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***Research article***

# State estimation and optimal control of an inverted pendulum on a cart system with stochastic approximation approach

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**Abstract:** In this paper, optimal control of an inverted pendulum on a cart system is studied. Since the nonlinear structure of the system is complex, and in the presence of random disturbances, optimization and control of the motion of the system become more challenging. For handling this system, a discrete-time stochastic optimal control problem for the system is described, where the external force is considered as the control input. By defining a loss function, namely, the mean squared errors to be minimized, the stochastic approximation (SA) approach is applied to estimate the state dynamics. In addition, the Hamiltonian function is defined, and the first-order necessary conditions are derived. The gradient of the cost function is determined so that the SA approach is employed to update the control sequences. For illustration, considering the values of the related parameters in the system, the discrete-time stochastic optimal control problem is solved iteratively by using the SA algorithm. The simulation results show that the state estimation and the optimal control law design are well performed with the SA algorithm, and the motion of the inverted pendulum cart is addressed satisfactorily. In conclusion, the efficiency of the SA approach for solving the inverted pendulum on a cart system is verified.

**Keywords:** inverted pendulum; nonlinear optimal control; stochastic approximation approach; state estimation; simulation result

## 1. Introduction

In nonlinear control modeling, the example of controlling an inverted pendulum on a cart system is a popular model because the system presents the nonlinear dynamics of its motion with the angle and the position of the displacement. With a single control input, which is an external force, using this two-degree-of-freedom control task to handle the pendulum cart system in staying its steady state is not easy [1]. Moreover, in the presence of random disturbances in the system, the task of controlling the movement of the pendulum cart system becomes more challenging.

From past studies, the importance of studying the pendulum cart system has been clearly proven in various areas, e.g., rocket or missile guidance [2], robotics [3], circuit design [4], power system [5] and engine design [6]. A practical example is that a controller based on the linear quadratic Gaussian model was designed for balancing the inverted pendulum mobile robot [7]. In addition, the comparison of time specification performance of the controllers between the linear quadratic regulator and the proportional–integral–derivative controller for an inverted pendulum system [8] was made. By using the linear matrix inequality [9], the inverted pendulum on a cart system was controlled.

In this paper, the stochastic approximation (SA) approach [10] is proposed to handle the pendulum cart system, which is disturbed by the random Gaussian white noise. Since the information on the state of the system is incomplete, estimating the state trajectory would be the prominent work before controlling the movement of the system is addressed. From this point of view, we aim to investigate the practicality of the SA approach for state estimation of the pendulum cart system. So, a loss function is introduced, where the differences between the actual output information and the estimated output are minimized. Considering the necessary conditions, the state estimate updating equation, which is associated with the stochastic gradient of the loss function, is derived to capture the original state dynamics. Thus, the optimal control problem is formulated, and the Hamiltonian function is defined. The optimality conditions are derived such that the optimal control law is designed by updating the stochastic gradient of the cost function. As a result, the computational procedure is summarized as an iterative algorithm and is known as the SA for state-control (SASC) algorithm. For illustration, the values of the related parameters in the system are considered, and the simulation results are presented. Then, the performance of the approach proposed is observed, the optimal trajectories are discussed and the optimal cost is determined.

The rest of the paper is organized as follows. In Section 2, the mathematical model of the inverted pendulum on a cart system is discussed, and in the presence of the random disturbance, the discrete-time nonlinear stochastic optimal control problem of the system is described. In Section 3, the state estimation and the control law design based on the SA algorithm are discussed, and the calculation procedure is summarized as an iterative algorithm for solving the problem. In Section 4, the simulation results are presented, where the optimal trajectories of the state estimate and control are expressed to show the efficiency of the SA algorithm. Finally, some concluding remarks are made.

## 2. Inverted pendulum on a cart system

An inverted pendulum on a cart system is an inverted pendulum mounted on a motor-driven cart [11], as shown in Figure 1. Some assumptions for modeling this inverted pendulum system are given as follows:

- The pendulum rod is massless.
- The hinge is frictionless between the pendulum and the cart.
- The friction between the cart and the surface is neglected.

Denote $M$ as the cart mass and $m$ as the ballpoint mass at the upper end of the inverted pendulum, respectively, where $M > m$. There is an externally $x$-directed force $F$ on the cart, and a gravity force $mg$ always acts on the ballpoint mass.

