Review

On modeling arterial blood flow with or without solute transport and in presence of atherosclerosis

Daniel N. Riahi[1,2] and Saulo Orizaga[3,*]

[1] Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801; d-riahi@illinois.edu

[2] School of Mathematical & Statistical Sciences, University of Texas Rio Grande Valley, Brownsville, Texas 78520; riahid@oal.com

[3] Department of Mathematics, New Mexico Institute of Mining and Technology, Socorro, NM 87801

* Correspondence: Email: saulo.orizaga@nmt.edu.

Abstract: In this article, we review previous studies of modeling problems for blood flow with or without transport of a solute in a section of arterial blood flow and in the presence of atherosclerosis. Moreover, we review problems of bio-fluid dynamics within the field of biophysics. In most modeling cases, the presence of red blood cells in the plasma is taken into account either by using a two-phase flow approach, where blood plasma is considered as one phase and red blood cells are counted as another phase, or by using a variable viscosity formula that accounts for the amount of hematocrit within the blood. Both analytical and computational methods were implemented to solve the governing equations for blood flow in the presence of solute transport, which, depending on the type of the investigated problem, could contain momentum, mass conservation, and solute concentration, which were mostly subjected to reasonable approximations. The form of atherosclerosis implemented in the modeling system either was either based on the experimental data for an actual human or was due to a reasonable mathematical modeling for both steady and unsteady atherosclerosis cases. For the wall of the artery itself, which is elastic in nature, modeling equations for the displacement of the artery wall has previously been used, even though their effects on the blood flow inside the artery were shown to be rather small. In some cases, thermal effects were also taken into account by including a temperature equation in the investigation. In the case of the presence of solutes in the blood, various blood flow parameters such as blood pressure force, blood speed, and solute transport were mostly determined or approximated for different values of the parameters that could represent hematocrit, solute diffusion, atherosclerosis height, solute reaction, and pulse frequency. In some studies, available experimental

results and data were used in the modeling system that resulted in a more realistic outcome for the blood flow parameters, such as blood pressure force and blood flow resistance. Results have been found for variations of blood flow parameters and solute transport when compared to different values of the parameters. Effects that can increase or decrease the blood flow parameters and solute transport in the artery have mostly been determined, with particular applications for further understanding efforts to improve the patients' health care.

Keywords: solute concentration; blood flow; atherosclerosis; artery; hematocrit

1. Introduction

For many decades, it has been known that serious diseases in blood vessels and in the heart can cause heart attacks and strokes, which are major causes of death. The main cause for these diseases is the formation atherosclerosis; for example, fatty materials within arteries reduces the cross sectional area of blood passage, thereby preventing an adequate blood supply to the distal bed [1−4]. In addition, clots can form as emboli and occlude the smaller vessels, which can result in an interruption of the blood supply to the distal bed. Plaques formed in coronary arteries can lead to heart attacks, whereas clots in the cerebral circulation can result in a stroke.

There are a number of risk factors for the presence of atherosclerotic lesions. The common sites for the formation and development of atherosclerosis include the coronary arteries, the branching of the subclavian and common carotids in the aortic arch, the bifurcation of the common carotid to the internal and external carotids, especially in the carotid sinus region distal to the bifurcation, the renal arterial branching in the descending aorta, and in the bifurcations of the descending aorta. The common feature in the location for the development of these lesions is the presence of a curvature, branching, and a bifurcation [1−4]. The fluid dynamics at these sites is anticipated to be vastly different from other segments of the arteries that are relatively straight and devoid of any branching segments [3]. Hence, several investigators (who are discussed in the next sections) have attempted to link the fluid dynamically induced stress with the formation of atherosclerotic lesions in the human circulation.

By assuming the artery to be circularly cylindrical in shape, Misra et al. [5] studied the flow of blood in the artery in presence of atherosclerosis for a Casson-type fluid. In 1973, Young and Tsai [6] discussed some characteristics of blood flow in stented arteries. The blood vessels carried blood from the heart to all the organs and tissues of the body, including the brain, kidneys, gut, muscles, and the heart itself. Venkateswarlu and Rao [7] studied an assumed oscillatory form of blood flow through an indented tube in the presence of steady atherosclerosis with a very simple shape. They used the so-called Einstein model for the viscosity of the blood with a variable volume flow rate and a prescribed value for the magnitude of the blood pressure force.

Srivastava et al. [8] studied arterial blood flow in the presence of atherosclerosis [5] using a Casson-type fluid flow. They calculated the blood pressure force, blood speed, impedance, and shear stress for different atherosclerosis heights. All the aforementioned investigations considered cases where no catheter was inserted into the artery, although there have been a number of studies of the blood flow systems in catheterized arteries [9−12]. A catheter is an important medical tool that is used to facilitate blood flow circulation within arteries and gives patients an opportunity for further treatment on their given medical condition.

There have been a number of studies where the blood was represented by a two-phase macroscopic model [13,14]. In [14], Riahi applied experimentally obtained data from [15] for the atherosclerosis shape in a coronary-type artery [15], where the blood was represented by a two-phase macroscopic model in [14]; moreover, the elastic aspect of the artery wall was included in this model, where Riahi performed modeling for elastic arterial blood flow. His considered elastic artery contained atherosclerosis that was experimentally studied within [15], whose authors collected data from an actual casting of a man with atherosclerosis. Riahi [14] made use of the axial extent L_0 property of the atherosclerosis, which was large in comparison to the radius R_0 of the artery. He applied a perturbation technique for the small aspect ratio γ of the artery's radius to the stenosis's axial extent in the artery, which was clearly in agreement with the experimental finding of the male artery casting within [15] and determined results for the leading order terms in γ of the blood flow and the elastic wall quantities. In addition to determining blood pressure force and blood speed, he found, in particular, that the radial and axial displacements of the elastic wall oscillated in time and the oscillation grew with an increasing hematocrit, volume flow rate, blood pressure force, and viscoelastic wall stress; however, such oscillations decayed with an increase in the elastic wall stiffness and thickness.

