AIMS Mathematics, 11(11): 38771–38817.
DOI: 10.3934/math.20261496
Received: 04 August 2026
Revised: 15 September 2026
Accepted: 19 October 2026
Published: 20 November 2026

https://www.aimspress.com/journal/Math

Review

Fractional physics-informed neural networks for fractional PDEs: an operator-aware review of error, stability, cost, and reliability

Muhammad Asim Khan[1], **Ayesha Ayub**[2], **Naglaa F. Soliman**[3,*] **and Abeer D. Algarni**[3]

[1] School of Mathematical Sciences, Universiti Sains Malaysia, Penang 11800, Malaysia
[2] Advanced Telecommunication Research Center (ATRC), Faculty of Electrical and Electronic Engineering, Universiti Tun Hussein Onn Malaysia, Parit Raja 86400, Malaysia
[3] Department of Information Technology, College of Computer and Information Sciences, Princess Nourah Bint Abdulrahman University, Riyadh 11671, Saudi Arabia

* **Correspondence:** Email: nfsoliman@pnu.edu.sa.

Abstract: Fractional physics-informed neural networks (fPINNs) combine neural approximation with numerical or analytical treatments of fractional operators, making their reliability dependent on both network training and operator construction. This preferred reporting items for systematic reviews and meta-analyses (PRISMA) 2020-adapted review synthesizes 37 primary studies and classifies them by their dominant residual-construction mechanism: discretization, quadrature or integral, spectral or basis, Monte Carlo, analytical-prior, variable- or distributed-order, and operator-learning methods. The audit identified formal convergence theory in 7 studies, direct classical-solver comparisons in 11, operator-level ablations in 10, detailed cost reporting in 8, and full reproducibility packages in 9. Evidence is strongest for L1/L2 Caputo formulations, whereas Monte Carlo, variable- and distributed-order, and operator-learning approaches require stronger analyses of uncertainty, identifiability, and generalization. Across classes, operator accuracy, stability, boundary treatment, computational cost, and reproducibility remain decisive. Current evidence therefore supports fPINNs primarily as complementary hybrid solvers for inverse, data-constrained, high-dimensional, and repeated-query fractional partial differential equation problems rather than universal replacements for classical methods.

Keywords: fractional partial differential equations; fractional physics-informed neural networks; operator-aware taxonomy; systematic review; fractional operator approximation; error analysis; computational efficiency; reproducibility
Mathematics Subject Classification: 35R11, 35R30, 65M12, 65M15, 68T07

This supplementary file supports the preferred reporting items for systematic reviews and meta-analyses (PRISMA) 2020-adapted selection summary, operator-aware taxonomy, methodological appraisal, and quantitative evidence audit used in the review. Supplementary Table S1 provides the literature-extraction matrix and reference-role classification. Supplementary Table S2 provides the complete search documentation, Supplementary Tables S3 and S4 report the selection and exclusion process, Supplementary Table S5 reports corpus statistics, Supplementary Table S6 defines the methodological-appraisal criteria, and Supplementary Table S7 provides the study-level quantitative evidence audit. Supplementary Table S8 reports the hybrid-method primary/secondary class crosswalk used for the classification sensitivity analysis, and Supplementary Table S9 provides the exact class-level inputs and scores displayed in Figure 5. Supplementary Table S10 presents the re-audit of 14 potential classical-solver comparisons, of which 11 met the direct-comparison criterion. It summarizes the comparison designs, eligibility decisions, reported outcomes, and methodological limitations, supporting the analysis in Section 2.6 of the main manuscript. A bibliographic-status audit was completed on 15 July 2026. Final publisher metadata were used where available; records without a final journal version remain explicitly identified as preprints or early-access items. The audit was not a new search or screening round and did not add or remove records; the 67-record review framework and the 37-primary-study quantitative denominator were unchanged.

A total of 67 records were retained within the complete review framework. The core evidence set contains 38 records, comprising 37 primary fPINN or fractional neural-solver studies and one fPINN-focused positioning review. The remaining 29 records form a contextual supporting layer used for fractional-calculus background, classical numerical comparison, PINN/SciML context, neural-operator positioning, and reproducibility interpretation. The two role counts are mutually exclusive and sum to 67. The contextual supporting references and the positioning review were excluded from the quantitative evidence-frequency calculations, whose denominator is therefore 37 primary studies.

Table S1. Literature-extraction matrix and reference-role classification. Core fPINN evidence-set records are distinguished from supporting references used for mathematical background, classical numerical comparison, PINN/SciML context, and neural-operator context.