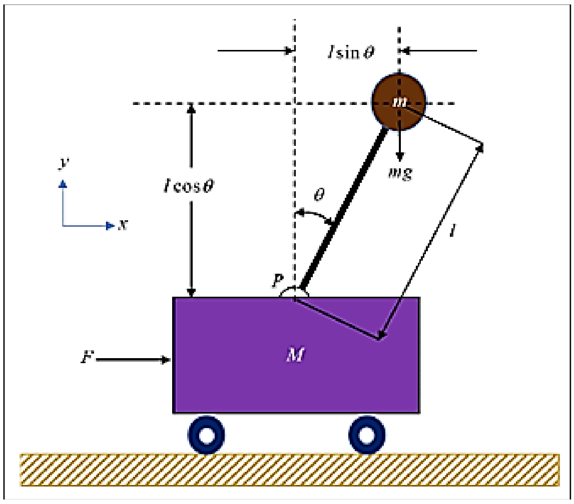


**Figure 1.** Inverted pendulum on a cart system.

Suppose $x(t)$ represents the cart position along the $x$ direction at time $t$, and $\theta(t)$ represents the tilt angle referenced to the vertically upward direction at time $t$. Let $(x_m, y_m)$ be the coordinate of the center of gravity of the point mass of the pendulum, and let $l$ be the length of the pendulum rod. The kinematics of the pendulum system is given by

$$x_m = x + l\sin\theta \text{ and } y_m = l\cos\theta, \tag{1}$$

while the velocity of the pendulum system is presented by

$$\dot{x}_m = \dot{x} + \dot{\theta}l\cos\theta \text{ and } \dot{y}_m = -\dot{\theta}l\sin\theta. \tag{2}$$

Consider the kinetic energy of the pendulum system

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2), \tag{3}$$

and substitute (2) in (3) to yield

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{x}\cos\theta, \tag{4}$$

while the potential energy of the pendulum system is

$$V = mgl\cos\theta. \tag{5}$$

Then, the Lagrangian, which summarizes the dynamics of the entire system, is defined by

$$L_1 = T - V. \tag{6}$$

According to the D'Alambert's principle [12], the Euler-Lagrange equations, which are used to derive the motion equations of the system, are written on the $x$ axis and the $\theta$ axis, respectively, by

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{x}}\right)-\frac{\partial L_1}{\partial x}=F\ , \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{\theta}}\right)-\frac{\partial L_1}{\partial \theta}=0\ , \tag{8}$$

where $F$ is the generalized force considered. By referring to (4), (5) and (6), the partial derivatives

$$\frac{\partial L_1}{\partial \dot{x}}=\frac{\partial T}{\partial \dot{x}}-\frac{\partial V}{\partial \dot{x}}=(M+m)\dot{x}+ml\dot{\theta}\cos\theta-0\ ,$$

$$\frac{\partial L_1}{\partial x}=\frac{\partial T}{\partial x}-\frac{\partial V}{\partial x}=0\ ,$$

$$\frac{\partial L_1}{\partial \dot{\theta}}=\frac{\partial T}{\partial \dot{\theta}}-\frac{\partial V}{\partial \dot{\theta}}=ml^2\dot{\theta}+ml\dot{x}\cos\theta-0\ ,$$

$$\frac{\partial L_1}{\partial \theta}=\frac{\partial T}{\partial \theta}-\frac{\partial V}{\partial \theta}=-ml\dot{\theta}\dot{x}\sin\theta+mgl\sin\theta$$

are resulted. Then, we expand (7) and (8) to obtain the following motion equations:

$$(M+m)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta=F, \tag{9}$$

$$ml^2\ddot{\theta}+ml\ddot{x}\cos\theta=mgl\sin\theta, \tag{10}$$

where (9) is a force balance on the system and (10) is a torque balance on the system. Taking some algebraic manipulations on (9) and (10), the dynamics of motion is given as follows:

(a) Cart position dynamics

$$\ddot{x}=\frac{F+ml\dot{\theta}^2\sin\theta-mg\cos\theta\sin\theta}{M+m-m(\cos\theta)^2}\ . \tag{11}$$

(b) Pendulum angle dynamics

$$\ddot{\theta}=\frac{F\cos\theta-(M+m)g\sin\theta+ml\dot{\theta}^2(\cos\theta\sin\theta)}{ml(\cos\theta)^2-(M+m)l}\ . \tag{12}$$

In the mathematical modeling of the inverted pendulum on a cart system, we aim to balance the pendulum in its equilibrium while the cart is moving. In this study, we consider the presence of random disturbance, such as the unsmooth path that is passed by the cart and the air blowing on the pendulum. The external force might be applied several times to the pendulum on a cart system so it can stay in a balanced situation in movement. The initial state is not necessarily a balanced situation because the initial state is a random vector, and the mean of the initial state is preferred to be used. Therefore, to optimize and control the inverted pendulum on a cart system in a stochastic environment, we shall define it as a stochastic optimal control problem.