In regard to the notable experimental work of Back et al. [15], which was referred to in the previous paragraph, it is worthwhile to describe such work in detail within this paragraph. Back et al. [15] performed experimental studies to measure the cross-sectional area of the inner arterial surface, which was found to be circular to a very good approximation, and the blood flow in a patient's coronary artery along the axis of the artery. This artery had a mild atherosclerosis within the patient's main coronary artery. The authors obtained this artery casting from a human cadaver from the University of Southern California, School of Medicine. They used sugar-water solutions to simulate the blood viscosity and developed a flow in the artery by maintaining a volume flow rate. Their results were found to be consistent for a straight axisymmetric model of the artery.

The elastic aspect of the wall in arterial blood flow was also studied by Misra et al. [5], Srivastava et al. [8], Chakravary [16], and Riahi [17]. In 2017, Riahi [17] investigated cases for arbitrary values of the atherosclerosis aspect ratio γ and was able to uncover results for more fully tethering elastic walls. His study focused on cases of low values of the pulse frequency. His subsequent results focused more on blood pressure force and blood flow resistance (impedance). These results can be applicable to the cases of bradycardia patients; whose hearts beat very slowly.

There have been studies of solute, heat, and mass transport in blood flow arteries and tissues that were performed over the last three decades [12,18−22]. In 1999, Lee and Vafai [18] developed a model for heat transfer within an artery with blood flow and tissue, where they included the source and sink terms in both heat equations for the fluid and tissue and solved the modeling system numerically.

Valencia and Villanueva [20] numerically investigated fully coupled models for the fluid and structure of unsteady non-Newtonian flow of blood in arteries and a model of mass transfer for type stenotic arteries, taking fluid-structure interaction into account, where the interface between fluid flow and arteries was assumed to move with the artery wall. Their investigation was based on a finite element package that was used to determine the results. They found that the geometry of the stenosis they used had effects on the wall stress.

Roy and Beg [22] developed a two-fluid blood flow model for transport of a reactive [23] and diffusive species in a rigid artery with an assumed constant axial blood pressure gradient. The two-fluid case that they assumed was composed of a micro-polar fluid in the core region of the artery and a Newtonian fluid near the boundary of the artery. They analytically solved their modeling system by

calculating the blood flow parameters and found that the axial and transverse concentrations of the species were enhanced with an increasing reactive rate. Additionally, they stated the applicability of their results in hemodynamics. Rana and Murthy [24] investigated unsteady solute dispersion in a blood flow vessel using a two-phase Casson model. They found that the mean value of solute transport was reduced with an increasing radius of the vessel.

Orizaga et al. [12] studied a steady case of drug transportation in a catheterized arterial blood flow with atherosclerosis, which was determined from experimental data determined by [15] from the casting of a human. They proposed a simple model for drug evolution that only contained convection and diffusion terms. They calculated important blood flow parameters, in particular, the blood pressure force and the blood speed. Their determined blood pressure force, whose form was greatly influenced by the experimentally collected data for the atherosclerosis of coronary artery of an actual human [15], was more realistic as compared with other blood pressure results by previous studies that used a mathematical model with no human data for the structure of the atherosclerosis [5−8]. In addition, they numerically solved the equation for the drug by an implicit finite difference method and found that convective drug transport was very effective when the diffusivity coefficient was very small. In the next section, we shall provide a review of the work in [12] in some details, but the readers are referred to the full original paper for more details on the subject matter.

Ponalagusmy et al. [25] studied solute dispersion in a non-Newtonian blood flow in a circular tube that was based on an empirical modeling formula for the blood flow. They included the effect of chemical reactions [23] due to presence of solutes in blood. They assumed uniform blood pressure force with the blood velocity as a function of a radial variable. They calculated a solute dispersion coefficient and found that it decreased as the reactive parameter increased. This later result indicated that solute transport decreases with an increasing reactive rate.

Additionally, there have been laboratory experiments of solute transport across the interface of a media. Chen et al. [26] performed experiments for solute transport as it crossed an interface of porous media. They found that for a low Reynolds number flow, the solute transport across such an interface was mainly due to convective transport that dominated the diffusive processes.

Very recently, we investigated the unsteady blood flow and transport of a reactive and diffusive property, which can be the case of a drug in a porous artery with atherosclerosis [27]. As in [14], we considered the length of atherosclerosis in the case of a small aspect ratio γ of the artery radius to be small, which was consistent with the experimental results of Back et al. [15]. In addition, in contrast to Sharma et al. [28] and Riahi [14,17], the time-averaged of our experimental blood flow solutions were non-zero [15]. We treated the section of the artery with atherosclerosis to be a porous medium. We implemented a variable viscosity formula known as the Einstein model for the blood viscosity [7,9] to take the number of red cells in the blood fluid of the considered arterial flow system into account. Moreover, we studied an unsteady arterial blood flow with no restriction on the magnitude of the blood pressure force. In section 3, we briefly describe the work presented in [27], which included the main governing equations, modeling, and results for various blood flow parameters, such as blood pressure force and blood speed, as well as results for the solute transport. However, the readers are referred to the original work [27] for details of the modeling, analysis, and results of the studied system.