No.	Citation	Year	Reference role/category	Fractional operator/scope	Operator approximation/method	Role in the review	Verification note
A. Core fPINN and fractional neural-solver studies included in the qualitative synthesis							
1	[1]	2019	Core fPINN/fractional neural-solver study	GL/vector-GL; space-time fractional operators	Hybrid residual: AD for integer-order terms + numerical approximation for fractional terms	Foundational fPINN framework; supports taxonomy and hybrid-residual argument	Verified
2	[2]	2022	Core fPINN/fractional neural-solver study	Caputo time-fractional derivative	Spectral collocation with fPINN residual	Time-fractional phase-field forward/inverse benchmark	Verified
3	[3]	2022	Core fPINN/fractional neural-solver study	Fractional Laplacian/nonlocal spatial operators	Monte Carlo estimator inside residual	High-dimensional forward/inverse FPDEs	Verified
4	[4]	2023	Core fPINN/fractional neural-solver study	Stochastic fractional PDE setting	Bi-orthogonal stochastic representation + weak residual + grid fractional discretization	Uncertainty and weak-form extension	Verify metadata

Continued on next page

No.	Citation	Year	Reference role/category	Fractional operator/scope	Operator approximation/method	Role in the review	Verification note
5	[5]	2024	Core fPINN/fractional neural-solver study	Time-fractional diffusion/Caputo-type operator	Laplace-transform residual + numerical inverse transform	Analytical-prior fPINN class; forward/inverse time-fractional problems	Verified
6	[6]	2023	Core fPINN/fractional neural-solver study	Variable-order space-fractional operator	Power-series approximation + dual-network inverse identification	Variable-order forward/inverse FPDEs	Verified
7	[7]	2023	Core fPINN/fractional neural-solver study	General fractional DE/PDE setting	Improved nonlinear propagation in neural architecture	Architecture-enhanced fPINN branch	Verified
8	[8]	2024	Core fPINN/fractional neural-solver study	Caputo derivatives and fractional integro-differential terms	Auxiliary outputs + adaptive loss weighting + fractional derivative scheme	FPIDE residual construction and adaptive-loss branch	Verified
9	[9]	2024	Core fPINN/fractional neural-solver study	Caputo time-fractional derivative	L1 approximation + hard/soft constraints	Time-fractional Huxley benchmark	Verified
10	[10]	2025	Core fPINN/fractional neural-solver study	Caputo time-fractional derivative	Fast L1 approximation on graded meshes + adaptive activation	Memory-aware high-dimensional reaction-diffusion problems	Publisher metadata verified (15 July 2026)
11	[11]	2025	Core fPINN/fractional neural-solver study	Fractional operators in spectral/Jacobi setting	Jacobi-polynomial spectral discretization with error discussion	Spectral approximation, stability, and efficiency	Publisher metadata verified (15 July 2026)
12	[12]	2026	Core fPINN/fractional neural-solver study	General fractional PDE operators	RBF-enhanced neural approximation space	Basis-function design for fPINNs	Publisher metadata verified (15 July 2026)
13	[13]	2023	Core fPINN/fractional neural-solver study	Distributed-order time-fractional derivative	Composite numerical integration over order + time-difference PINN	Order-space quadrature and multi-output design	Verified
14	[14]	2025	Core fPINN/fractional neural-solver study	Distributed-order derivative	Hybrid AD + numerical discretization for distributed-order residuals	Forward/inverse distributed-order problems	Publisher metadata verified (15 July 2026)
15	[15]	2024	Core fPINN/fractional neural-solver study	Fractional and tempered fractional operators	MC-fPINN + quadrature-enhanced Monte Carlo residuals	Curse-of-dimensionality and tempered operators	Verified
16	[16]	2025	Core fPINN/fractional neural-solver study	Time-fractional Black-Scholes operator	PINN residual with fractional derivative approximation	Financial FPDE benchmark	Publisher metadata verified (15 July 2026)
17	[17]	2025	Core fPINN/fractional neural-solver study	Time-space fractional Black-Scholes; GL operator	Stable matrix/GL formulation inside loss	Space-time fractional finance benchmark	Publisher metadata verified (15 July 2026)
18	[18]	2025	Core fPINN/fractional neural-solver study	Caputo time-fractional derivative	L1 approximation + adaptive sampling/weighting + two-stage optimization	Fractional Allen-Cahn/Cahn-Hilliard benchmark	Publisher metadata verified (15 July 2026)
19	[19]	2025	Core fPINN/fractional neural-solver study	Caputo multi-term/variable-order time-fractional operators	DeepONet + physics-informed residual + L2 Caputo approximation	Parametric fractional diffusion-wave equations	Publisher metadata verified (15 July 2026)
20	[20]	2023	Core fPINN/fractional neural-solver study	Fractional Laplacian	Modified trapezoidal-rule evaluation in residual	Nonlocal Fokker-Planck/fractional Laplacian comparison	Verified
21	[21]	2023	Core fPINN/fractional neural-solver study	Integral fractional Laplacian	Neural Ritz formulation with weighted Sobolev analysis	Energy-norm convergence and error analysis	Verified
22	[22]	2020	Core fPINN/fractional neural-solver study	Fractional-order PDE operators	Sine/cosine neural network on sample points	Pre-/parallel fPINN neural-solver context	Verified

