Supplementary Material

Detailed Proofs of theorems in the paper

A multi-strain sequential super-infection model for dengue fever with antibody dependent enhancement

Contents

1 Existence of the Endemic Equilibrium Point

1.1 Case 1: Considering Primary Infection Dynamics

We now consider the primary infection dynamics only as the initial case where $S_p \neq 0$, $A_p \neq 0$, $I_p \neq 0$ and $R_p \neq 0$ then the system admits an endemic equilibrium point given by:

$$S_p^* = \frac{\pi}{\mu + \lambda_p^*}, \quad A_p^* = \frac{\theta\lambda_p^*\pi}{(\mu+\gamma)(\mu+\lambda_p^*)}, \quad I_p^* = \frac{\rho\lambda_p^*\pi}{(\mu+\sigma)(\mu+\lambda_p^*)}, \quad R_p^* = \frac{\gamma A_p^* + \sigma I_p^*}{(\mu+\nu)}$$

Now, considering the force of infection for the vector population we have,

$$\lambda_v^* = \frac{b}{N_h^*}\Big(\beta_{vp}(I_p^* + \eta A_p^*)\Big) \tag{1.1}$$

Substituting for I_p^* and A_p^* in the force of infection yields,

$$\lambda_v^* = \frac{b\beta_{vp}}{N_h^*}\left(\frac{\rho\pi\lambda_p^*}{(\mu+\nu)(\mu+\lambda_p^*)} + \frac{\eta\theta\pi}{(\mu+\gamma)(\mu+\lambda_p^*)}\right) \tag{1.2}$$

but we know that the primary force of infection in human beings at the endemic equilibrium is given by $\lambda_p^* = \frac{b\beta_{hp}I_v^*}{N_h^*}$.

Back substitution for λ_p^* in the above expression yields:

$$\lambda_v^* = \frac{b^2\rho\pi\beta_{vp}\beta_{hp}I_v^*}{N_h^*(\mu+\sigma)(\mu N_h^* + b\beta_{hp}I_v^*)} + \frac{b^2\eta\theta\pi\beta_{vp}\beta_{hp}I_v^*}{N_h^*(\mu+\gamma)(\mu N_h^* + b\beta_{hp}I_v^*)} \tag{1.3}$$

From the two equations in vector dynamics we establish that

$$S_v^* = \frac{B_v}{\mu_v + \lambda_v^*} \quad \text{and} \quad I_v^* = \frac{\lambda_v^* B_v}{\mu_v(\mu_v + \lambda_v^*)}$$

$$\mu N_h^*\lambda_v^* + b\beta_{hp}I_v^*\lambda_v^* = \frac{b^2\beta_{vp}}{N_h^*}\left(\frac{\pi\rho(\mu+\gamma) + \pi\theta\eta(\mu+\sigma)}{(\mu+\sigma)(\mu+\gamma)}\right)\beta_{hp}I_v^* \tag{1.4}$$

Substituting with I_v^* in the above expression yields the following:

$$\mu(\mu_v + \lambda_v^*)N_h^*\lambda_v^* + \frac{b\beta_{hp}\lambda_v^{*2}B_v}{\mu_v} = \frac{b^2\beta_{vp}\beta_{hp}B_v}{N_h^*\mu(\mu+\sigma)(\mu+\gamma)}\Big(\pi\rho(\mu+\gamma) + \pi\theta\eta(\mu+\sigma)\Big)\lambda_v^* \tag{1.5}$$

Further algebra and simplification yields $\lambda_v^* = 0$ and

$$\lambda_v^* = \left(\frac{\mu\mu_v N_h^*}{\mu N_h^* + \dfrac{b\beta_{hp}B_v}{\mu_v}}\right)\left(\frac{b^2\beta_{vp}\beta_{hp}B_v\pi}{\mu\mu_v^2(\mu+\sigma)(\mu+\gamma)} \cdot \frac{[\rho(\mu+\gamma) + \eta\theta(\mu+\sigma)]}{N_h^{*2}} - 1\right) \tag{1.6}$$

but we established that

$$R_0 = \sqrt{\frac{b^2\mu\beta_{hp}\beta_{vp}B_v[\rho(\mu+\gamma)+\eta\theta(\mu+\sigma)]}{\pi\mu_v^2(\mu+\gamma)(\mu+\sigma)}}$$

thus

$$\lambda_v^* = \left(\frac{\mu\mu_v N_h^*}{\mu N_h^* + \dfrac{b\beta_{hp}B_v}{\mu_v}}\right)(R_0 - 1) \tag{1.7}$$

1.2 Case 2: Assuming Negligible deaths in overall Disease dynamics

Now we consider the infection dynamics for the whole model and we note that in addition to the primary infection equilibrium, the model will now admit both primary and secondary infections as the endemic state with secondary infections given as $S_s \neq 0$, $A_s \neq 0$, $V \neq 0$, $I_s \neq 0$ and $R_s \neq 0$, then the system admits an endemic equilibrium point given by:

$$S_s^* = \frac{(1-\omega)\nu R_p^* + \kappa V^*}{\lambda_s^* + \mu}, \quad A_s^* = \frac{\theta\lambda_s^* S_s^*}{(\mu+\gamma)}, \quad I_s^* = \frac{\rho\lambda_s^* S_s^*}{(\mu+\alpha+\delta)}, \quad V^* = \frac{\omega\nu R_p^*}{(\mu+\kappa)}$$

Now considering force of infection for the vector population we have,

$$\lambda_v^* = \frac{b\beta_{vp}(I_p^* + \eta A_p^*) + b\beta_{vs}(I_s^* + \eta A_s^*)}{N_h^*} \tag{1.8}$$