Now, define the state variable $x_p=(x_1,x_2,x_3,x_4)^{\mathrm{T}}$ with $x_1=\theta$, $x_2=\dot{\theta}$, $x_3=x$ and $x_4=\dot{x}$, and let the control variable $u = F$ ; the dynamics of motions in (11) and (12) can be formulated in the state space representation given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ \dfrac{u\cos x_1 - (M+m)g\sin x_1 + mlx_2^2(\cos x_1 \sin x_1)}{ml(\cos x_1)^2 - (M+m)l} \\ x_4 \\ \dfrac{u + mlx_2^2(\sin x_1) - mg\cos x_1 \sin x_1}{M + m - m(\cos x_1)^2} \end{pmatrix}. \tag{13}$$

Therefore, the equivalent discrete-time state equation for (13) is written as

$$x_p(k+1) = f(x_p(k), u(k)), \tag{14}$$

where the system dynamics $f = (f_1, f_2, f_3, f_4)^{\mathrm{T}} : \Re^4 \times \Re \to \Re^4$ is given by

$$f_1 = x_1 + \tau \cdot x_2, \tag{15}$$

$$f_2 = x_2 + \tau \cdot \frac{u\cos x_1 - (M+m)g\sin x_1 + ml(\cos x_1 \sin x_1)x_2^2}{ml\cos^2 x_1 - (M+m)l}, \tag{16}$$

$$f_3 = x_3 + \tau \cdot x_4, \tag{17}$$

$$f_4 = x_4 + \tau \cdot \frac{u + ml(\sin x_1)x_2^2 - mg\cos x_1 \sin x_1}{M + m - m\cos^2 x_1}, \tag{18}$$

where $\tau$ is the sampling time, and the output variable $y_p = (y_1, y_2)^{\mathrm{T}}$ is defined by

$$y_p(k) = h(x_p(k)), \tag{19}$$

where the output measurement channel $h = (h_1, h_2)^{\mathrm{T}} : \Re^4 \to \Re^2$ is given by

$$h_1 = x_1 \text{ and } h_2 = x_3, \tag{20}$$

which represents the solution for the tilt angle $\theta$ and the cart position $x$, respectively.

In addition, considering the presence of random noises, the discrete-time system that consists of (14) and (19) is expressed by

$$x_p(k+1) = f(x_p(k), u(k), k) + G\omega(k), \tag{21}$$

$$y_p(k) = h(x_p(k)) + \eta(k), \tag{22}$$

where $G$ is a 4 × 4 coefficient matrix, whereas $\omega(k) \in \Re^4$, $k = 0,1,\cdots,N-1$, and $\eta(k) \in \Re^2$, $k = 0,1,\cdots,N$, are the additive Gaussian white noises with a zero mean and their respective covariance matrices are respectively given by $Q_\omega \in \Re^{4\times4}$ and $R_\eta \in \Re^{2\times2}$. Note that this discrete-time system is commonly known as the discrete-time stochastic dynamical system [13].

The initial state

$$x_p(0) = x_0 \tag{23}$$

is a random vector with the mean and state error covariance matrix that are, respectively, given by

$$E[x_0] = \bar{x}_0 \text{ and } E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^{\mathrm{T}}] = M_0, \tag{24}$$

where $M_0 \in \Re^{4\times4}$ is a positive definite matrix and $E[\cdot]$ is the expectation operator. It is assumed that the initial state, process noise and measurement noise are statistically independent.

Here, the aim is to determine a set of admissible control sequences $u(k) \in \Re,\ k = 0,1,\cdots,N-1,$ such that the expected cost function

$$J(u) = E\left[\varphi(x_p(N)) + \sum_{k=0}^{N-1} L(x_p(k),u(k))\right] \tag{25}$$

is minimized over the dynamical system given in (21) and (22), where $\varphi : \Re^4 \to \Re$ is the terminal cost; $L : \Re^4 \times \Re \to \Re$ is the cost under summation. This problem is regarded as the discrete-time nonlinear stochastic optimal control problem and is referred to as Problem (P).

## 3. Stochastic approximation approach

Now, consider the following recursive equation:

$$\theta^{(i+1)} = \theta^{(i)} - a_i \cdot g^{(i)}, \tag{26}$$

where $\theta^{(i)}$ is the set of the parameters to be estimated at the iteration $i$, $g^{(i)} = g(\theta^{(i)})$ is the stochastic gradient and $a_i$ is the gain sequence. Equation (26) is known as the SA approach [10]. Here, the state estimation and the optimal control design based on the SA approach will be further discussed.