Continued on next page

No.	Citation	Year	Reference role/category	Fractional operator/scope	Operator approximation/method	Role in the review	Verification note
23	[23]	2022	Core fPINN/fractional neural-solver study	Conformable derivative	DNN/PINN-style framework with conformable operator	Distinguishes local conformable derivatives from nonlocal fPINNs	Verified
24	[24]	2023	Core fPINN/fractional neural-solver study	Discrete Caputo derivative	Discrete Caputo derivative + AD for integer terms	Hybrid residual construction beyond canonical fPINNs	Verified
25	[25]	2023	Core fPINN/fractional neural-solver study	Time-fractional derivative	Interpolation-based finite-difference temporal iteration	Model-driven neural fractional solver	Verified
26	[26]	2024	Core fPINN/fractional neural-solver study	Caputo time-fractional derivative	High-order Hermite explicit fractional approximation	Higher-order residual approximation	Verified
27	[27]	2025	Core fPINN/fractional neural-solver study	L1/L2/Hermite-type fractional residuals	Systematic comparison of fractional neural solver variants	Benchmarking and method-selection evidence	Publisher metadata verified (15 July 2026)
28	[28]	2025	Core fPINN/fractional neural-solver study	Subdiffusion/time-fractional operators	Multiprecision multistage fPINN computation	Precision, stability, and high-accuracy regimes	Publisher metadata verified (15 July 2026)
29	[29]	2026	Core evidence-set review/positioning record	Caputo-based formulations	Implementation and efficiency survey	Related fPINN review and practical implementation reference	Publisher metadata verified (15 July 2026)
30	[30]	2026	Core fPINN/fractional neural-solver study	Fractional Laplacian/fractional diffusion	PINN convergence analysis with boundary-compatible treatment	Convergence theory and boundary compatibility	Preprint; arXiv metadata verified (15 July 2026)
31	[31]	2025	Core fPINN/fractional neural-solver study	Distributed-order/time-fractional operator	Adaptive fractional polynomial basis	Basis-enhanced/distributed-order method	Publisher metadata verified (15 July 2026)
32	[32]	2025	Core fPINN/fractional neural-solver study	Fractional PDE operators	Fractional spectrally adapted PINN	Basis-enhanced fPINN class	Publisher metadata verified (15 July 2026)
33	[33]	2026	Core fPINN/fractional neural-solver study	Time-fractional PDE with initial singularity	Decomposed singularity-capturing PINN	Initial singularity and regularity-aware design	Publisher metadata verified (15 July 2026)
34	[34]	2025	Core fPINN/fractional neural-solver study	Psi-Caputo-type derivative	Generalized PINN solution representation	Generalized fractional operator branch	Publisher metadata verified (15 July 2026)
35	[35]	2025	Core fPINN/fractional neural-solver study	Fractional PDE operators	Monte Carlo PINN + cuckoo search optimization	Stochastic residual and metaheuristic optimization	Publisher metadata verified (15 July 2026)
36	[36]	2025	Core fPINN/fractional neural-solver study	Fractional PDE operators	Fractional-order basis-function multi-step NN	Other fractional neural solver	Publisher metadata verified (15 July 2026)
37	[37]	2025	Core fPINN/fractional neural-solver study	Fractional integro-differential equations	Universal fractional integro-differential equation solver	General solver and reproducibility context	Publisher metadata verified (15 July 2026)
38	[38]	2026	Core fPINN/fractional neural-solver study	Caputo-type FPDE	Laplace-transform physics-informed neural networks	Analytical-prior fPINN class	Publisher metadata verified (15 July 2026)
B. Supporting references used for background, comparison, and positioning; not counted as core fPINN studies							
39	[39]	2019	Supporting PINN/SciML reference	Integer-order PDE residuals	Automatic differentiation + data/BC/IC losses	PINN background and baseline for fPINN extension	Verified
40	[40]	2021	Supporting neural-operator reference	General operator learning	DeepONet maps between function spaces	Foundation for fractional operator-learning discussion	Verified