Substituting for I_p^*, A_p^*, I_s^* and A_s^* in the force of infection and carrying out some algebraic simplifications we get:

$$\begin{aligned}\lambda_v^* =& \frac{b}{N_h^*}\left[\beta_{vp}\left(\frac{\rho\lambda_p^*\pi}{(\mu+\sigma)(\mu+\lambda_p^*)} + \frac{\eta\theta\pi\lambda_p^*}{(\mu+\gamma)(\mu+\lambda_p^*)}\right)\right] \\ &+ \frac{b}{N_h^*}\left[\beta_{vs}\left(\frac{\rho\psi\lambda_p^* S_s^*}{(\mu+\alpha+\delta)} + \frac{\eta\theta\psi\lambda_p^* S_s^*}{(\mu+\gamma)}\right)\right]\end{aligned} \tag{1.9}$$

where

$$\begin{aligned}S_s^* =& \frac{(1-\omega)\nu}{\psi\lambda_p^*+\mu}\left(\frac{\gamma\theta\pi\lambda_p^*}{(\mu+\nu)(\mu+\gamma)(\mu+\lambda_p^*)}\right) + \frac{(1-\omega)\nu}{\psi\lambda_p^*+\mu}\left(\frac{\sigma\rho\pi\lambda_p^*}{(\mu+\nu)(\mu+\sigma)(\mu+\lambda_p^*)}\right) \\ &+ \frac{\omega\kappa\nu}{(\psi\lambda_p^*+\mu)(\mu+\kappa)}\left(\frac{\gamma\theta\pi\lambda_p^*}{(\mu+\nu)(\mu+\gamma)(\mu+\lambda_p^*)}\right) + \frac{\kappa}{\psi\lambda_p^*+\mu}\left(\frac{\omega\nu\sigma\rho\pi\lambda_p^*}{(\mu+\kappa)(\mu+\nu)(\mu+\sigma)(\mu+\lambda_p^*)}\right)\end{aligned} \tag{1.10}$$

and

$$\frac{\lambda_s^*}{\lambda_p^*} = \frac{b\phi\beta_{hs}I_v^*}{b\beta_{hp}I_v^*} = \frac{\phi\beta_{hs}}{\beta_{hp}} \quad \Rightarrow \quad \lambda_s^* = \frac{\phi\beta_{hs}}{\beta_{hp}}\lambda_p^* = \psi\lambda_p^* \quad \text{where} \quad \psi = \frac{\phi\beta_{hs}}{\beta_{hp}}$$

We note that $\lambda_p^* = \frac{b\beta_{hp}I_v^*}{N_h^*}$.

From the two equations in vector dynamics we establish that $S_v^* = \frac{B_v}{\mu_v+\lambda_v^*}$ and $I_v^* = \frac{\lambda_v^* B_v}{\mu_v(\mu_v+\lambda_v^*)}$. Further simplification of λ_p^* after back substitution of S_v^* and I_v^* yields:

$$\lambda_p^* = \frac{b\beta_{hp} B_v \lambda_v^*}{N_h^* \mu_v(\mu_v + \lambda_v^*)} \tag{1.11}$$

Rearranging the above equation and making λ_v^* the subject of the formula yields:

$$\lambda_v^* = \frac{N_h^* \mu_v^2 \lambda_p^*}{(b\beta_{hp} B_v - N_h^* \mu_v \lambda_p^*)} \tag{1.12}$$

Substituting Equation 1.10 into Equation 1.9 and also substituting Equation 1.12 into Equation 1.9 yields a polynomial in terms of λ_p^*.

Simplification of the terms we get the equation:

$$\begin{aligned}\frac{A\lambda_p^*}{B - P\lambda_p^*} &= \frac{C\lambda_p^*}{\mu + \lambda_p^*} + \frac{D\lambda_p^*}{\mu + \lambda_p^*} \\ &+ E\lambda_p^*\left(\frac{G\lambda_p^* + H\lambda_p^* + M\lambda_p^*}{(\psi\lambda_p^* + \mu)(\mu + \lambda_p^*)}\right) \\ &+ F\lambda_p^*\left(\frac{G\lambda_p^* + H\lambda_p^* + M\lambda_p^*}{(\psi\lambda_p^* + \mu)(\mu + \lambda_p^*)}\right)\end{aligned} \tag{1.13}$$

With coefficients:

$$\begin{aligned}&A = \mu_v^2 N_h^*, \quad B = b\beta_{hp}B_v, \quad C = \frac{b\beta_{vp}\rho\pi}{N_h^*(\mu+\sigma)}, \quad D = \frac{b\eta\theta\pi}{N_h^*(\mu+\gamma)}, \\ &E = \frac{b\beta_{vs}\rho\psi}{N_h^*(\mu+\alpha+\delta)}, \quad F = \frac{b\beta_{vs}\eta\theta\psi}{N_h^*(\mu+\gamma)}, \quad G = \frac{(1-\omega)\nu\gamma\theta\pi}{(\mu+\nu)(\mu+\gamma)}, \\ &H = \frac{\sigma\rho\pi(1-\omega)\nu}{(\mu+\nu)(\mu+\sigma)}, \quad M = \frac{\omega\nu\gamma\theta\pi\kappa}{(\mu+\kappa)(\mu+\nu)(\mu+\gamma)}, \\ &N = \frac{\omega\nu\sigma\rho\pi\kappa}{(\mu+\kappa)(\mu+\nu)(\mu+\sigma)}, \quad P = N_h^*\mu_v\end{aligned}$$

Further simplification of Equation 1.13 yields:

$$\begin{aligned}&(A\psi + PEG + PEH + PFG + PFH + PEM + PFM)\lambda_p^{*3} \\ &+ (A\mu\psi + A\mu + PC\psi + PC\mu + PD\psi + PD\mu \\ &\quad - BC\psi - BD\psi - BEG - BEH - BEM - BFG - BFH - BFM)\lambda_p^{*2} \\ &+ (A\mu^2 - BC\mu - BD\mu)\lambda_p^* = 0\end{aligned} \tag{1.14}$$

which is analogous to:

$$\lambda_p^*(A_0\lambda_p^{*2} + A_1\lambda_p^* + A_2) = 0 \tag{1.15}$$

With the coefficients given as follows:

$$\begin{aligned}A_0 &= \frac{\mu_v^2\pi}{\mu} + \frac{b\mu_v\beta_{vs}\rho\psi(1-\omega)\nu\theta\gamma\pi}{(\mu+\alpha+\delta)(\mu+\nu)(\mu+\gamma)} + \frac{b\mu_v\beta_{vs}\rho^2\psi\sigma\pi(1-\omega)\nu}{(\mu+\alpha+\delta)(\mu+\nu)(\mu+\sigma)}\\ &+ \frac{b\mu_v\beta_{vs}\eta\theta\psi(1-\omega)\nu\gamma\theta\pi}{(\mu+\gamma)^2(\mu+\nu)} + \frac{b\mu_v\beta_{vs}\eta\theta\psi\sigma\rho\pi\nu(1-\omega)}{(\mu+\gamma)(\mu+\nu)(\mu+\sigma)}\\ &+ \frac{b\mu_v\beta_{vs}\rho\psi\omega\nu\gamma\theta\pi\kappa}{(\mu+\alpha+\delta)(\mu+\kappa)(\mu+\nu)(\mu+\gamma)} + \frac{b\mu_v\beta_{vs}\eta\theta\psi\omega\nu\gamma\theta\pi\kappa}{(\mu+\gamma)^2(\mu+\kappa)(\mu+\nu)}\end{aligned}$$

$$\begin{aligned}A_1 = &\frac{(\mu_v^2\pi\psi + \mu_v^2\pi)(\mu+\gamma)(\mu+\sigma) + \mu_v\psi b\beta_{hp}\rho\pi + \mu\mu_v b\beta_{vp}\rho\pi + \mu_v\psi b\eta\theta\pi + \mu\mu_v b\eta\theta\pi}{(\mu+\sigma)(\mu+\gamma)}\\ &- \mu b^2\beta_{hp}\psi B_v\left(\frac{\beta_{vp}\rho(\mu+\gamma) + \eta\theta(\mu+\sigma)}{(\mu+\sigma)(\mu+\gamma)}\right)\\ &- \frac{\mu b^2\beta_{hp}\nu\gamma\theta\beta_{vs}\rho\psi B_v}{(\mu+\alpha+\delta)(\mu+\nu)(\mu+\gamma)}\cdot\frac{\omega\kappa(\mu+\gamma)+(1-\omega)(\mu+\kappa)}{(\mu+\kappa)(\mu+\gamma)}\\ &- \frac{b^2\beta_{hp}\beta_{vs}\eta\theta\psi\nu B_v}{(\mu+\nu)(\mu+\gamma)}\cdot\frac{\sigma\rho(1-\omega)(\mu+\kappa)+\omega\gamma\kappa\theta(\mu+\sigma)}{(\mu+\sigma)(\mu+\kappa)}\end{aligned}$$

$A_2 = (1 - R_0)$

2 Global Stability Analysis

Definition 1. *We define the Volterra-Lyapunov function as:*

$$L = \sum_{i=1}^{12} b_i(x_i - x_i^* \ln x_i) \tag{2.1}$$

Which in extended form is given as:

$$\begin{aligned}L = &\, b_1\left(S_p - S_p^*\ln S_p\right) + b_2\left(A_p - A_p^*\ln A_p\right) + b_3\left(I_p - I_p^*\ln I_p\right)\\ &+ b_4\left(R_p - R_p^*\ln R_p\right) + b_5\left(S_s - S_s^*\ln S_s\right) + b_6\left(A_s - A_s^*\ln A_s\right)\\ &+ b_7\left(I_s - I_s^*\ln I_s\right) + b_8\left(I_h - I_h^*\ln I_h\right) + b_9\left(R_s - R_s^*\ln R_s\right)\\ &+ b_{10}\left(V - V^*\ln V\right) + b_{11}\left(S_v - S_v^*\ln S_v\right) + b_{12}\left(I_v - I_v^*\ln I_v\right)\end{aligned} \tag{2.2}$$

where $b_i > 0$ are positive constants to be determined.

Now computing the time derivative, $\frac{dL}{dt}$ along the trajectories of Model 2.1 we obtain:

$$\begin{aligned}\frac{dL}{dt} &= b_1\left(1-\frac{S_p^*}{S_p}\right)\frac{dS_p}{dt} + b_2\left(1-\frac{A_p^*}{A_p}\right)\frac{dA_p}{dt} + b_3\left(1-\frac{I_p^*}{I_p}\right)\frac{dI_p}{dt}\\ &+ b_4\left(1-\frac{R_p^*}{R_p}\right)\frac{dR_p}{dt} + b_5\left(1-\frac{S_s^*}{S_s}\right)\frac{dS_s}{dt} + b_6\left(1-\frac{A_s^*}{A_s}\right)\frac{dA_s}{dt}\\ &+ b_7\left(1-\frac{I_s^*}{I_s}\right)\frac{dI_s}{dt} + b_8\left(1-\frac{I_h^*}{I_h}\right)\frac{dI_h}{dt} + b_9\left(1-\frac{R_s^*}{R_s}\right)\frac{dR_s}{dt}\\ &+ b_{10}\left(1-\frac{V^*}{V}\right)\frac{dV}{dt} + b_{11}\left(1-\frac{S_v^*}{S_v}\right)\frac{dS_v}{dt} + b_{12}\left(1-\frac{I_v^*}{I_v}\right)\frac{dI_v}{dt} \qquad (2.3)\end{aligned}$$

Now using Model 2.1 equations, we substitute into the derivative and we get:

$$\begin{aligned}\frac{dL}{dt} &= b_1\left(1-\frac{S_p^*}{S_p}\right)[b-\lambda_p S_p-\mu S_p] + b_2\left(1-\frac{A_p^*}{A_p}\right)[\theta\lambda_p S_p-(\mu+\gamma)A_p]\\ &+ b_3\left(1-\frac{I_p^*}{I_p}\right)[\rho\lambda_p S_p-(\mu+\sigma)I_p] + b_4\left(1-\frac{R_p^*}{R_p}\right)[\gamma A_p+\sigma I_p-(\mu+\nu)R_p]\\ &+ b_5\left(1-\frac{S_s^*}{S_s}\right)[(1-\omega)\nu R_p+\kappa V-\lambda_s S_s-\mu S_s]\\ &+ b_6\left(1-\frac{A_s^*}{A_s}\right)[\theta\lambda_s S_s-(\mu+\gamma)A_s] + b_7\left(1-\frac{I_s^*}{I_s}\right)[\rho\lambda_s S_s-(\mu+\alpha+\delta)I_s]\\ &+ b_8\left(1-\frac{I_h^*}{I_h}\right)[\delta I_s-(\mu+\epsilon+d)I_h]\\ &+ b_9\left(1-\frac{R_s^*}{R_s}\right)[\gamma A_s+\alpha I_s+\epsilon I_h-\mu R_s] + b_{10}\left(1-\frac{V^*}{V}\right)[\omega\nu R_p-(\mu+\kappa)V]\\ &+ b_{11}\left(1-\frac{S_v^*}{S_v}\right)[B_v-\lambda_v S_v-\mu_v S_v] + b_{12}\left(1-\frac{I_v^*}{I_v}\right)[\lambda_v S_v-\mu_v I_v] \qquad (2.4)\end{aligned}$$

2.1 Coefficients of the Lyapunov Function

The coefficients are chosen so as to guarantee the cancellation of terms and achieve $\frac{dL}{dt} \leq 0$, which is a necessary condition for the Lyapunov function. Thus we choose the Lyapunov coefficients as follows:

Lemma 1. *The coefficients b_i are chosen to satisfy:*

$$b_1 = 1 \tag{2.5}$$

$$b_2 = \frac{b_1 \theta \lambda_p^*}{\mu + \gamma} \tag{2.6}$$

$$b_3 = \frac{b_1 \rho \lambda_p^*}{\mu + \sigma} \tag{2.7}$$

$$b_4 = \frac{b_2 \gamma + b_3 \sigma}{\mu + \omega + \nu} \tag{2.8}$$

$$b_5 = \frac{(1-\omega)\nu b_4 + \kappa b_{10}}{\lambda_s^* + \mu} \tag{2.9}$$

$$b_6 = \frac{b_5 \theta \lambda_s^*}{\mu + \gamma} \tag{2.10}$$

$$b_7 = \frac{b_5 \rho \lambda_s^*}{\mu + \alpha + \delta} \tag{2.11}$$

$$b_8 = \frac{b_7 \delta}{\mu + \epsilon + d} \tag{2.12}$$

$$b_9 = \frac{b_6 \gamma + b_7 \alpha + b_8 \epsilon}{\mu} \tag{2.13}$$

$$b_{10} = \frac{\omega \nu b_4}{\mu + \kappa} \tag{2.14}$$

$$b_{11} = \frac{b_{12} \lambda_v^*}{\lambda_v^* + \mu_v} \tag{2.15}$$

$$b_{12} = \frac{b[\beta_{vp}(b_3 + \eta b_2) + \beta_{vs}(b_7 + \eta b_6)]}{\mu_v N_h^*} \tag{2.16}$$

where the equation in b_{12} incorporates the form of λ_v^*.

Using Model 2.1 equilibrium conditions and the chosen coefficients above, many terms in $\frac{dL}{dt}$ cancel out. After sufficient algebraic manipulation, we obtain:

$$\begin{aligned}\frac{dL}{dt} = &-b_1\mu\left(\frac{S_p - S_p^*}{S_p}\right)^2 S_p - b_2(\mu+\gamma)\left(\frac{A_p - A_p^*}{A_p}\right)^2 A_p - b_3(\mu+\sigma)\left(\frac{I_p - I_p^*}{I_p}\right)^2 I_p \\ &- b_4(\mu+\nu)\left(\frac{R_p - R_p^*}{R_p}\right)^2 R_p - b_5\mu\left(\frac{S_s - S_s^*}{S_s}\right)^2 S_s - b_6(\mu+\gamma)\left(\frac{A_s - A_s^*}{A_s}\right)^2 A_s \\ &- b_7(\mu+\alpha+\delta)\left(\frac{I_s - I_s^*}{I_s}\right)^2 I_s - b_8(\mu+\epsilon+d)\left(\frac{I_h - I_h^*}{I_h}\right)^2 I_h - b_9\mu\left(\frac{R_s - R_s^*}{R_s}\right)^2 R_s \\ &- b_{10}(\mu+\kappa)\left(\frac{V - V^*}{V}\right)^2 V - b_{11}\mu_v\left(\frac{S_v - S_v^*}{S_v}\right)^2 S_v - b_{12}\mu_v\left(\frac{I_v - I_v^*}{I_v}\right)^2 I_v\end{aligned} \tag{2.17}$$

Following this analysis, we clearly see that $L > 0$ and $\frac{dL}{dt} < 0$ at only the Endemic Equilibrium Point (EEP); therefore $L(S_p, A_p, I_p, R_p, S_s, A_s, I_s, I_h, R_s, V, S_v, I_v)$ is a Lyapunov function since all the state variables are continuous and bounded with derivatives in the space Ω. We summarise this conclusion by stating the following theorem:

Theorem 2.1 (Global Asymptotic Stability)**.** *The endemic equilibrium point*

$$\left(S_p^*, A_p^*, I_p^*, R_p^*, S_s^*, A_s^*, I_s^*, I_h^*, R_s^*, V^*, S_v^*, I_v^*\right)$$

of the model is globally asymptotically stable in the interior of Ω*, since all state variables are continuous and bounded with derivatives in* L^∞*.*

Proof. The defined Volterra-Lyapunov function $L = \sum_{i=1}^{12} b_i(x_i - x_i^* \ln x_i)$ establishes global stability through the following properties:

(i) Positive Definite. The Volterra-Lyapunov function satisfies the properties of a Lyapunov function, which emanate from the properties of each term $x - x^* \ln x$.

(ii) Negative Time Derivative. We have shown that:

$$\frac{dL}{dt} = -\sum_{i=1}^{12} b_i \left(\frac{x_i - x_i^*}{x_i}\right)^2 x_i \leq 0$$

where $b_i > 0$ are positive constants.

(iii) The Invariant Region. The set where $\frac{dL}{dt} = 0$ is:

$$E = \left\{x \in \Omega : \frac{x_i - x_i^*}{x_i} = 0, \ \forall\, i = 1, 2, \ldots, 12\right\}$$

The set above implies that $x_i = x_i^*$, $\forall\, i$, so $E = \{x^*\}$.