### *3.1. State estimation*

Consider the state mean propagation [13] for (21) and (22), given by

$$\bar{x}_p(k+1) = f(\bar{x}_p(k),u(k)), \tag{27}$$

$$\bar{y}_p(k) = h(\bar{x}_p(k)), \tag{28}$$

where $\bar{x}_p(k)$ and $\bar{y}_p(k)$ are the expected state sequence and the expected output sequence, respectively. To find the optimal state estimate, we introduce the following weighted least-squares problem [14]:

$$\min_{x_p} J_{sse}(x_p) = \frac{1}{2}(x_p(k) - \bar{x}_p(k))^{\mathrm{T}}(M_0)^{-1}(x_p(k) - \bar{x}_p(k))$$

$$+\frac{1}{2}(y_p(k)-h(x_p(k)))^{\mathrm{T}}(R_\eta)^{-1}(y_p(k)-h(x_p(k))), \tag{29}$$

where $J_{sse}$ is the sum of square errors, $M_0$ is the state error covariance and $R_\eta$ is the output noise covariance. By taking the first-order derivative, the gradient of the sum of square errors is defined by

$$\nabla_{x_p} J_{sse}(x_p) = (M_0)^{-1}(x_p(k)-\bar{x}_p(k)) - C^{\mathrm{T}}(R_\eta)^{-1}(y_p(k)-h(x_p(k))), \tag{30}$$

where $C = \partial h/\partial x_p$ . Thus, using the SA approach in (26), the optimal state estimate

$$\hat{x}_p(k)^{(i+1)} = \hat{x}_p(k)^{(i)} - a_{1,i}\cdot\nabla_x J_{sse}(\hat{x}_p(k)^{(i)}) \tag{31}$$

is obtained, where $a_{1,i} > 0$ is the learning rate and the optimal output estimate is determined by

$$\hat{y}_p(k)^{(i)} = h(\hat{x}_p(k)^{(i)}). \tag{32}$$

**Remark**: For evaluating the gradient (30), the equation of the state error covariance matrix [13] is not required, and we only use the initial state error covariance.

*3.2. Optimality conditions*

Referring to Problem (P), the expected cost function (25) can be simply reformulated as

$$J(u) = \varphi(\bar{x}_p(N)) + \sum_{k=0}^{N-1} L(\bar{x}_p(k), u(k)). \tag{33}$$

So, it is measurable in the state mean propagated sequences. Define the Hamiltonian function [15]:

$$H(k) = L(\bar{x}_p(k), u(k)) + p(k+1)^{\mathrm{T}} f(\hat{x}_p(k), u(k)), \tag{34}$$

where $p(k) \in \Re^4$, $k = 0,1,\cdots,N$, is the costate sequence to be determined later. Then, the augmented cost function becomes

$$J'(u) = \varphi(\bar{x}_p(N)) + \sum_{k=0}^{N-1}(H(k) - p(k+1)^{\mathrm{T}}\bar{x}_p(k+1)). \tag{35}$$

Taking the first-order derivative of the Hamiltonian function (34) and the augmented cost function (35), the following optimality conditions [16,17] are derived.

(a) Stationary condition

$$\nabla_{u(k)} L(\bar{x}_p(k), u(k)) + \nabla_{u(k)} f(\hat{x}_p(k), u(k))^{\mathrm{T}} p(k+1) = 0. \tag{36}$$

(b) State equation

$$\bar{x}_p(k+1) = f(\hat{x}_p(k), u(k)). \tag{37}$$

(c) Costate equation

$$p(k) = \nabla_{x_p(k)} L(\bar{x}_p(k), u(k)) + \nabla_{x_p(k)} f(\hat{x}_p(k), u(k))^{\mathrm{T}} p(k+1). \tag{38}$$

(d) Output equation

$$\bar{y}_p(k) = h(\bar{x}_p(k)). \tag{39}$$

(e) Boundary conditions

$$\hat{x}_p(0) = \bar{x}_0 \text{ and } p(N) = \nabla_{x_p(N)} \varphi(\bar{x}_p(N)). \tag{40}$$

**Remark:** For convenience, the following standard cost function with the quadratic criterion [13,15] could be calculated when the proper cost function is not given:

$$\phi(\bar{x}_p(N)) = \frac{1}{2} \bar{x}_p(N)^{\mathrm{T}} S(N) \bar{x}_p(N), \tag{41}$$

$$L(\bar{x}_p(k), u(k)) = \frac{1}{2} (\bar{x}_p(k)^{\mathrm{T}} Q \bar{x}_p(k) + u(k)^{\mathrm{T}} R u(k)). \tag{42}$$