Continued on next page

No.	Citation	Year	Reference role/category	Fractional operator/scope	Operator approximation/method	Role in the review	Verification note
41	[41]	2019	Supporting classical numerical-method reference	Caputo time-fractional derivative	Crank-Nicolson-Galerkin finite element scheme	Classical baseline for time-fractional diffusion	Verified
42	[42]	2021	Supporting PINN/SciML reference	General PINN residuals	Gradient-flow/pathology analysis	Optimization warning for fPINNs	Verified
43	[43]	2021	Supporting PINN/SciML reference	General PINN/PIDL losses	Multi-objective loss balancing	Loss weighting context	Verify metadata
44	[44]	2020	Supporting PINN/SciML reference	General neural PDE solver	Adaptive activation functions	Training-stability context	Verified
45	[45]	2023	Supporting PINN/SciML reference	General PDE residuals	Generalization-error estimates	Sampling/generalization context for fPINNs	Verified
46	[46]	2021	Supporting PINN/SciML reference	General PDE residuals	cPINN/XPINN parallel domain decomposition	Scalability and domain decomposition context	Verified
47	[47]	2021	Supporting PINN/SciML reference	General PDE residuals	Finite-basis PINN with overlapping subdomains	Basis/domain-decomposition context	Verified
48	[48]	2021	Supporting PINN/SciML reference	General PINN/SciML	DeepXDE software ecosystem	Reproducibility/software context	Verified
49	[49]	2021	Supporting PINN/SciML reference	General physics-informed learning	Models + data + differential constraints	Broad SciML context	Verified
50	[50]	2022	Supporting PINN/SciML reference	General PINN methods	Survey of PINN variants and applications	Related review context	Verified
51	[51]	2025	Supporting PINN/SciML reference	General SciML	Survey of physics-informed machine learning	Related review context	Publisher metadata verified (15 July 2026)
52	[52]	2025	Supporting PINN/SciML reference	General PINN methods	Review of PINN variants, applications, and challenges	Related review context	Publisher metadata verified (15 July 2026)
53	[53]	2020	Supporting PINN/SciML reference	General DE solver	NeuroDiffEq package	Software/reproducibility context	Verified
54	[54]	2018	Supporting PINN/SciML reference	Variational PDE setting	Deep Ritz variational neural method	Weak/variational neural-solver background	Verified
55	[55]	2018	Supporting PINN/SciML reference	General PDE solver	DGM deep learning algorithm	Neural PDE solver background	Verified
56	[56]	Pre-2019	Supporting classical fractional-calculus reference	Caputo/RL and fractional operators	Analytical definitions and solution theory	Mathematical background	Verified
57	[57]	Pre-2019	Supporting classical fractional-calculus reference	Fractional derivatives/integrals	Analytical definitions and theory	Mathematical background	Verified
58	[58]	Pre-2019	Supporting classical fractional-calculus reference	Fractional differential equations	Analytical/numerical foundations	Mathematical background	Verified
59	[59]	Pre-2019	Supporting classical fractional-calculus reference	Fractional relaxation/asymptotics	Mittag-Leffler functions and fractional dynamics	Asymptotic priors and benchmarks	Verified
60	[60]	Pre-2019	Supporting classical fractional-calculus reference	Memory and nonlocal transport	Random-walk/anomalous diffusion theory	Physical motivation for FPDEs	Verified
61	[61]	Pre-2019	Supporting classical fractional-calculus reference	Fractional diffusion and waves	Analytical solutions and Mittag-Leffler behavior	Benchmark and asymptotic background	Verified
62	[62]	Pre-2019	Supporting classical numerical-method reference	Fractional calculus operators	Discretized fractional calculus / convolution methods	Numerical baseline for fPINN residuals	Verified

Continued on next page

No.	Citation	Year	Reference role/category	Fractional operator/scope	Operator approximation/method	Role in the review	Verification note
63	[63]	Pre-2019	Supporting classical numerical-method reference	Fractional evolution equations	Convolution quadrature	Time-memory baseline	Verified
64	[64]	Pre-2019	Supporting classical numerical-method reference	Caputo subdiffusion	Graded-mesh L1 analysis	Weak initial singularity and L1 accuracy	Verified
65	[65]	Pre-2019	Supporting classical numerical-method reference	Time-fractional diffusion	Finite element analysis	Classical baseline for time-fractional diffusion	Verified
66	[66]	Pre-2019	Supporting classical numerical-method reference	Fractional Laplacian	Finite element treatment of nonlocal operator	Boundary/nonlocal baseline	Verified
67	[67]	Pre-2019	Supporting classical numerical-method reference	Fractional powers of elliptic operators	Numerical approximation of fractional powers	Classical baseline for fractional Laplacian and nonlocal operators	Verified

Note: References labelled as "Core fPINN / fractional neural-solver study" constitute the main qualitative synthesis. Supporting references are included to provide mathematical background, classical numerical comparison, PINN/SciML context, and neural-operator context. They are not interpreted as core fPINN studies unless explicitly labelled as such.

Preprint counting rule: At the 15 July 2026 audit cutoff, record 29 was a preprint positioning review and record 30 was a preprint primary study. Record 29 was excluded from all 37-study frequency denominators. Record 30 was included in the raw descriptive audit because it satisfied the primary-study eligibility criteria, but its preprint status was retained as a separate appraisal factor and it was not treated as independent peer-reviewed validation in the class-level maturity assessment.