(iv) Largest Invariant Subset. Since $E = \{x^*\}$ and the invariant point satisfies $\frac{dx^*}{dt} = 0$, the largest invariant subset of E is $M = \{x^*\}$.

(v) LaSalle's Invariance Principle. Every trajectory $x(t)$ of Model 2.1, starting in the interior of Ω, approaches $M = \{x^*\}$.

Conclusion. Since this reasoning satisfies all the initial conditions within the interior of Ω, convergence is global within the feasible region. Therefore, the endemic equilibrium x^* is globally asymptotically stable in the interior of Ω. □

3 Bifurcation Analysis

To investigate the local dynamics of the endemic equilibrium point, we apply the center manifold theorem by [**?**]. We denote the state variables as follows

$(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}) = (\frac{dS_p}{dt}, \frac{dA_p}{dt}, \frac{dI_p}{dt}, \frac{dR_p}{dt}, \frac{dS_s}{dt}, \frac{dA_s}{dt}, \frac{dI_s}{dt}, \frac{dI_h}{dt}, \frac{dR_s}{dt}, \frac{dV}{dt}, \frac{dS_v}{dt}, \frac{dI_v}{dt})$
and we change the model variables

$(S_p, A_p, I_p, R_p, S_s, A_s, I_s, I_h, R_s, V, S_v, I_v) = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$

$$\begin{cases} f_1 = \dfrac{dx_1}{dt} = \pi - \lambda_p x_1 - \mu x_1 \\ f_2 = \frac{dx_2}{dt} = \theta\lambda_p x_1 - (\mu+\gamma)x_2 \\ f_3 = \dfrac{dx_3}{dt} = \rho\lambda_p x_1 - (\mu+\sigma)x_3 \\ f_4 = \dfrac{dx_4}{dt} = \gamma x_2 + \sigma x_3 - (\mu+\nu)x_4 \\ f_5 = \dfrac{dx_5}{dt} = (1-\omega)\nu x_4 + \kappa V - \lambda_s x_5 - \mu x_5 \\ f_6 = \dfrac{dx_6}{dt} = \theta\lambda_s x_5 - (\mu+\gamma)x_6 \\ f_7 = \dfrac{dx_7}{dt} = \rho\lambda_s x_5 - (\mu+\alpha+\delta)x_7 \\ f_8 = \dfrac{dx_8}{dt} = \delta x_7 - (\mu+\epsilon+d)x_8 \\ f_9 = \dfrac{dx_9}{dt} = \gamma x_6 + \alpha x_7 + \epsilon x_8 - \mu x_9 \\ f_{10} = \dfrac{dx_{10}}{dt} = \omega\nu x_4 - (\mu+\kappa)x_{10} \\ f_{11} = \dfrac{dx_{11}}{dt} = B_v - \lambda_v x_{11} - \mu_v x_{11} \\ f_{12} = \dfrac{dx_{12}}{dt} = \lambda_v x_{11} - \mu_v x_{12} \end{cases} \tag{3.1}$$

Choosing the bifurcation point $\beta_{hp} = \beta^*_{hp}$ and $R_0 = 1$, we solve for $R_0 = 1$ and we get;

$$\beta_{hp} = \beta^*_{hp} = \frac{\pi\mu_v^2(\mu+\gamma)(\mu+\sigma)}{b^2\mu\beta_{vp}[\rho(\mu+\gamma)+\eta\theta(\mu+\sigma)]}$$

The Jacobian of the system above evaluated at the disease free equilibrium point with $\beta_{hp} = \beta^*_{hp}$ is given as below;

$$J(E_0) = \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B^* \\ 0 & -(\mu+\gamma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D^* \\ 0 & 0 & -(\mu+\sigma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F^* \\ 0 & \gamma & \sigma & -(\mu+\nu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\mu+\sigma) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\mu+\alpha+\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & -(\mu+\epsilon+d) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & \alpha & \epsilon & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & -(\mu+\kappa) & 0 & 0 \\ 0 & M^* & N^* & 0 & 0 & W^* & P^* & 0 & 0 & 0 & -\mu_v & 0 \\ 0 & Q^* & R^* & 0 & 0 & S^* & T^* & 0 & 0 & 0 & 0 & -\mu_v \end{pmatrix}$$

With eigenvalues of the above matrix at $R_0 = 1$ given as follows;

$$\begin{cases} \lambda_1 = -\mu \\ \lambda_2 = -\mu \\ \lambda_3 = -\mu \\ \lambda_4 = -\mu_v \\ \lambda_5 = -(\gamma + \mu) \\ \lambda_6 = -(\mu + \kappa) \\ \lambda_7 = -(\mu + \nu) \\ \lambda_8 = -(\gamma + \delta + \mu) \\ \lambda_9 = -(d + \epsilon + \mu) \end{cases} \tag{3.2}$$

and for the bifurcation point $R_0 = 1$ the equation in Π reduces to

$\lambda[-\lambda^2 - \lambda(\gamma + \sigma + 2\mu + \mu_v) + (R_0 k - 1)] = 0$

Further simplification of the polynomial yields;

$\lambda_{10} = \frac{(\gamma+\sigma+2\mu+\mu_v)+\sqrt{(\gamma+\sigma+2\mu+\mu_v)^2+4(R_0k-1)}}{-2}$ and $\lambda_{11} = \frac{(\gamma+\sigma+2\mu+\mu_v)-\sqrt{(\gamma+\sigma+2\mu+\mu_v)^2+4(R_0k-1)}}{-2}$

From the above equation it is clear that there exist a simple eigenvalue $\lambda_{12} = 0$ at the bifurcation point $R_0 = 1$ while the rest of the eigenvalues have negative real parts.

