -\frac{10\pi^2(-16 + 15(cosec[\pi(l_1 - x)/(l_1 - l_2)])^2)}{(l_1 - l_2)^2(3 + 2cos[2\pi(l_1 - x)/(l_1 - l_2)])^2} \\ &\lambda_k^{(2)} = \lambda_k^{(0)} = \left[\frac{\pi(k + 1)}{(l_2 - l_1)}\right]^2, k \neq 1, 2 \\ &\psi_k^{(2)} = \sqrt{\tfrac{1}{2(l_2-l_1)^5}}(6\pi^2(3 + cos[\tfrac{2\pi(l_1-x)}{(l_1-l_2)}] + cos[\tfrac{4\pi(l_1-x)}{(l_1-l_2)}])cosec^2[\tfrac{\pi(l_1-x)}{(l_1-l_2)}]sin[(-l_1 + x)\sqrt{\lambda_k^{(0)}}] \\ &+10(l_1 - l_2)\pi(1 + 2cos[\tfrac{2\pi(l_1-x)}{(l_1-l_2)}])cos[(-l_1 + x)\sqrt{\lambda_k^{(0)}}]cot[\tfrac{\pi(l_1-x)}{(l_1-l_2)}]\sqrt{\lambda_k^{(0)}} \\ &-2(l_1 - l_2)^2(3 + 2cos[\tfrac{2\pi(l_1-x)}{(l_1-l_2)}])sin[(-l_1 + x)\sqrt{\lambda_k^{(0)}}]\lambda_k^{(0)})/(3 + 2cos[\tfrac{2\pi(l_1-x)}{(l_1-l_2)}]), k \neq 1, 2 \end{aligned} \tag{7.6}$$

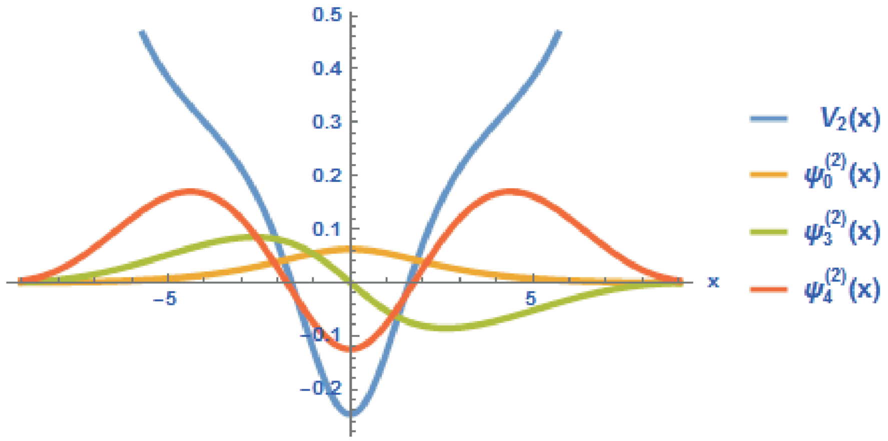

Figure 2. graphical representation of the alternative potential $V_2(x)$ and the wave functions $\psi_2^{(2)}, \psi_3^{(2)}, \psi_4^{(2)}$.

Case III : We begin with the bound states $\phi_{15}^{(0)}(x)$ and $\phi_{16}^{(0)}(x)$ of the potential $V_0(x)$ corresponding to energies $\lambda_{15}^{(0)}$ and $\lambda_{16}^{(0)}$ respectively in this example. Then from Eq. (5.5) we have

$$\begin{aligned} \lambda_{15}^{(0)} &= \left[\frac{16\pi}{(l_2 - l_1)}\right]^2 \\ \lambda_{16}^{(0)} &= \left[\frac{17\pi}{(l_2 - l_1)}\right]^2 \\ \phi_{15}^{(0)}(x) &= \sqrt{\frac{2}{l_2 - l_1}}\, sin[\sqrt{\lambda_{15}^{(0)}}(x - l_1)] \\ \phi_{16}^{(0)}(x) &= \sqrt{\frac{2}{l_2 - l_1}}\, sin[\sqrt{\lambda_{16}^{(0)}}(x - l_1)] \end{aligned} \tag{7.7}$$