*3.3. Optimal control design*

Define an equivalent stochastic optimization problem [10,14] for Problem (P), and denote this problem as Problem (Q), as follows:

$$\text{Minimize} \qquad J'(u), \tag{43}$$

where the necessary conditions (37) and (38) are satisfied. Hence, the control law would be designed through solving Problem (Q). By virtue of this, the gradient of the objective function (43) is indeed expressed by

$$\nabla_u J'(u) = \nabla_u H(k), \tag{44}$$

where

$$\nabla_u H = \nabla_u L(\bar{x}(k), u(k)) + \nabla_u f(\hat{x}(k), u(k))^{\mathrm{T}} p(k+1) \tag{45}$$

is the derivative of the Hamiltonian function with respect to the control variable $u$, and the necessary condition for Problem (Q) is actually given by (36). Hence, the control law can be updated through the following recursive equation:

$$u(k)^{(i+1)} = u(k)^{(i)} - a_{2,i} \cdot \nabla_u J(u(k)^{(i)}), \tag{46}$$

where $a_{2,i} > 0$ is the learning rate.

**Remark:** The principle of separation is assumed to be satisfied when applying the SA approach for state estimation and optimal control design. This statement is true for solving stochastic optimal control problems.

### *3.4. Stochastic approximation algorithm*

From the discussion above, the computational procedure of the SA approach to state estimation and control law design is summarized as an iterative algorithm and has been named the SASC algorithm.

**The SASC algorithm**

Data Given $f$, $h$, $G$, $\varphi$, $L$, $N$, $M_0$, $Q_\omega$, $R_\eta$, $a_1$, $a_2$, $y_p$.

Determine the initial control $u(k)^{(0)} = u_0$ for $k = 0,1,\cdots,N-1$, and the initial state $\hat{x}_p(k)^{(0)} = x_0$ for $k = 0,1,...,N$. Set the tolerance $\varepsilon$ and the iteration $i$ := 0.

REPEAT

// **State estimation**

1 Calculate the sum squares of error $J_{sse}(\hat{x}_p(k)^{(i)})$ from (29), and the stochastic gradient $\nabla_{x_p} J_{sse}(\hat{x}_p(k)^{(i)})$ from (30), respectively.

2 Update the state estimate $\hat{x}_p(k)^{(i+1)}$ from (31).

3 Compute the output estimate $\hat{y}_p(k)^{(i)}$ from (32).

// **Solving two-state boundary value problem**

4 Solve the state equation forward in time from (37) with the given initial state $\bar{x}_0$ to obtain the state solution $\bar{x}_p(k)^{(i)}$.

5 Solve the costate equation backward in time from (38) with the given final costate $p(N)$ to provide the costate solution $p(k)^{(i)}$.

6 Compute the output measurement $\bar{y}_p(k)^{(i)}$ from (39).

**Optimal control law**

7 Calculate the augmented cost function $J(u(k)^{(i)})$ from (33) and calculate the stochastic gradient $\nabla_u J(u(k)^{(i)})$ from (45).

8 Update the control law $u(k)^{(i+1)}$ from (46).

Iteration $i := i + 1$;

UNTIL $|\hat{x}_p(k)^{(i+1)} - \hat{x}_p(k)^{(i)}| < \varepsilon$ and $|u(k)^{(i+1)} - u(k)^{(i)}| < \varepsilon$.

**Remark:** For simplification, the initial value of control and state can be set to a zero vector.

## 4. Illustrative example

Consider the model parameters in Problem (P), for which their values were taken from studies in [11,12] and given in Table 1, while the parameters that were set for estimation and control are provided in Table 2, where the term *diag* represents the diagonal matrix. Since the initial state is random and is not necessary to be the balanced situation of the system, the mean of the initial state is preferred to be used. In the beginning, the tilt angle deviates to the right side with 0.01 units and the cart position is moved to the left side with 1 unit before having any movement. This is given in the initial mean value. After that, the tilt angle starts to swing from the right side to the left side, while the cart is moving to the right side. Moreover, the presence of random disturbance, which can be the unsmooth path that is passed by the cart and the air blowing on the pendulum, was considered. The external force might be applied several times to the pendulum on a cart system so that it is in a balanced situation in movement. In this situation, we aim to balance the pendulum and the cart in their respective equilibriums. So, this inverted pendulum on a cart system was formulated as a stochastic optimal control problem, aiming at finding an optimal control law that is the optimal external force for balancing the pendulum and the cart at a minimum cost of the control effort.