It can be noted that Model 2.1 has at least 1 zero eigenvalue while the rest have negative real parts. The right eigenvector associated with the zero eigenvalue of $J(\beta^*_{hp})$ at $R_0 = 1$ is obtained from $J(\beta^*_{hp})w = 0$ and is given by $w = (w_1, w_2, w_3, w_4, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12})^T$, where

$$\begin{cases} w_1 = \frac{-bw_{12}S_p\beta_{hp}}{\mu N_h} \\ w_2 = \frac{bw_{12}\theta S_p\beta_{hp}}{(\gamma+\mu)N_h} \\ w_3 = \frac{bw_{12}\rho S_p\beta_{hp}}{(\delta+\mu)N_h} \\ w_4 = \frac{w_2\gamma+w_3\sigma}{\mu+\nu} \\ w_5 = \frac{w_{10}\kappa+w_4\nu-w_4\nu\omega}{\mu} \\ w_6 = 0 \\ w_7 = 0 \\ w_8 = 0 \\ w_9 = 0 \\ w_{10} = \frac{w_4\omega}{\kappa+\mu} \\ w_{11} = \frac{-bS_v(w_3\beta_{vp}+w_2\eta\beta_{vp})}{\mu_v N_h} \\ w_{12} = \frac{bS_v(w_3\beta_{vp}+w_2\eta\beta_{vp})}{\mu_v N_h} \end{cases} \tag{3.3}$$

The left eigenvector associated with the zero eigenvalue is obtained from $vJ(\beta^*_{hp}) = 0$ which must satisfy the condition $vw = 1$, is found to be $v = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12})$ where

$$\begin{cases} v_1 = 0 \\ v_2 = \frac{bv_{12}\eta S_v\beta_{pv}}{(\gamma+\mu)N_h} \\ v_3 = \frac{bv_{12}S_v\beta_{pv}}{(\sigma+\mu)N_h} \\ v_4 = 0 \\ v_5 = 0 \\ v_6 = \frac{bv_{12}\eta S_v\beta_{vs}}{(\sigma+\mu)N_h} \\ v_7 = \frac{bv_{12}S_v\beta_{vs}}{(\alpha+\delta+\mu)N_h} \\ v_8 = 0 \\ v_9 = 0 \\ v_{10} = 0 \\ v_{11} = 0 \\ v_{12} = \frac{b(v_2\theta+v_3\rho)S_p\beta_{hp}}{N_h\mu_v} \end{cases} \tag{3.4}$$

Since $v1 = 0$, $v_4 = 0$, $v_5 = 0$, $v_8 = 0$, $v_9 = 0$, $v_{10} = 0$ and $v_{11} = 0$, we do not need derivatives of f_1, f_4, f_5, f_8, f_9, f_{10} and f_{11}

The non-vanishing second order partial derivatives of f_2, f_3, f_6, f_7 and f_{12} are:

$$\begin{aligned}
\frac{\partial^2 f_2(E^0,\phi^*)}{\partial x_1 \partial x_{12}} &= \frac{\partial^2 f_2(E^0,\phi^*)}{\partial x_{12} \partial x_1} = \frac{b\theta\beta_{hp}}{N_h} \\
\frac{\partial^2 f_3(E^0,\phi^*)}{\partial x_1 \partial x_{12}} &= \frac{\partial^2 f_3(E^0,\phi^*)}{\partial x_{12} \partial x_1} = \frac{b\rho\beta_{hp}}{N_h} \\
\frac{\partial^2 f_6(E^0,\phi^*)}{\partial x_5 \partial x_{12}} &= \frac{\partial^2 f_6(E^0,\phi^*)}{\partial x_{12} \partial x_5} = \frac{b\theta\phi\beta_{hs}}{N_h} \\
\frac{\partial^2 f_7(E^0,\phi^*)}{\partial x_5 \partial x_{12}} &= \frac{\partial^2 f_7(E^0,\phi^*)}{\partial x_{12} \partial x_5} = \frac{b\rho\phi\beta_{hs}}{N_h} \\
\frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_2 \partial x_{11}} &= \frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_{11} \partial x_2} = \frac{b\eta\beta_{hs}}{N_h} \\
\frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_3 \partial x_{11}} &= \frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_{11} \partial x_3} = \frac{b\beta_{hs}}{N_h} \\
\frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_6 \partial x_{11}} &= \frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_{11} \partial x_6} = \frac{b\eta\beta_{hs}}{N_h} \\
\frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_7 \partial x_{11}} &= \frac{\partial^2 f_{12}(E^0,\phi^*)}{\partial x_{11} \partial x_7} = \frac{b\beta_{hs}}{N_h}
\end{aligned}$$

Thus, considering the assumptions of the Center Manifold Theorem, we are guaranteed that the nature of the endemic equilibrium point of the system near the chosen bifurcation point can be guaranteed by the signs of the constants a and b whose formulae is given by

$$a = \sum_{k,i,j=1}^{n} V_k W_i W_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E^0, \phi^*) \quad \text{and} \quad b = \sum_{k,i=1}^{n} V_k W_i \frac{\partial^2 f_k}{\partial x_i \partial \beta_{hp}}(E^0, \phi^*)$$

Hence we obtain the constants a and b as:

$a = v_2[2(w_1 w_{12})\frac{b\theta\beta_{hp}}{N_h}]+v_3[2(w_1 w_{12})\frac{b\rho\beta_{hp}}{N_h}]+v_6[2(w_{12}w_5)\frac{b\theta\phi\beta_{hs}}{N_h}]+v_7[2(w_{12}w_5)\frac{b\rho\phi\beta_{hs}}{N_h}]+v_{12}[2(w_2 w_{11})\frac{b\eta\beta_{hs}}{N_h}+ 2(w_3 w_{11})\frac{b\beta_{hs}}{N_h}]$

Substituting for the obtained values of v_2, v_3, v_6, v_7, v_{12}, w_1, w_5, w_{11}, w_{12} and evaluating the value of b at the disease-free equilibrium point yields;