Fractional-ODE transferability rule: A FODE-only record was classified as transferable only if the fractional operator and neural-residual approximation were mathematically explicit, the approximation could be coupled to spatial differential or nonlocal terms without changing its formula or consistency argument, operator-specific validation was reported through a consistency/error result, controlled resolution study, or matched classical comparison, and the paper did not treat ODE performance as proof of FPDE performance. Records satisfying all four conditions were eligible for the methodological synthesis and quantitative audit with their FODE scope labelled explicitly. Records failing any condition were excluded from the counted corpus and, when methodologically useful, were cited only as external scope-boundary examples.

Table S2. Search concepts and keyword groups used to construct the source-specific queries.

Search group	Representative keywords and combinations
Core fPINN terms	"fractional physics-informed neural network", "fractional physics informed neural network", "fractional PINN", "fPINN", "physics-informed neural network" AND "fractional PDE"
Fractional operators	"Caputo", "Riemann-Liouville", "Riesz", "Grünwald-Letnikov", "fractional Laplacian", "tempered fractional", "variable-order", "distributed-order", "conformable derivative"
Operator approximation	"L1 scheme", "L2 scheme", "Grünwald", "quadrature", "spectral method", "finite element", "finite difference", "Monte Carlo", "Laplace transform", "Hermite interpolation", "operator approximation"
Specific fPINN variants	"spectral fPINN", "Monte Carlo fPINN", "Laplace-fPINN", "RBF-fPINN", "distributed-order PINN", "variable-order PINN", "fractional DeepONet"
Operator learning	"DeepONet", "neural operator", "operator learning", "physics-informed operator learning" AND "fractional PDE"
Reliability and efficiency	"stability", "convergence", "error analysis", "computational efficiency", "fast convolution", "memory compression", "benchmark", "reproducibility", "loss balancing"
Inverse and data-driven problems	"inverse problem", "parameter identification", "fractional-order identification", "coefficient identification", "data-driven" AND "fractional PINN"
Classical FPDE comparators	"fractional PDE" AND "finite difference", "fractional PDE" AND "finite element", "fractional Laplacian" AND "finite element", "Caputo derivative" AND "graded mesh", "fractional derivative" AND "fast convolution"

Table S3. PRISMA 2020-adapted selection counts and composition of the retained review framework.

Selection item	Count	Notes
Records identified from Zotero CSV export	107	Bibliographic records exported from the working Zotero library
Additional records identified through verified literature mapping and citation searching	34	Records identified through backward and forward citation searching, manual bibliographic verification, and literature mapping
Total records before duplicate removal	141	Combined record pool before deduplication
Duplicate records removed	47	Duplicates identified using DOI, normalized-title, and author-year matching
Records after duplicate removal	94	Unique records available for title and abstract screening
Records screened by title and abstract	94	All deduplicated records underwent title and abstract screening
Records excluded after title and abstract screening	18	Excluded for insufficient relevance to FPDEs, fractional neural solvers, or fractional-operator approximation
Reports sought for retrieval	76	Records retained after title and abstract screening
Reports not retrieved	0	All 76 reports sought for retrieval were successfully obtained for eligibility assessment
Reports assessed for eligibility	76	Eligibility was assessed using full articles, preprints, publisher records, DOI records, or sufficiently detailed verified metadata
Reports excluded after eligibility assessment	9	Excluded after detailed application of the eligibility criteria; reasons are reported in Supplementary Table S4
Records retained in the complete review framework	67	Records remaining after screening and eligibility assessment
Composition of the 67-record review framework		
Core evidence-set records	38	Comprises 37 primary fPINN or fractional neural-solver studies and one fPINN-focused positioning review
Contextual supporting references	29	Used for mathematical background, classical numerical comparison, PINN/SciML context, neural-operator positioning, and reproducibility interpretation
Primary studies entering the quantitative audit	37	Subset of the 38-record core evidence set; the positioning review and all contextual references were excluded from quantitative denominators

Note: The core and contextual categories are mutually exclusive and sum to 67. The quantitative-audit count of 37 is a subset of the 38-record core evidence set and therefore should not be added to the other category counts.

Table S4. Primary reasons for exclusion across title/abstract screening and eligibility assessment. A total of 27 records were excluded: 18 during title/abstract screening and 9 during eligibility assessment. Each excluded record was assigned one mutually exclusive primary reason.

Reason for exclusion	Count	Notes
Unrelated machine-learning or application field	10	Not relevant to FPDEs, fPINNs, or fractional neural solvers
Integer-order or generic PDE/PINN paper without fractional-operator relevance	10	No fractional operator or nonlocal residual construction
Non-neural or non-fPINN numerical/physics paper	4	Not used unless required as a classical comparator
Outside FPDE/fPINN review scope	2	Did not match the methodological scope
Incomplete or unverifiable bibliographic record	1	Metadata, source, or relevance could not be verified

Table S5. Corpus statistics for the 67 records included in the final qualitative synthesis. Counts were calculated directly from the study roles and publication years recorded in Supplementary Table S1.