The present review article mostly focuses controlling and maintaining normal functioning of the cardiovascular system of a living human. Such a system relies on the blood flow in the arteries that can carry nutrients or other species and circulates passage through the heart. The main bio-fluid dynamics parameters that need to be taken into account for the modeling of an actual arterial blood

flow in a human for a particular patient are the hematocrit parameter, whose very low value causes anemia and very large value causes hypertension in patients, the atherosclerosis peak parameter, whose high values can cause hypertension, the diffusion parameter in the presence of species, whose very small value can cause hypertension, and the frequency of pulsation, whose very low value can indicate a patient with bradycardia.

In the next 4 sections, we shall point out the importance of blood pressure and how it influences and affects all of the associated blood flow parameters on a solute transport blood flow model.

2. Brief review of the work in [12]

In this section, we briefly review the problem for the case of blood flow and transport of a solute in the form of a drug in a coronary-type artery. In this experimental case study, a catheter was inserted into an artery to protect the corresponding patient who has had atherosclerosis in his/her artery, as compared to the case study [15] of the coronary artery casting of a patient had a mild atherosclerosis. Similar to the work in [14], we performed modeling of the problem using a two-phase blood flow system that carried drug. As we described in the previous section, such a two-phase blood flow was composed of a red blood cells phase and a blood plasma phase.

Due to the experimental verifications [15], the authors in [12] considered blood flow in a section of a straight annulus-type tube whose inner boundary was that of an inserted catheter in the artery. The outer boundary of the tube was that of the atherosclerosis form studied in [15]. The authors in [12] considered two-phase blood flow in the annulus, where one phase represented the plasma and another phase represented the particles (red cells). The axial length of the annulus section was taken to be sufficiently large in comparison to the radius of the artery; therefore, the end effects were neglected. Moreover, the described two-phase blood flow transported a drug.

The investigation in [12] was based on the original governing system of equations for the two-phase forms [14] of the momentum, mass conservation, and drug concentration. The governing system for the two-phase blood flow and drug were simplified using reasonable approximations and assumptions [14], such as the dominant axial velocity as compared to that in the radial direction, whose axis was perpendicular to that of the axial direction, and a small Reynolds number implying small inertial forces that were neglected. Under reasonable boundary conditions, the simplified forms of the two-phase blood flow equations were first solved by both analytical and computational methods. The collected experimental data for the atherosclerosis [15] were used by the authors in [12] to produce a graphical form, as shown in Figure 1. Then, the authors determined important blood flow parameters such as blood pressure force, which was used to determine the blood speed because the blood pressure force is the driving force of the blood in the artery; moreover, there is a strong interrelation between these two blood flow parameters in the flow domain. It should be noted that the calculated data for blood flow parameters were significantly influenced by the form of atherosclerosis. This implied a broader implication to much more realistic forms for the computed blood flow parameters in [12], since the implemented form of the atherosclerosis was completely based on experimental data [15] for the coronary artery casting of a human, as described in some details earlier in Section 1.
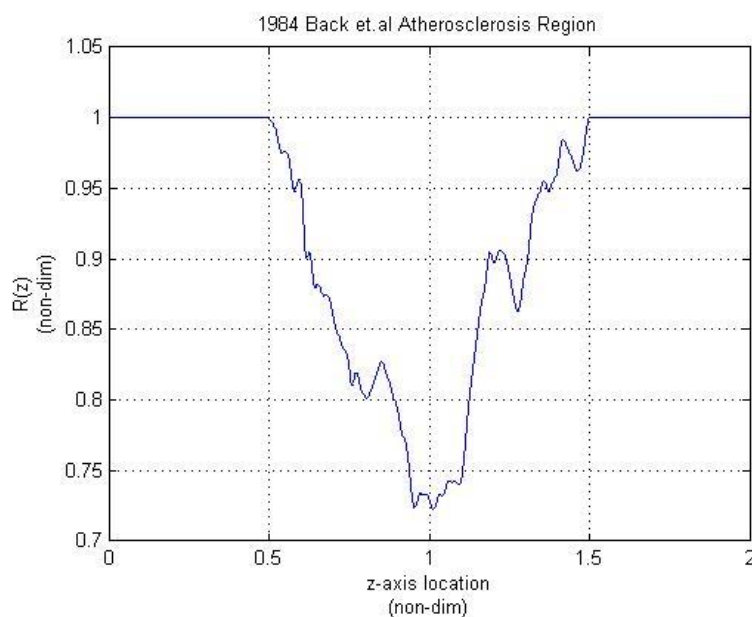
Figure 1. Atherosclerosis zone on the artery given by experimental data (Back *et al*. [15]).

The results shown in Figure 2 correspond to a catheter size of 0.2 and three different values of hematocrit C. The results indicated how the presence of a realistic atherosclerosis form [15] significantly increased the blood pressure force and affected its form. Such results are notably more realistic as compared with the previous study presented in [17], which was based on a mathematical model for the atherosclerosis shape. This is because the form of the atherosclerosis (Figure 1) used to draw Figure 2 was fully based on experimental data [15] for an actual human.
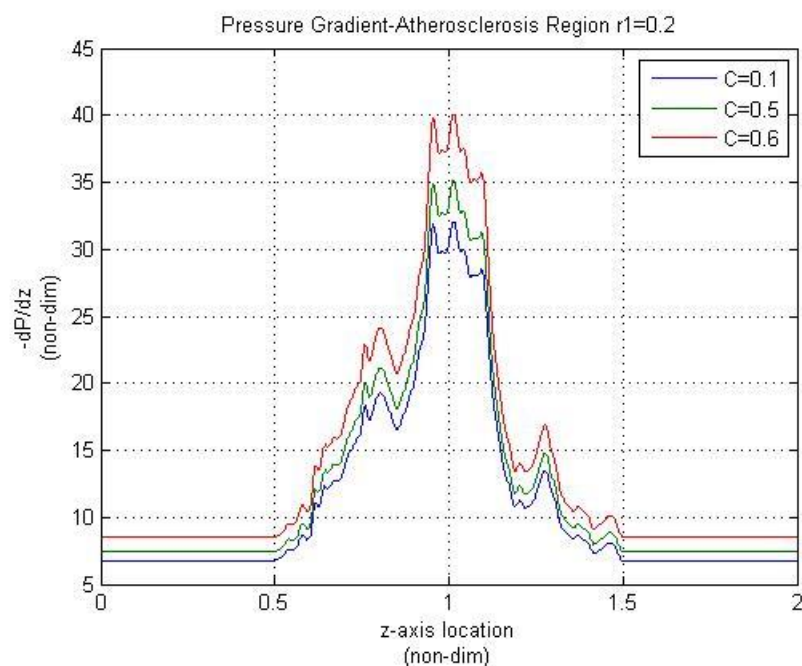