**Table 1.** Model parameters and their values.

| Model Parameters | Values |
|---|---|
| Mass of the cart | $M = 0.6$ kg |
| Mass of the pendulum | $m = 0.3$ kg |
| Length of the pendulum | $l = 0.35$ m |
| Acceleration of gravity | $g = 9.81$ ms$^{-2}$ |

**Table 2.** Parameters for estimation and control.

| Estimation and Control Parameters | Values |
|---|---|
| Sampling time | $\tau = 0.05\,s$ |
| Covariance matrices | $M_0 = 0.2I_{4\times4},\ Q_\omega = 10^{-5}I_{4\times4},\ R_\eta = 10^{-5}I_{2\times2}$ |
| Weighting matrices | $Q = diag(10,\ 1,\ 100,\ 1)$ , $R = 0.1$ |
| Initial condition | $\bar{x}_1(0) = 0.01,\ \bar{x}_2(0) = 0,\ \bar{x}_3(0) = -1,\ \bar{x}_4(0) = 0$ |
| Final time step | $N = 100$ |

Simulation results obtained using the SASC algorithm are shown in Table 3. The optimal cost of 700.611 given by the SASC algorithm is the cost of designing the optimal control law to optimize and control the pendulum on a cart system. This optimal cost indicates that the SASC algorithm could handle the control problem of the system effectively. Moreover, the state trajectories are well-estimated using the SASC algorithm since the mean squared error (MSE) of $2.8634 \times 10^{-6}$ reveals that the estimated output trajectory is closely related to the actual output trajectory. In addition, the SASC algorithm spent 45 iterations to reach convergence, with an elapsed time of 1.439 seconds. Therefore, this demonstrates the efficiency of the SASC in handling the pendulum on a cart system with random disturbance.

**Table 3.** Simulation result.

| Algorithm | Optimal cost | MSE | Iterations | Elapsed time (s) |
|---|---|---|---|---|
| SASC | 700.6111 | $2.8634 \times 10^{-6}$ | 45 | 1.439 |

Figure 2 shows the final output trajectories for the solutions of the tilt angle $\theta(t)$ and the cart position $x(t)$, which are represented by $y_1$ and $y_2$, respectively. From Figure 2(a), the tilt angle deviated to the positive direction with an amplitude of 0.5 units and then moved to the negative direction with an amplitude of 0.3 units. After 20 seconds, movement of the tilt angle $\theta(t)$ started to move toward the equilibrium position, and the pendulum stayed at the equilibrium position for 60 seconds with a slight fluctuation. Referring to Figure 2(b), in the beginning, the cart moved toward the left side for the first 5 seconds, and then it started to move to the right side. After 20 seconds, the cart was reaching a stable position.