$a = \frac{2b^3 S_v(\eta w_2+w_3)\beta_{vp}}{N_h^4\mu_v^2}[-(\frac{b\theta(\gamma+\mu+\eta(\delta+\mu))S_p^2(\theta v_2+\rho v_3)w_{12}\beta_{hp}^2\beta_{hs}}{(\gamma+\mu)(\delta+\mu)} + \frac{bS_p S_v v_{12}w_{12}\beta_{hp}^2\beta_{vp}\mu_v(\rho(\gamma+\mu)+\eta\theta(\sigma+\mu))}{\mu(\gamma+\mu)(\sigma+\mu)}) + \frac{\eta\theta N_h S_v v_{12}(v(1-\omega)w_4+w_{10}\kappa)\beta_{hs}\beta_{vs}\mu_v(\theta(\alpha+\delta+\mu)+\rho(\sigma+\mu))}{\mu(\alpha+\delta+\mu)(\sigma+\mu)}]$

and

$b = \frac{ab^3 S_v^2 v_{12}(\eta w_{12}+w_3)\beta_{hp}\beta_{vp}^2(\rho(\gamma+\mu)+\eta\theta(\sigma+\mu))}{\mu\mu_v N_h^3(\gamma+\mu)(\sigma+\mu)} > 0$

From the above expressions of a and b, we note the b is always positive thus the bifurcation of the given system depends on the sign of a. Following this analysis, we will summarize the conclusions by stating the following theorem

Theorem 3.1. *Following the bifurcation analysis,*

1. *When $a > 0$, meaning when*

$$\left[\frac{\eta\theta N_h S_v v_{12}(v(1-\omega)w_4+w_{10}\kappa)\beta_{hs}\beta_{vs}\mu_v(\theta(\alpha+\delta+\mu)+\rho(\sigma+\mu))}{\mu(\alpha+\delta+\mu))(\sigma+\mu)}\right] > \left[\frac{b\theta(\gamma+\mu+\eta(\delta+\mu))S_p^2(\theta v_2+\rho v_3)w_{12}\beta_{hp}^2\beta_{hs}}{(\gamma+\mu)(\delta+\mu)} + \frac{bS_p S_v v_{12}w_{12}\beta_{hp}^2\beta_{vp}\mu_v(\rho(\gamma+\mu)+\eta\theta(\sigma+\mu))}{\mu(\gamma+\mu)(\sigma+\mu)}\right],$$

   thus the system will undergo backward bifurcation when $R_0 = 1$ and $\beta_{hp} = \beta_{hp}^*$

2. *When $a < 0$, meaning when*

$$\left[\frac{\eta\theta N_h S_v v_{12}(v(1-\omega)w_4+w_{10}\kappa)\beta_{hs}\beta_{vs}\mu_v(\theta(\alpha+\delta+\mu)+\rho(\sigma+\mu))}{\mu(\alpha+\delta+\mu))(\sigma+\mu)}\right] < \left[\frac{b\theta(\gamma+\mu+\eta(\delta+\mu))S_p^2(\theta v_2+\rho v_3)w_{12}\beta_{hp}^2\beta_{hs}}{(\gamma+\mu)(\delta+\mu)} + \frac{bS_p S_v v_{12}w_{12}\beta_{hp}^2\beta_{vp}\mu_v(\rho(\gamma+\mu)+\eta\theta(\sigma+\mu))}{\mu(\gamma+\mu)(\sigma+\mu)}\right],$$

   thus the system will undergo forward bifurcation when $R_0 = 1$ and $\beta_{hp} = \beta_{hp}^*$

4 Jacobian Matrix Coefficients

Expressions of coefficients used in the Jacobian Matrix:

$$\begin{cases} B^* = \dfrac{-bS_p\beta_{hp}}{N_h}, & D^* = \dfrac{b\theta S_p\beta_{hp}}{N_h}, \\ F^* = \dfrac{b\rho S_p\beta_{hp}}{N_h}, & M^* = \dfrac{-b\eta S_v\beta_{vp}}{N_h}, \\ N^* = \dfrac{-bS_v\beta_{vp}}{N_h}, & W^* = \dfrac{-b\eta S_v\beta_{vs}}{N_h}, \\ P^* = \dfrac{-bS_v\beta_{vs}}{N_h}, & Q^* = \dfrac{b\eta S_v\beta_{vp}}{N_h}, \\ R^* = \dfrac{bS_v\beta_{vp}}{N_h}, & S^* = \dfrac{b\eta S_v\beta_{vs}}{N_h}, \\ T^* = \dfrac{bS_v\beta_{vs}}{N_h}. \end{cases} \tag{4.1}$$

5 Full Model PRCC Value Table

Table 1: PRCC and P-Values for S_p, A_p, I_p, and R_p

Parameter	S_p		A_p		I_p		R_p	
	PRCC	P-Val	PRCC	P-Val	PRCC	P-Val	PRCC	P-Val
b	-0.428	6.84e-45	0.415	5.16e-42	0.409	8.46e-41	0.583	4.67e-90
β_{hp}	-0.139	1.31e-5	0.059	0.064	0.087	6.67e-3	0.189	2.61e-9
π	0.423	8.82e-44	-0.022	0.486	-0.017	0.590	0.034	0.289
μ	-0.700	2.74e-145	0.273	3.49e-18	0.275	1.77e-18	0.128	6.30e-5
θ	-0.124	1.05e-4	0.276	1.63e-18	0.155	1.10e-6	0.213	1.77e-11
γ	0.263	6.79e-17	-0.517	6.79e-68	-0.256	4.49e-16	-0.212	2.07e-11
σ	0.089	5.24e-3	-0.189	2.54e-9	-0.485	6.25e-59	-0.085	7.92e-3
ρ	-0.061	0.055	0.018	0.582	0.135	2.20e-5	0.069	0.031
ω	0.041	0.199	-0.020	0.529	-0.040	0.217	-0.043	0.177
ν	-0.057	0.073	-0.045	0.163	-0.054	0.090	0.008	0.809
κ	-0.003	0.917	-0.054	0.092	-0.052	0.101	-0.016	0.628
ϕ	0.008	0.810	-0.049	0.128	-0.048	0.136	-0.017	0.585
β_{hs}	-0.043	0.181	-0.044	0.172	-0.040	0.217	0.012	0.719
α	0.029	0.360	-0.010	0.764	-0.011	0.730	-0.037	0.248
δ	0.001	0.987	0.010	0.751	0.026	0.414	0.032	0.314
ϵ	0.019	0.553	0.045	0.162	0.052	0.106	0.042	0.188
d	-0.015	0.631	0.011	0.731	0.018	0.583	-0.019	0.557
B_v	-0.364	5.36e-32	0.348	2.83e-29	0.382	1.87e-35	0.468	1.57e-54
β_{vp}	-0.370	4.80e-33	0.336	2.87e-27	0.342	2.56e-28	0.454	7.34e-51
η	-0.110	5.63e-4	0.106	8.73e-4	0.102	1.35e-3	0.124	9.46e-5
β_{vs}	-0.016	0.614	0.064	0.047	0.063	0.047	0.042	0.193
μ_v	0.674	1.28e-130	-0.539	7.71e-75	-0.556	2.07e-80	-0.653	3.07e-120