Figure 2. Blood pressure force for different Hematocrit values.

The structure of the drug delivery problem was modeled with a partial differential equation that was developed in the form of a drug concentration, which included terms representing convection and diffusion. Such terms depended on the blood plasma velocity, the hematocrit value, and the diffusion coefficient of the drug. In particular, the boundary conditions depended on the catheter radius. It is important to note that as we previously explained, the blood velocity strongly depends on the blood pressure force, and consequently, the convection term in the drug equation affects the outcome for the results for the drug. A conservative-implicit finite difference scheme was developed to numerically solve the drug concentration in the aforementioned annulus model.

Here, we briefly describe and discuss the main results of the work presented in [12], and we refer the reader to the original paper for more details on the data generation, computation, and the subsequent results. The results for the blood pressure force for different values of the axial variable, which was associated with the atherosclerosis, catheter radius, and the hematocrit showed that the blood pressure force was enhanced with an increasing catheter radius and hematocrit. It indicated that the blood pressure force intensified with an increasing number of red blood cells, which facilitated higher rates of change for the blood pressure. The blood pressure force intensified notability in the atherosclerosis zone, which was due to the fact that the artery was narrowed down in the damaged zone and, thus, a greater pressure driven force was required to drive the blood flow.

The results for the blood pressure versus the axial variable and for different values of the hematocrit indicated that the magnitude of the blood pressure increased with the hematocrit. This result indicates that a greater percentage of red cells in a patient's blood may indicate possible hypertension [1−3,15]. Additionally, it should be emphasized that as we referred to before, a realistic form of a mild atherosclerosis was used in this investigation, which was based on experimental data [15]. It is hoped that the applicability of such a type of realistic mild atherosclerosis can be used more in future modeling studies to provide more useful medical information for other patients with coronary artery disease due to the presence of atherosclerosis.

The authors in [12] also determined results for the velocity of the blood plasma compared to the axial variable and for different values of the catheter radius. Since blood pressure force was very strong in the atherosclerosis zone, the blood velocity and its speed had a high value in such a zone.

The authors found that the plasma velocity in the axial direction was positive since the blood flow was driven from left to right by the blood pressure force. The value of the plasma velocity was constant outside the damaged zone, though it was variable, with significantly higher values inside the atherosclerosis regime. This result was reasonable due to significantly higher values of pressure driven force. Additionally, the velocity of the plasma increased with the catheter radius since a higher value of such a radius reduces the cross-sectional area of the blood flow in the annulus region due to a fixed value of the flow rate [15]. Moreover, they calculated the plasma velocity for different values of the hematocrit and found that the plasma velocity slightly increased with an increasing hematocrit. Their calculated value for the red blood cells velocity qualitatively indicated the same value as that of the plasma velocity.

Next, the authors [12] reported their computational results for drug transport in the blood flow region for a given value of the diffusion coefficient in the drug equation. First, they considered the case of particular drug with a moderate value and for given values of the other parameters. They found that the drug was mostly delivered in the central part of the blood flow region after a short distance from which it was initiated. In the regions close to the catheter surface and close to the artery surface (and especially near the stenosis zone), there was very little amount of drug to be delivered, since

diffusion was more significant than the convection effect in the drug equation. Then, they computed drug transport in the blood flow region for a rather small, but not too small, value of the diffusion coefficient, though they kept the values of the other parameters the same as those for the previous computation. They found that the drug was delivered in more places in the blood flow region. This indicated that convection had more active role to deliver drug to more parts of the blood flow region mostly in its central location.

Next, the authors took a significantly small value of the diffusion coefficient, but kept the values of the other parameters the same as before. They found that the regions before and at the atherosclerosis zone were fully covered by the drug. This result indicated that the convection effect was now more significant and dominated over diffusion to deliver the drug. Additionally, it should be noted that the role of the blood pressure force drove the flow velocity and consequently affected the convection to play a significant role for transporting drug if diffusion was insignificant.

Related studies have been previously performed to model two-phase arterial blood flow in the presence of a catheter, as shown in the work of Srivastave and Rastogi [11]. These authors studied blood flow through a narrow, catheterized artery in the presence of axially non-symmetrical atherosclerosis, whose form was based on a mathematical model. They used a two-phase macroscopic model for the blood flow to represent plasma as one phase and red blood cells as another phase. They determined the solutions for the plasma velocity, particle velocity, flow rate, impedance (i.e., blood flow resistance) and wall shear stress, which were based on the simplified governing equations for the plasma and the particle (red blood cells) analytically. They found that impedance and the wall shear stress strongly increased with an increasing size of the catheter.

Comparing the results of the described models [11,12], which were under a condition of mild atherosclerosis and were based on the actual biological system, it should be noted that a number of important results were obtained from the described models, such as an increasing blood pressure force and its variations with respect to the presence of atherosclerosis, a variation of the blood pressure force with respect to the hematocrit parameter, and variations of the impedance and blood pressure force versus the catheter size, are all qualitatively in agreement with the results based on the corresponding biological system [1−3,15]. In addition, the important role of convection versus diffusion that was described in [12] agrees with the role of a biological system under similar conditions described for the model.