and consequently,

$$W_{15,16}(x) = -\frac{\pi(-33sin[\pi(l_1 - x)/(l_1 - l_2)] + sin[33\pi(l_1 - x)/(l_1 - l_2)])}{(l_1 - l_2)^2} \tag{7.8}$$

fron Eq. (6.4) and (6.5),

$$\begin{aligned} V_2(x) &= \frac{132\pi^2(33 - 289cos[32\pi(l_1 - x)/(l_1 - l_2)] + 256cos[34\pi(l_1 - x)/(l_1 - l_2)])}{(l_1 - l_2)^2(-33sin[\pi(l_1 - x)/(l_1 - l_2)] + sin[33\pi(l_1 - x)/(l_1 - l_2)])^2} \\ \lambda_k^{(2)} &= \lambda_k^{(0)} = \left[\frac{\pi(k+1)}{(l_2 - l_1)}\right]^2, k \neq 15, 16 \\ \psi_k^{(2)} &= -\sqrt{\frac{2}{(l_2-l_1)}}\lambda_k^{(0)} sin[(-l_1 + x)\sqrt{\lambda_k^{(0)}}] \\ &- \frac{66\sqrt{\frac{2}{(l_2-l_1)}}\pi\sqrt{\lambda_k^{(0)}}cos[(-l_1+x)\sqrt{\lambda_k^{(0)}}]sin[\frac{16\pi(l_1-x)}{(l_1-l_2)}]sin[\frac{17\pi(l_1-x)}{(l_1-l_2)}]}{(l_1-l_2)(-33sin[\pi(l_1-x)/(l_1-l_2)]+sin[33\pi(l_1-x)/(l_1-l_2)])} \\ &- \frac{272\sqrt{\frac{2}{(l_2-l_1)}}\pi^2(33sin[\pi(l_1-x)/(l_1-l_2)]+sin[33\pi(l_1-x)/(l_1-l_2)])sin[(-l_1+x)\sqrt{\lambda_k^{(0)}}]}{(l_1-l_2)^2(-33sin[\pi(l_1-x)/(l_1-l_2)]+sin[33\pi(l_1-x)/(l_1-l_2)])}, k \neq 15, 16 \end{aligned} \tag{7.9}$$

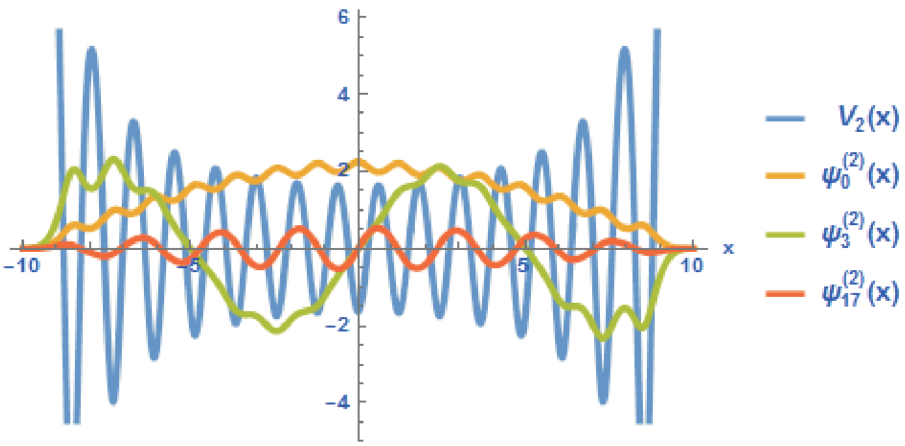

Figure 3. graphical representation of the alternative potential $V_2(x)$ and the wave functions $\psi_0^{(2)}, \psi_3^{(2)}, \psi_{17}^{(2)}$.