Category	Count	Notes
Publication year		
Pre-2019	14	Twelve records labelled pre-2019 and two records published in 2018
2019	3	Records published in 2019
2020	3	Records published in 2020
2021	7	Records published in 2021
2022	4	Records published in 2022
2023	9	Records published in 2023
2024	5	Records published in 2024
2025	17	Records published in 2025
2026	5	Records published in 2026
Total	**67**	Final qualitative-synthesis corpus
Corpus role		
Primary fPINN/fractional neural-solver studies	37	Studies included in the quantitative evidence audit
Positioning review	1	Core evidence-set review excluded from quantitative denominators
Supporting references	29	Mathematical, numerical, PINN/SciML, and operator-learning context
Total	**67**	Complete qualitative-synthesis corpus
Supporting-reference composition		
PINN/scientific-machine-learning references	15	General PINN methods, optimization, software, and surveys
Neural-operator reference	1	General operator-learning background
Fractional-calculus references	6	Definitions, analytical theory, and fractional-model background
Classical numerical-method references	7	Discretization, convolution-quadrature, finite-difference, and finite-element foundations
Total supporting references	**29**	Exclusive supporting-reference role count

Table S6. Quality-assessment criteria used to evaluate included studies.

Criterion	Guiding question	Relevance to this review
Operator transparency	Does the study clearly define the fractional operator, derivative type, fractional order, kernel, normalization, and boundary interpretation?	Essential for determining whether the residual is consistent with the intended FPDE model
Numerical consistency	Is the embedded operator approximation mathematically justified or connected to a known fractional numerical scheme?	Distinguishes reliable hybrid solvers from ad hoc residual constructions
Residual-construction clarity	Does the study explain how the fractional term is evaluated inside the neural loss?	Central to the operator-based taxonomy of fPINNs
Stability and error discussion	Does the study provide theoretical error estimates, empirical convergence tests, stability analysis, or clear discussion of error sources?	Supports reliability, convergence, and error-decomposition analysis
Computational-cost reporting	Does the study report residual-evaluation cost, auxiliary points, memory use, hardware, wall-clock time, or comparison with classical solvers?	Needed to assess computational efficiency beyond final accuracy
Benchmark quality	Are the benchmark problems standard, reproducible, and sufficiently diverse in operator type, fractional order, dimension, and difficulty?	Supports fair comparison across fPINN classes
Reproducibility	Are code, data, random seeds, optimizer settings, architecture, hyperparameters, and reference-solver settings reported?	Determines whether reported results can be checked or reproduced
Hybridization relevance	Does the study clarify the interaction between analytical structure, numerical approximation, and neural learning?	Directly supports the central thesis of hybrid analytical-numerical-learning solvers
Classical-solver comparison	Does the study compare with finite-difference, finite-element, spectral, quadrature, Monte Carlo, analytical, or semi-analytical baselines?	Helps evaluate whether fPINNs complement or outperform established FPDE solvers
Limitations and uncertainty	Does the study discuss limitations, parameter sensitivity, stochastic variance, or uncertainty in inverse problems?	Important for reliability, robustness, and methodological maturity

Table S7. Study-level quantitative evidence audit of the 37 primary studies.

ID	Study	Theory	Classical baseline	Public code	Cost reporting	Operator ablation	Uncertainty	Reproducibility
1	[1]	No	Yes	Yes	Basic	Yes	Yes	Full
2	[2]	No	Yes	No	Detailed	No	Yes	Partial
3	[3]	No	No	No	None	No	No	Partial
4	[4]	No	No	No	Detailed	No	Yes	Partial
5	[5]	No	No	No	None	No	Yes	Partial
6	[6]	Yes	Yes	No	Detailed	Yes	No	Partial
7	[7]	No	No	No	None	No	N/A	Limited
8	[8]	No	No	No	None	No	N/A	Partial
9	[9]	No	Yes	Yes	Basic	No	N/A	Full
10	[10]	No	No	No	Basic	Yes	Yes	Partial
11	[11]	Yes	No	No	Basic	Yes	Yes	Partial
12	[12]	No	Yes	No	Basic	No	N/A	Partial
13	[13]	No	No	No	Detailed	No	N/A	Partial
14	[14]	No	Yes	No	Basic	No	No	Partial