3.   Brief review of the work in [27]

Figure 3 presents a section of a very simple arterial blood flow model for the investigated work [27], where L_0 is the length of the atherosclerosis, r is the radial axis, and z is the axial axis of an assumed axisymmetric artery section, which we explained before to be reasonable.

The medium of the artery section was assumed to be porous due to the presence of fatty materials and artery clogging blood [19,28,29], which also contributed to the presence of atherosclerosis. The artery length was assumed to be very large, thus the end effects were neglected.

The unsteady blood flow in the artery carried solutes and the whole system was based on the equations for the solute concentration and the blood flow equations for continuity and Darcy-momentum [30,31]. Due to a very complex nature of the actual blood flow system and its circulatory motion in the human's artery, it is a formidable task to be able to analyze the actual blood flow motion [11]. Thus, a number of assumptions that were reasonably justified [32] were used to develop
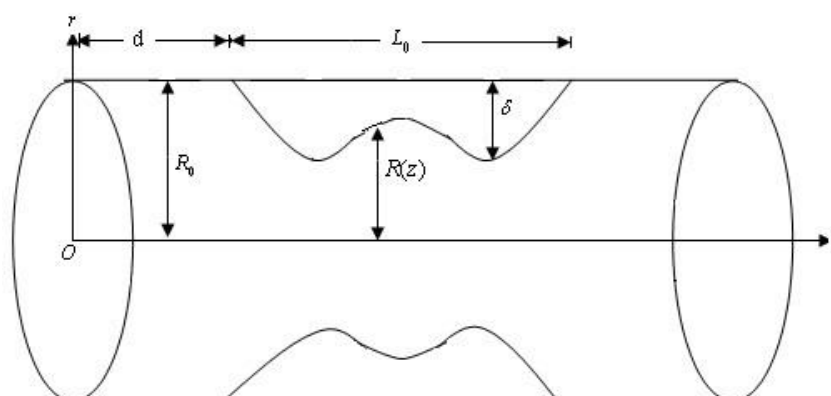
the studied model.

Figure 3. Modeling section of arterial blood flow system.

The governing system of equations for the blood flow and solute concentration was assumed to be in the axisymmetric form, as was justified in the experimental work [15]; the reader is referred to the original paper [27] for the actual original governing equations. The authors in [27] made such equations dimensionless, and made their forms as simple as possible under justifiable conditions for mild atherosclerosis, unidirectional flow, and a convective assumption [33], where the axial velocity component and the axial solute convection dominated over the radial velocity component and the radial solute convection term, and was subjected to the assumptions that the inertial terms in the momentum equations were small.

Following experimental evidence [15], it was assumed the aspect ratio parameter γ of the artery radius to the length of atherosclerosis was very small; therefore, the authors treated γ as a small parameter and applied a power series expansion of dependent variables of the blood flow and solute concentration using a perturbation procedure with γ as the perturbation parameter to determine solutions to such dependent variables of the unsteady system up to the order γ^2.

Following the earlier stated reasonable approximations and assumptions for the governing equations, the authors reduced those equations to simple forms involving only equations for dominant quantities for axial blood speed, blood pressure, and solute concentration. Then, these equations were solved, subjected to a constant volume blood flow rate up to the second order in the small parameter γ, and subjected to reasonable boundary conditions. Both analytical and computational methods including separation of variables, Simpson's rule, and the Runge-Kutta (RK) method of fourth order [34] were used to determine the unsteady solutions for axial blood flow speed, blood pressure, and solute concentration. The results depended on the values of the non-dimensional parameters of their final system; however, they include diffusivity and a chemical reaction parameter due the presence of solutes in the blood artery, the frequency of blood flow oscillation, hematocrit, and the atherosclerosis height. The shape of the internal surface of the atherosclerosis in the artery was based on a mathematical model that was further based on a polynomial of fourth order in an axial variable, which qualitatively represented the atherosclerosis adjacent to the inside surface of the artery.

An important aspect of the studied flow system was a strong dependence of the blood speed to

the blood pressure force as physically understandable due to the fact the blood flow in the artery was driven by the blood pressure force. In addition, the blood pressure force strongly influenced the blood velocity, which consequently affected the convection term in the solute equation; thus, the solution for the solute was dependent on the blood pressure force for cases where the convection dominated the diffusion effect [12].

Here, we briefly present the main results of their work. They investigated the unsteady blood flow and transport of a solute in the presence of the atherosclerosis effect. Their formulation was based on a variable viscosity formula to take the presence of red cells in the fluid plasma with the artery into account. For the experimental case of a small ratio value of the artery radius and axial extent of the atherosclerosis [15], they determined important time-dependent quantities in the porous artery. These quantities included solute transport, blood pressure force, blood speed, impedance, which is the magnitude of blood pressure difference at the two ends of the artery section, and wall shear stress force, which is the force acted by the blood flow on the internal surface of the artery.

They calculated the above-described quantities versus various values of time and space variables, as well different values of the parameters that represented effects due to the pulsation period, atherosclerosis, hematocrit, convection/diffusion of solute, and chemical reaction response due to the presence of solutes in the blood flow. In particular, they found that the chemical reaction reduced solute transport. An increased blood pulsation period increased the solute concentration, while higher amount of hematocrit reduced the transport of solute. In the local regions, solute transport was higher in the artery with peak atherosclerosis. Various quantities oscillated with the pulse frequency and were magnified for either a high convective transport, a weak reactive response, or low hematocrit. Increasing the value of the frequency of the pulsation decreased solute transport.