8. Managerial insight

According to our formalism, one can generate several potential, i.e., several path dependent options by choosing consecutive two bound states corresponding to the starting potential but from the graphical representation of different cases it is observed that for more excited consecutive states the newly generated potential becomes more complicated. It would be noted that the eigenfunctions $f(x), g(x)$ in 6.6 (i.e., which states are used to construct new potential for different cases) are not acceptable because they do not satisfy the boundary conditions for the bound states and in any case they are not SUSY partners of the corresponding states in the original potential. Additionally, SUSY's beauty makes sure that any cases may be solved exactly.

9. Conclusion

One of the most important characteristics of supersymmetric quantum mechanics (SUSYQM) is that it offers insight into factorization techniques, which helps in the comprehension of potential issues that can be solved analytically [26]. Here, we employ higher order supersymmetry, which has a number of advantages over first order supersymmetry in terms of building multiple partners of the BS Hamiltonian beginning with the double knock out barrier potential.

A new approach to finance called ”green finance” can help the economy grow sustainably. The development of green bonds is similarly a gradual process. The purpose of green bonds is to finance environmental-friendly projects. If green bonds surpass other debt securities, more investors will get involved in green energy projects. Therefore, the design of green bonds is essential to the growth of the market for green bonds. In reference [4], it is indicated that the double knock out barrier option plays an important role to design green bonds. So we believe that a number of precisely solvable partners of the BS Hamiltonian formed via double knock out barrier optin will be essential to model path-dependent options and defining novel financial instruments for sustainable financial management. However, in quantum physics, there are also different strategies for constructing isospectral partners [29, 30]. We believe it would be fascinating to use some of these strategies to build alternative BS Hamiltonians and investigate their various features.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix A:

We have

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} + rC \tag{A.1}$$

The transformations are

$$C(S,t) = e^{\epsilon t}\psi(S) \quad \text{and} \quad S = e^x, \quad -\infty < x < \infty \tag{A.2}$$

So,

$$\frac{\partial C}{\partial t} = \epsilon e^{\epsilon t}\psi(S)$$

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{e^x}\frac{\partial C}{\partial x}$$

and

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}[\frac{1}{e^x}\frac{\partial C}{\partial x}] = \frac{\partial}{\partial x}[\frac{1}{e^x}\frac{\partial C}{\partial x}]\frac{\partial x}{\partial S} = \frac{1}{e^x}[-e^{-x}\frac{\partial C}{\partial x} + e^{-x}\frac{\partial^2 C}{\partial x^2}]$$

Substituting these values in (A.1), we get

$$H_{BS}\psi = \epsilon\psi$$

Appendix B:

We have from Eq.(4.4)

$$[-\frac{\sigma^2}{2}\frac{d^2}{dx^2}+\left(\frac{\sigma^2}{2}-r\right)\frac{d}{dx}+r+V]\psi=\epsilon\psi \tag{B.1}$$

The transformations are

$$\psi=\rho\phi,\quad \rho=exp[(\frac{1}{2}-\frac{r}{\sigma^2})x] \tag{B.2}$$

Then

$$\frac{d\psi}{dx}=\frac{d\rho}{dx}\phi+\rho\frac{d\phi}{dx}=[\frac{1}{2}-\frac{r}{\sigma^2}]\rho\phi+\rho\frac{d\phi}{dx}$$

and

$$\frac{d^2\psi}{dx^2}=\frac{d}{dx}[\frac{d\phi}{dx}]=\frac{d}{dx}[(\frac{1}{2}-\frac{r}{\sigma^2})\rho\phi+\rho\frac{d\phi}{dx}=\rho\frac{d^2\phi}{dx^2}+2\rho(\frac{1}{2}-\frac{r}{\sigma^2})\frac{d\phi}{dx}+(\frac{1}{2}-\frac{r}{\sigma^2})^2\rho\phi$$

Substituting these values in (B.1), we get

$$-\frac{d^2\phi}{dx^2}+\frac{2V}{\sigma^2}\phi=\lambda\phi,\ where\ \lambda=\frac{2\epsilon}{\sigma^2}-(\frac{r}{\sigma^2}+\frac{1}{2})^2$$