Continued on next page

ID	Study	Theory	Classical baseline	Public code	Cost reporting	Operator ablation	Uncertainty	Reproducibility
15	[15]	No	No	Yes	Detailed	Yes	No	Full
16	[16]	No	No	No	None	No	N/A	Limited
17	[17]	No	No	No	None	No	N/A	Partial
18	[18]	No	No	No	Detailed	No	N/A	Partial
19	[19]	No	No	Yes	None	No	No	Full
20	[20]	No	No	Yes	Basic	No	N/A	Full
21	[21]	Yes	Yes	No	None	No	N/A	Partial
22	[22]	Yes	No	No	None	No	N/A	Partial
23	[23]	No	No	No	Basic	No	N/A	Partial
24	[24]	No	Yes	Yes	Basic	No	Yes	Full
25	[25]	No	No	Yes	Basic	Yes	N/A	Full
26	[26]	No	Yes	Yes	Detailed	Yes	N/A	Full
27	[27]	No	No	No	None	Yes	N/A	Partial
28	[28]	No	No	No	None	Yes	N/A	Partial
29	[30]	Yes	No	No	None	No	N/A	Partial
30	[31]	Yes	No	No	Basic	No	N/A	Partial
31	[32]	No	No	No	None	No	N/A	Partial
32	[33]	No	Yes	No	Detailed	No	N/A	Partial
33	[34]	Yes	Yes	No	None	Yes	N/A	Partial
34	[35]	No	No	Yes	Basic	No	No	Full
35	[36]	No	No	No	None	No	Yes	Partial
36	[37]	No	No	Yes	Basic	No	No	Partial
37	[38]	No	No	No	Basic	Unclear	No	Partial

Coding protocol. Coding was based on the full article, available supplementary material, and identifiable public code repositories. Positive evidence was not inferred solely from article titles or citations to external work. "Theory" required a study-specific theorem, proof, or rigorous convergence or error bound for the neural method; empirical convergence alone was coded No. "Classical baseline" required a quantitative same-problem comparison with an identified non-neural fractional solver using compatible equations, parameters, domains, and error measures. A classical solution used only to generate reference data was coded No.

"Public code" required a functioning public repository or executable implementation. Cost reporting was coded Basic when at least one meaningful cost item, such as runtime, hardware, memory, complexity, or residual-evaluation cost, was reported. It was coded Detailed when runtime and hardware information was supplemented by memory, operator-residual or inference timing, resolution scaling, or cost-error evidence. "Operator ablation" required variation of the fractional-operator approximation or its resolution under fixed or reasonably matched learning settings.

Quantitative uncertainty required dispersion estimates, uncertainty intervals, posterior summaries, or repeated-run evidence. It was coded N/A when uncertainty analysis was not methodologically applicable. "Reproducibility" was coded Full only when public code was accompanied by sufficiently complete operator, architecture, optimization, sampling, boundary-condition, and reference-solver specifications. Partial and Limited indicate progressively more consequential omissions. "Unclear" indicates that the available evidence did not support a defensible categorical assignment.

Table S8. Primary/secondary mechanism crosswalk for the 10 hybrid studies included in the simultaneous alternative-assignment stress test. Primary classes follow the operator-first decision rules. Stress-test classes represent plausible alternative assignments obtained by foregrounding the methodological novelty, implementation, representation, or problem class.

Study	Primary class	Secondary mechanism	Stress-test class	Decision rationale
[2]	Discretization-based	Spectral collocation for the integer-order spatial terms	Spectral/basis	Operator-first coding selects the numerical approximation used for the Caputo time-fractional derivative. The alternative spectral/basis assignment foregrounds the spectral collocation treatment of the spatial terms in the time-fractional phase-field models.
[4]	Discretization-based	Bi-orthogonal stochastic expansion and weak-form constraints	Spectral/basis	The fractional derivative is discretized on a grid. The alternative applies a representation-first rule that foregrounds the bi-orthogonal stochastic basis and associated weak formulation.
[5]	Analytical-prior	Gaver-Stehfest numerical inverse Laplace transform	Quadrature/integral	The Laplace transform reformulates the time-fractional operator and removes its direct time-domain evaluation from the neural residual. The alternative broadly treats numerical inversion of the inverse-transform integral as the dominant implementation step.
[8]	Discretization-based	Auxiliary-output representation of integral terms and adaptive loss weighting	Quadrature/integral	Operator-first coding selects the numerical scheme used for the Caputo derivative. A problem-first alternative foregrounds the FPIDE integral terms, although these terms are represented by auxiliary network outputs rather than directly discretized by quadrature.
[13]	Variable/distributed-order	Composite order quadrature with BDF2/fractional Newton-Gregory temporal differences	Quadrature/integral	Distributed-order treatment is the principal task. The alternative applies an implementation-first rule that foregrounds the composite numerical integration over fractional order.
[14]	Variable/distributed-order	Order quadrature with L- or S-type fractional approximation formulas	Discretization-based	Distributed-order modelling is the principal task. The alternative foregrounds the quadrature-based numerical discretization embedded in the distributed-order residual.
[15]	Monte Carlo	Gauss–Jacobi radial quadrature for fractional operators and generalized Gauss-Laguerre radial quadrature for tempered operators, combined with Monte Carlo angular sampling	Quadrature/integral	The multidimensional operator is decomposed into a one-dimensional radial integral and an integral over the unit sphere. Deterministic Gaussian quadrature evaluates the radial component, whereas Monte Carlo sampling remains responsible for the angular component that enables high-dimensional scaling. The primary assignment is therefore Monte Carlo, while the alternative assignment foregrounds the deterministic radial-quadrature enhancement.
[19]	Operator-learning	L2 approximation of the Caputo fractional derivative	Discretization-based	The principal task is learning a family-to-solution operator map. The alternative applies a residual-first rule that foregrounds the embedded L2 approximation used to construct the physics-informed loss.
[31]	Spectral/basis	Distributed-order formulation with Legendre-Gauss order quadrature	Variable/distributed-order	The adaptive fractional Chebyshev basis and associated operational matrices constitute the proposed residual-construction mechanism. The alternative applies a problem-first rule that foregrounds the distributed-order model class.
[34]	Discretization-based	State-transition-matrix solution representation	Analytical-prior	The ψ-Caputo derivative is approximated using an extended Kumar-Ertürk method. The alternative applies a representation-first rule that foregrounds the analytical state-transition-matrix formulation. This study concerns fractional time-varying dynamical systems and is retained as a scope-boundary study rather than direct FPDE evidence.