For the case where the solute convection dominated over conduction, the magnitude of the solute transport was higher in the early stage of the blood flow pulsation period. A weaker chemical reaction in the blood flow system increased solute transport with higher values during the initial stage of each pulse oscillation period. An increased amount of hematocrit increased the blood pressure force and the wall shear stress, though it decreased the blood speed and solute transport. Increased values of the volume flow rate increased the blood pressure force, blood speed, and the convective transport of solute. Solute transport was higher in the central region of the porous artery. Increased amounts of hematocrit in the blood reduced the convective transport of solute. The radial rate of change of solute transport notably decreased for a radial distance above a small value. This rate of decrease was higher for either very low diffusivity, hematocrit, or a very weak reactive response. The axial rate of change of solute transport increased close to the axial level of the maximum atherosclerosis height.

In the bio-mechanics problem investigated in [27], the authors considered blood flow and solute transport in a porous artery, where the solute was treated as a general one. Thus, the present results focused on the general features of various blood flow parameters and solute transport; this can provide useful information to biomedical areas in cardiovascular system. In addition, their results were needed to have some understanding when more specific cases for the solute could be considered, such as drug, oxygen, carbon dioxide, etc., in the patient's arterial blood flow system that may have other medical deficiencies, such as a very low hematocrit, etc. These particular cases could make the chemical reaction of the solute more interrelated to the patients' body metabolic system. Additionally, the results of these cases could stimulate experimental studies that could lead to medical and experimental generated data to identify components of the patients' arterial blood flow and diseases (organs and tissues). Subsequently, the outcome may facilitate diagnostic treatment and a better health outcome for

such patients.

The readers are referred to the full original paper [27] for further details regarding this work, as well as into more important applications in various areas including those in biomechanics, health, chemical engineering, and thermodynamics etc. Additionally, a comparison of the authors' results with other related studies by different authors were provided in their original paper [27], which we refer interested readers to this reference for more details.

Related studies were mostly performed for the blood flow, such as the work performed by Sinha and Shit [29]. These authors investigated blood flow in a porous medium subjected to an imposed magnetic field and in the presence of different shapes of mathematically constructed atherosclerosis. They used governing equations for the steady blood flow in a two-dimensional case and analytically solved an implemented stream function approach to determine the blood velocity. In particular, they found that the blood velocity can be controlled by a suitable choice of the strength of the magnetic field. Moreover, the axial component of the blood flow was notably reduced by the magnetic field, whose strength was sufficiently strong. The blood flow velocity and the wall shear stress were increased with an increasing value of the atherosclerosis peak.

In relation to the studies described in this section and those in a biological system, it should be noted that some prominent results described in this section about the effects associated to the hematocrit parameter, atherosclerosis peak, and the oscillation frequency all qualitatively agree with those for a biological system [1−3,15] that is under similar conditions as those in the modeling cases such as, for example, the arterial blood flow under mild atherosclerosis level, etc.

4. Brief review of the work in [13]

This work presents a problem of two-phase blood flow in a catheterized elastic artery with atherosclerosis. The use of a catheter is important as a standard tool for diagnosis and treatment in modern medicine. Transducers attached to a catheter are a great method to measure blood pressure in arteries. A catheter is composed of a polyester material. To reduce atherosclerosis, a catheter with a tiny balloon attached at the end is inserted in the artery and the balloon is inflated to fracture fatty deposits and widen the narrowed portion in the artery.

In a large number of previous studies for the arterial blood flow, including those given in Section 1, the shape of the atherosclerosis was based on assumed simple analytical functions. Here, the form and extent of atherosclerosis in the artery was chosen based on the available experimental data for a human's artery with atherosclerosis [15]. The governing equations for the two-phase blood flow in the elastic artery system were solved assuming reasonable modeling conditions and approximations such as those described in [12].

Since actual arteries in the human body are elastic in nature and has their own motion, the author in [14] performed some modeling efforts partly due to some previous experimental studies to apply elastic effects for of the artery; moreover, this study contained blood flow. The governing equations of motion for the displacement component along the axial and radial directions of the elastic arterial wall motion were modeled based on the earlier formulations, respectively [35,36]. The original motivation for the development of this model, whose system turns out to be somewhat similar to a mass-spring type system in mechanics, was originally based on the experimental results obtained in [37], which showed that within the physiological range of the flow circulation, longitudinal tethering of arteries of dogs can be simulated by a mechanical model consisting of a spring, dashpot, and a lumped additional

mass. In such a mechanical model, there were terms associated with the acceleration of the wall, stiffness of the wall, friction due to the elastic wall displacement, surface forces, and the forces of constraints exerted by the surrounding connective tissues.

Important quantities such as blood pressure force, plasma velocity, red blood cell velocity, blood flow resistance, which is basically blood pressure difference across the artery section, wall shear stress, and the radial and axial elastic artery's wall displacements were computed. It was found that the magnitude of the blood pressure force and the flow resistance increased with the hematocrit and atherosclerosis effects. The magnitude of the radial and axial displacements of the wall increased with the hematocrit, catheter size, volume flow rate, circumferential, and axial viscoelastic wall stress; however, these values decrease with an increasing wall stiffness and thickness. The elastic motion due to the stated displacements of the artery was found to enhance the blood pressure force and the blood speed by a small amount (at most); however, in the study reported in [14], it was shown that, at least for the case where the aspect ratio of the artery radius to length of atherosclerosis was very small, such an elastic effect of the artery on the blood flow parameters was found to be negligible. The reader is referred to the full work given in [14] for more details on the governing equations for the elastic displacement of the artery and the results etc.