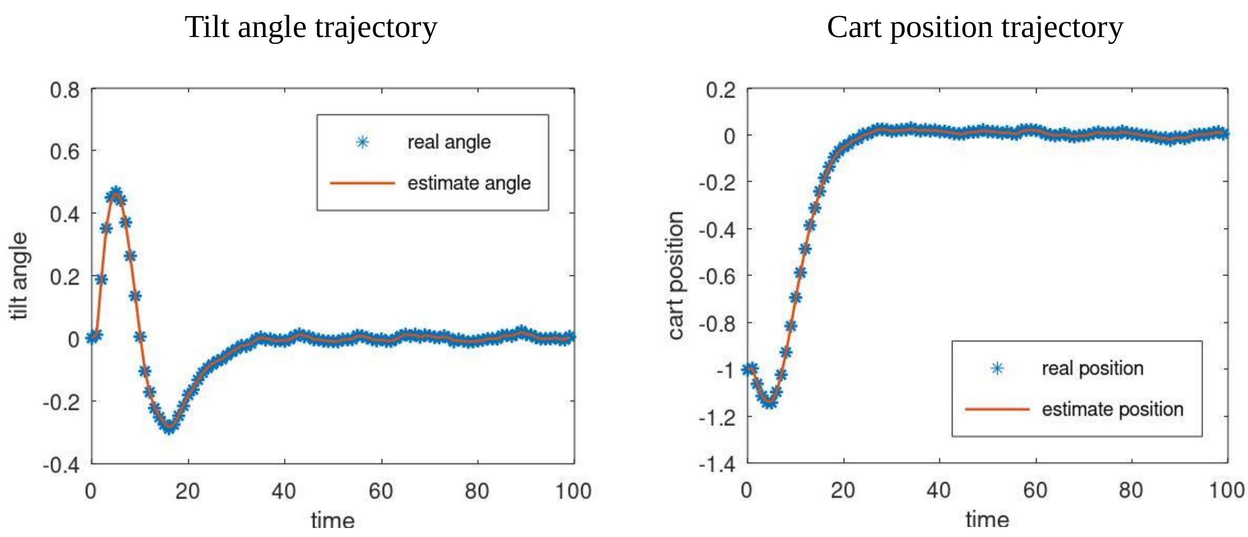


**Figure 2.** Final output trajectory and real output trajectory.

Figure 3 shows the final state estimate trajectories for tilt angle, angular velocity, cart position and cart velocity. The tilt angle $\theta(t)$ represented by $x_1$ had a positive amplitude of 0.5 units and was followed by a negative amplitude of 0.3 units in the first 20 seconds. After 40 seconds, the pendulum stayed at the equilibrium position with a small tilt angle. Thus, the state estimate of the tilt angle tracked accurately the real state of the tilt angle in the duration given. The angular velocity of the tilt angle $\dot{\theta}(t)$ is represented by $x_2$. In the first 20 seconds, the angular velocity reached the maximum positive velocity of 4 units per second, and then it reduced to the maximum negative velocity of 2.5 units per second. The pendulum was staying at the equilibrium position after 40 seconds.

In addition, the cart position $x$ represented by $x_3$ moved to the left at the position of 1.14 units in the first 6 seconds, and then it moved to the right toward the equilibrium position. The cart was estimated to reach the equilibrium position after 20 seconds. The cart velocity $\dot{x}$ represented by $x_4$ was estimated closely. In the first 20 seconds, the cart moved to the left with a maximum velocity of 1.2 units per second, and it moved to the right to reach a maximum velocity of 2.2 units per second. The cart was estimated to reach the equilibrium position with a velocity estimate similar to the actual cart velocity given after 20 seconds.