Table 2: PRCC and P-Values for S_s, A_s, I_s, and I_h

Parameter	S_s		A_s		I_s		I_h	
	PRCC	P-Val	PRCC	P-Val	PRCC	P-Val	PRCC	P-Val
b	0.402	2.70e-39	0.480	1.14e-57	0.468	2.37e-54	0.447	2.51e-49
β_{hp}	0.105	9.56e-4	0.077	0.016	0.100	1.77e-3	0.084	8.22e-3
π	0.004	0.903	-0.090	4.76e-3	-0.064	0.044	-0.074	0.020
μ	0.153	1.44e-6	0.340	5.75e-28	0.305	1.68e-22	0.166	1.79e-7
θ	0.250	2.07e-15	0.319	1.51e-24	0.256	4.17e-16	0.233	1.57e-13
γ	-0.103	1.28e-3	-0.467	3.65e-54	-0.265	3.11e-17	-0.142	8.32e-6
σ	-0.073	0.023	-0.184	6.21e-9	-0.162	3.24e-7	-0.029	0.357
ρ	0.059	0.066	0.056	0.078	0.148	3.35e-6	0.191	1.83e-9
ω	-0.154	1.27e-6	-0.111	5.10e-4	-0.126	8.01e-5	-0.074	0.021
ν	0.233	1.64e-13	0.155	1.14e-6	0.189	2.48e-9	0.224	1.20e-12
κ	-0.032	0.323	-0.032	0.312	-0.030	0.345	0.004	0.910
ϕ	-0.082	0.010	-0.007	0.826	-0.001	0.982	0.048	0.130
β_{hs}	-0.088	5.72e-3	-0.004	0.894	0.009	0.779	0.028	0.377
α	-0.021	0.512	-0.033	0.307	-0.235	8.54e-14	-0.167	1.60e-7
δ	0.043	0.179	0.017	0.589	-0.065	0.043	0.215	1.05e-11
ϵ	0.062	0.052	0.029	0.363	0.030	0.341	-0.419	8.52e-43
d	-0.017	0.599	-0.005	0.877	-0.001	0.971	-0.106	8.83e-4
B_v	0.296	2.89e-21	0.378	1.39e-34	0.386	3.65e-36	0.336	2.50e-27
β_{vp}	0.262	7.68e-17	0.390	6.41e-37	0.381	2.99e-35	0.372	1.91e-33
η	0.082	0.011	0.100	1.77e-3	0.106	8.55e-4	0.076	0.017
β_{vs}	0.060	0.059	0.052	0.107	0.053	0.098	0.033	0.304
μ_v	-0.237	5.58e-14	-0.588	3.70e-92	-0.571	9.53e-86	-0.522	1.41e-69

Table 3: PRCC and P-Values for R_s, V, S_v, and I_v

Parameter	R_s		V		S_v		I_v	
	PRCC	P-Val	PRCC	P-Val	PRCC	P-Val	PRCC	P-Val
b	0.478	5.08e-57	0.410	7.00e-41	-0.007	0.820	0.484	1.08e-58
β_{hp}	0.114	3.31e-4	0.067	0.036	-0.008	0.800	0.101	1.56e-3
π	-0.085	7.59e-3	-0.003	0.921	0.013	0.677	-0.122	1.26e-4
μ	0.155	1.11e-6	0.235	9.86e-14	0.043	0.174	0.406	3.20e-40
θ	0.317	3.20e-24	0.238	4.92e-14	0.004	0.901	0.211	2.33e-11
γ	-0.117	2.33e-4	-0.296	2.79e-21	0.102	1.40e-3	-0.404	8.74e-40
σ	-0.045	0.161	-0.220	3.67e-12	0.055	0.088	-0.257	2.79e-16
ρ	0.101	1.61e-3	0.062	0.054	-0.032	0.322	0.046	0.150
ω	-0.126	7.65e-5	-0.011	0.726	-0.006	0.854	-0.026	0.419
ν	0.232	2.03e-13	0.211	2.39e-11	-0.008	0.810	-0.033	0.299
κ	-0.010	0.762	-0.188	2.96e-9	-0.005	0.880	-0.034	0.283
ϕ	0.009	0.769	-0.042	0.189	0.019	0.560	-0.038	0.238
β_{hs}	0.026	0.408	-0.046	0.147	0.029	0.360	-0.023	0.470
α	-0.026	0.409	-0.020	0.529	0.059	0.064	-0.004	0.889
δ	-0.004	0.907	0.008	0.810	-0.019	0.546	0.012	0.706
ϵ	0.311	1.82e-23	0.039	0.218	-0.018	0.565	0.008	0.800
d	-0.058	0.070	0.013	0.695	-0.022	0.488	0.023	0.480
B_v	0.391	4.95e-37	0.343	2.05e-28	0.901	0	0.513	7.83e-67
β_{vp}	0.421	2.17e-43	0.329	3.60e-26	-0.057	0.077	0.501	2.41e-63
η	0.101	1.52e-3	0.102	1.40e-3	0.028	0.388	0.136	2.07e-5
β_{vs}	0.004	0.899	0.053	0.100	0.015	0.641	0.051	0.109
μ_v	-0.570	2.47e-85	-0.536	5.87e-74	-0.888	0	-0.784	6.06e-205