Here, we provide the two main results in [14], as shown by Figure 4 and 5 for the rate of change of blood pressure and for the blood pressure difference across the section of the blood flow artery.
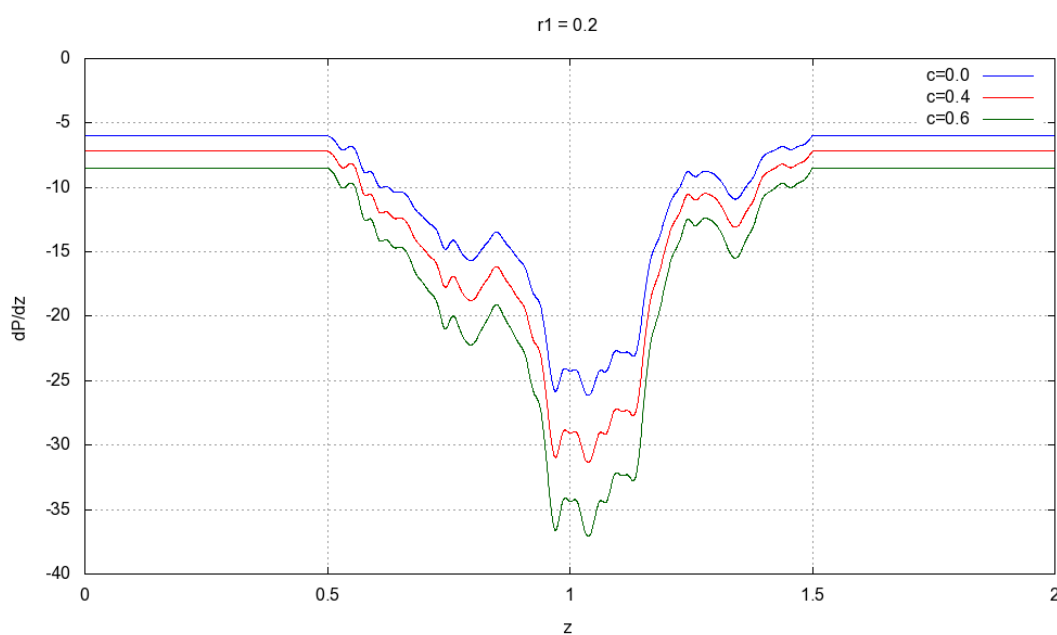

Figure 4. Axial rate of change of the blood pressure versus axial variable, where axial axis is along the central axis of the artery as shown in figure 3 in the artery section for given value of the catheter radius 0.2 and three different values of the hematocrit C.

From Figure 4, it is clear that the blood pressure force is very significant in the atherosclerosis zone, which is understandable due to the damaged zone in the artery, which is difficult to have normal blood circulation.
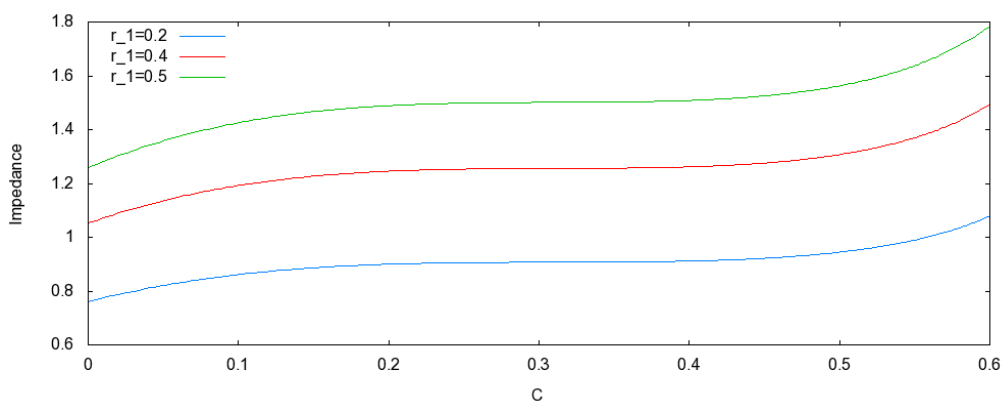
Figure 5. Blood pressure difference across the artery section versus hematocrit and for different values of catheter radius and given value of time.

From Figure 5, it can be seen that the blood pressure difference between the inlet and outlet sections of the artery increase with hematocrit and the catheter radius. These are due to the fact that the blood pressure force increases with an increasing hematocrit and such a force is stronger if the cross section of the artery becomes smaller for a given flow rate.

Related studies on the problem described in this section have very few articles, though here we briefly described a notable work [16] that is related to the elastic nature of the arteries. The author in [16] performed a theoretical investigation of blood flow in a section of arterioles. The author took the deformability of the vascular wall into account by considering an initially stressed elastic tube that contained an incompressible Newtonian viscous fluid. The author took the effect of connective tissues that surrounded the artery wall into account and formulated the governing equations for the blood flow and those for the displacement of the elastic boundary of the artery. Then, the author solved the governing equation for the stated problem and found that the amount of blood flow decreased with an increasing frequency of oscillation. The blood flow resistance was found to be periodic and reached a maximum at a low frequency of oscillation.

In regards to the modeling results and their relation to the biological system, it should be noted that there are important results obtained from the modeling system (described earlier in this section) which are in agreement with the physical problem. Modeling blood flow parameters, such as the hematocrit parameter, the frequency of oscillation, and the atherosclerosis nature, and its peak qualitatively agree with those from a biological system under a mild case of atherosclerosis [1−3,15].

In the next section, we provide a brief review of the work in [38], where we take the time dependent atherosclerosis and thermal effects in the elastic arterial blood flow problem into account. The roles that such effects could have on the blood pressure force and other blood flow parameters will be given in Section 5.