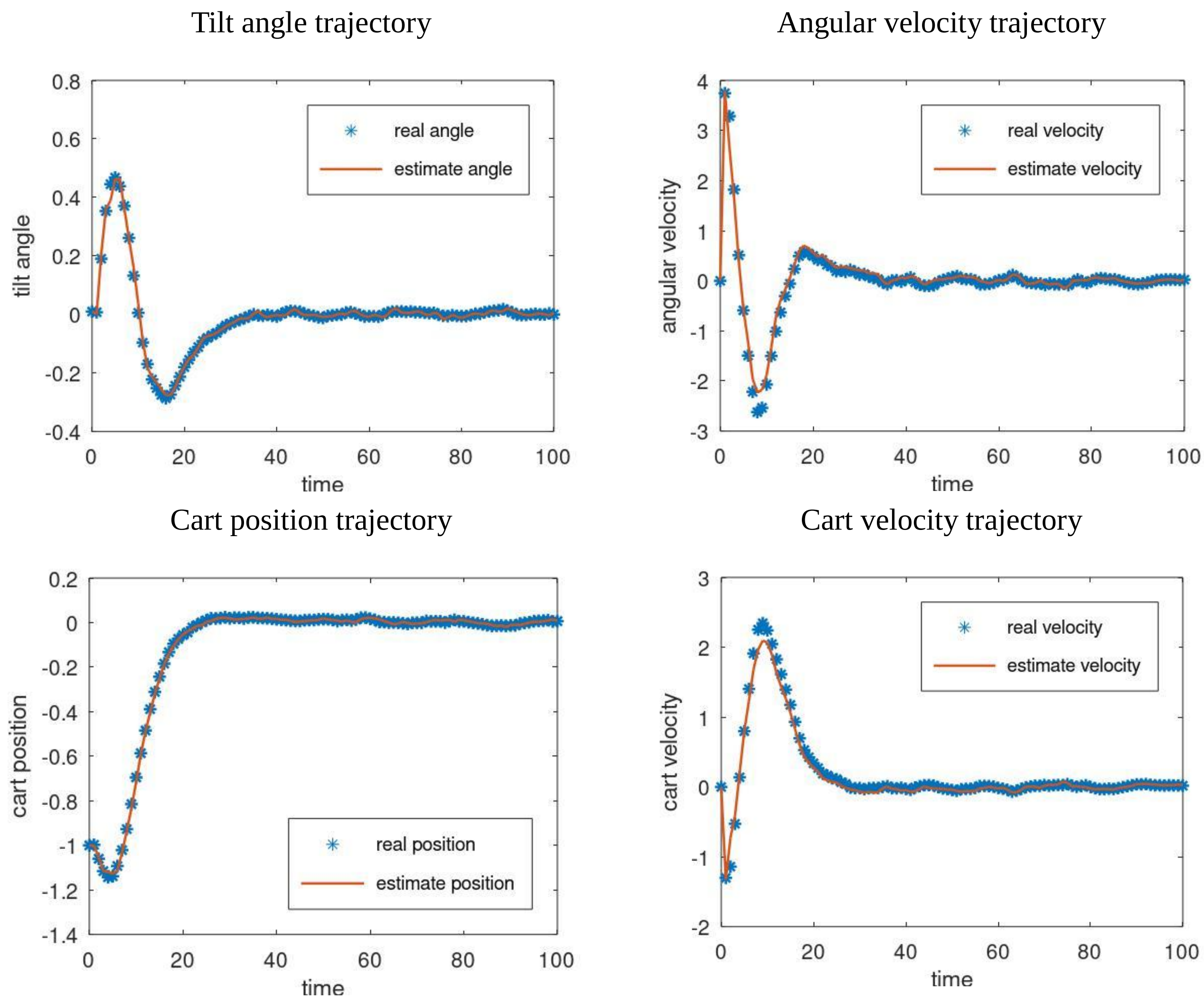


**Figure 3.** Final state trajectories and original state trajectories.

Figure 4 shows the final control trajectory given by $u$. The control effort was taken from the negative value of 15.9 units. It was increased to the positive value of 10.6 units in the first 5 seconds and reduced to the negative value of 4.5 units before 20 seconds. Later, the control effort started to increase toward the equilibrium position around zero, with a slight fluctuation. This reveals that the control effort was used to regulate the state trajectories to reach a stable position. Because of the presence of random white noises, the stationary condition, which is represented by the gradient of the Hamiltonian function as shown in Figure 5, showed a sharp reduction to the negative value of 1.45 units in the first 6 seconds. After that, the gradient tended to zero and approximated at zero along the $x$ axis, with a slight fluctuation after 10 seconds. With this, we have verified that the solutions to the tilt angle and cart position were optimal and the optimal cost is 700.6111 units using the SASC algorithm.

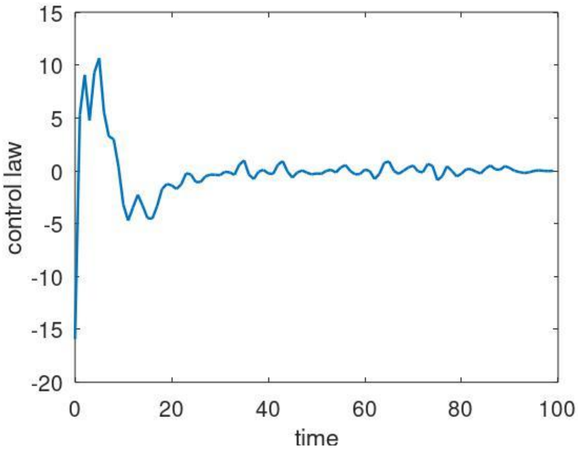


**Figure 4.** Final control trajectory.

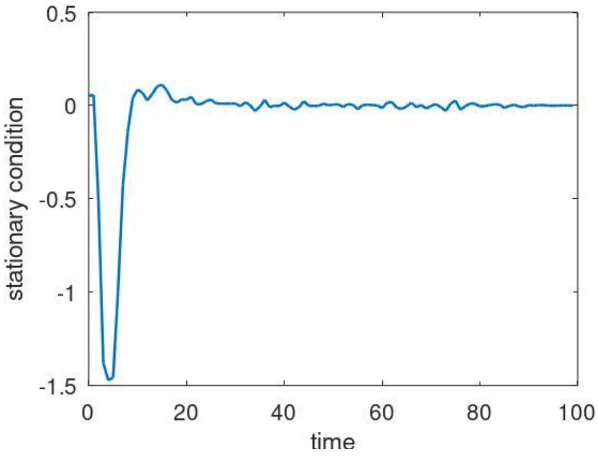


**Figure 5.** Stationary condition.

## 5. Concluding remarks

Optimal control of the inverted pendulum on a cart system with random disturbance was discussed in this paper. By considering random disturbances in the model, the inverted pendulum on a cart system was described as a stochastic optimal control problem. The SA method, which is termed the SASC algorithm, was discussed, aiming at state estimation and control law design for solving the problem formed. For illustration, the related value of model parameters has been used for the study, and the simulation results showed that the SASC algorithm proposed is efficient in handling the inverted pendulum on a cart system, where the trajectories of state, output and control were well demonstrated. In conclusion, the application of the SASC algorithm to the discrete-time nonlinear stochastic system is comprehensively presented. For future research, it is recommended to apply some latest stochastic gradient descent techniques to solve the stochastic optimal control problems.

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## Conflict of interest

All authors declare no conflict of interest regarding the publication of this paper.

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