5. Brief review of the work in [38]

In a majority of the studies that have been performed so far on the modeling of blood flow in elastic arteries, the thermal effect and time dependent atherosclerosis were not taken into account. In the investigation reported in [38], the authors attempted to further gain an understanding of the problems of blood flow in an elastic artery with some implication to biomedical problems [39,40].

They took an unsteady atherosclerosis with either tapered (diverging or converging) or non-tapered shapes into account and determined its effects on the blood flow parameters, such as the blood pressure force, etc. In addition, they studied the thermal effect with a prescribed temperature value on the surface of the artery in order to determine information about the effects of the presence of temperature on the blood flow parameters and the total heat flux in the arterial blood flow system. The temperature effect was studied in [41] and was further extended by the authors' work on the blood flow through a circular cylindrical artery with a rigid wall.

The authors in [38] performed an investigation of the blood flow problem and the thermal effects for cases of an elastic artery with either a steady or time variant atherosclerosis. The blood flow in the artery, which was assumed to be in an axisymmetric cylindrical form similar to that of Figure 3, was represented by a two-phase type model that was composed of a particle phase (red blood cells) and a fluid (plasma) phase. The coupled system of differential equations for the blood plasma, red blood cells, and temperature [41], which was essentially a balance between convection, diffusion, and viscous dissipation terms, were solved mostly by analytical methods, where the original governing system was first subjected to reasonable modeling and approximations such as those we described before [12,27].

The important blood flow parameters, such as the blood pressure force and the blood flow resistance, together with blood temperature referenced to its given artery surface temperature, were computed for different cases of tapered and non-tapered time-dependent atherosclerosis. The form of such atherosclerosis was based on some mathematical model [42]. Their computation was performed for different parameter values such as those for hematocrit and the Eckert number that represented the temperature effects. For more details on their work, the reader is referred to their work [38].

The authors in [38] found that the axial rate of change of the blood pressure and the blood pressure difference across the artery section were more significant in the case of the converging artery and in the atherosclerosis zone. The later result was due to the fact that converging atherosclerosis made the artery cross section in the atherosclerosis zone narrower, which consequently resulted in a higher blood pressure. For the time dependent atherosclerosis case, the blood flow parameters, such as the blood pressure force and the blood flow resistance, were evaluated as compared to the time that was observed to be oscillatory with an exponentially time decaying behavior. Additionally, the heat transfer calculation was performed and indicated that the heat was in the outward direction from the artery system.

Related studies to those described in this section are few, though an interesting one, namely Mekheimer et al. [42], is described here. These authors took the presence of an induced magnetic field in the blood flow into account through a tapered elastic artery with overlapping atherosclerosis, whose form was based on a mathematical model. The geometry of the artery was assumed to be similar to a section of an annulus due to the presence of a catheter in a straight cylinder. Using an axisymmetric two-dimensional case for the governing equations for the blood flow and under a stream function formulation, the authors determined the solutions to the blood flow parameters. In particular, they found that wall shear stress had higher values for divergent tapering than for convergent one. The blood flow resistance was found to have higher values for convergent tapering, as well as for a higher atherosclerosis peak value.

Regarding relations of the modeling results that were provided in this section and the corresponding ones for a biological system, it should be noted that the main results of the modeling system presented in this section are about the effects of the most effective presence of atherosclerosis,

as well as an increased value of the atherosclerosis peak that can lead to a higher impedance and a higher blood pressure are valid for a biological system [1−3,15], provided that such a biological system be under the same condition considered as those for the modeling system.

6.   Concluding remarks

The problem for modeling blood flow in a partially blocked artery has made numerous advances in both the theoretical and computational aspects. In the model used in [12], which took data generated by an experimental work for the atherosclerosis from the casting of a man into account [12], a numerical solution to a system of partial differential equations (PDEs) governing the blood flow was obtained with a conservative finite difference scheme. In their paper, the model accounted for the axial velocity of red blood cells; however, due to simplifying assumptions, the radial contributions (flow towards the artery walls) were ignored. The authors have plans to carry out further investigations by improving the complexity of the models (more realistic). These plans can include modeling the radial velocity, which can affect the blood pressure force. Changes in the blood pressure force have been previously reported and attributed to the implemented atherosclerosis due to the experimental generated data [12]. The plans also include the implementation of different boundary conditions (no flux and mixed boundary conditions) and an investigation of their interaction and impact for drug delivery effectiveness in a blocked artery. The results will be interesting to see to what extent the blood pressure force can affect both axial and radial convection terms in the drug equation, which can consequently affect the drug transport to the damaged zone of the artery. In addition to the mathematical and computational modeling involved in carrying the next mentioned investigation, the authors will also explore the idea of developing a further advanced numerical approach with a higher order spatial discretization and a higher time stepping integration. This could be explored by considering a compact finite difference approach coupled with different time-stepping techniques. Higher order numerical methods will serve a very important purpose to provide higher accuracy solutions; thus, medical decisions and predictions can be made with more confidence. Additional investigations will be related to a model for more than one artery. Our goal is to model a network of arteries [43,44]; ultimately, we will seek to collaborate with medical doctors to generate patient-specific models in which an adequate decision or treatment can be made for patients with illnesses or complications in the heart and arteries. The authors would like to emphasize the importance of mathematical modeling and simulations in blood flow problems [9,11−14,16,18,19,36,38−40,42], which is the main focus of this paper. In general terms, the importance of simulations in biophysical applications (for many diseases and medical conditions) relies on the ability to adjust and calibrate models and methods [45] so that patients could receive the benefit from optimized medical services or procedures, which may include less invasive treatments.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors declare no conflict of interest.

Author contributions

Both authors contributed equally in the preparation and completion of this manuscript.

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