Table S9. Underlying class-level data and deterministic evidence-coverage scores displayed in revised Figure 5. *B* denotes a direct classical baseline and *A* a fractional-operator ablation.

Dominant class	N_c	Theory T_c/N_c	Comparison/attribution $(B_c + A_c)/(2N_c)$	Detailed cost D_c/N_c	Full reproducibility R_c/N_c
Discretization-based	15	1/15 = 6.7%	(6 + 7)/30 = 43.3%	4/15 = 26.7%	5/15 = 33.3%
Quadrature/integral	4	1/4 = 25.0%	(1 + 0)/8 = 12.5%	0/4 = 0.0%	1/4 = 25.0%
Spectral/basis	7	4/7 = 57.1%	(1 + 1)/14 = 14.3%	0/7 = 0.0%	0/7 = 0.0%
Monte Carlo	3	0/3 = 0.0%	(0 + 1)/6 = 16.7%	1/3 = 33.3%	2/3 = 66.7%
Analytical-prior	3	0/3 = 0.0%	(1 + 0)/6 = 16.7%	1/3 = 33.3%	0/3 = 0.0%
Variable/distributed-order	3	1/3 = 33.3%	(2 + 1)/6 = 50.0%	2/3 = 66.7%	0/3 = 0.0%
Operator-learning	1	0/1 = 0.0%	(0 + 0)/2 = 0.0%	0/1 = 0.0%	1/1 = 100.0%
Scope-boundary conformable derivative	1	0/1 = 0.0%	(0 + 0)/2 = 0.0%	0/1 = 0.0%	0/1 = 0.0%

Aggregation note. No cross-domain total is calculated. Comparison/attribution coverage is the only composite cell and gives equal weight to baseline comparison and operator ablation. The other three columns are direct proportions. "Unclear" remains in the class denominator and is not counted as positive. The displayed percentages are descriptive coverage measures; they are not validated maturity thresholds and do not adjust for the substantial sampling uncertainty of small classes.

Table S10. Re-audit of 14 potential classical-solver comparisons, of which 11 met the criterion defined in main-text Table 5 and documented in Supplementary Table S7. Reported outcomes reflect the original studies and do not represent pooled or independently validated results.

Study	Comparison design	Eligible	Reported comparative outcome	Qualification
[1]	Head-to-head	Yes	Comparable space-fractional convergence and favourable performance for sparse black-box forcing; optimization error saturated under refinement.	Training cost was generally greater than FDM; the advantage was regime dependent.
[2]	Head-to-head	Yes	Lower reported errors than two classical methods on the tested time-fractional phase-field examples.	Smooth selected benchmarks; no matched wall-clock or memory comparison.
[4]	Reference only	No	QMC-BO and QMC solutions generated reference modes and statistics.	The reference solver was not evaluated as a competitor under common error and cost criteria.
[6]	Head-to-head	Yes	Moderate neural error was reached quickly, whereas a refined classical method attained lower error at greater runtime.	One equation family and unmatched resolution/training budgets.
[9]	Head-to-head	Yes	Lower reported error than FDM on all tested grids, with the largest difference on sparse grids.	No matched FDM runtime or memory comparison.
[10]	Complexity only	No	Asymptotic complexity was contrasted with FDM and comparable accuracy was asserted.	No quantitative same-problem classical error result was reported.
[12]	Head-to-head	Yes	RBF-fPINN error remained below plain-fPINN and FDM errors as the tested solution became sharper.	Point count rather than computational budget was matched.
[14]	Head-to-head	Yes	dPINN and FDM produced comparable errors, with their ordering changing across fractional orders and temporal resolutions.	The authors note that non-convex PINN optimization can be slower, but no matched wall-clock comparison was reported; the claimed irregular-domain and inverse-problem advantages were not isolated in the FDM comparison.
[20]	Reference only	No	FDM output served as the numerical reference for the neural error.	No independent truth or symmetric evaluation of both solvers.
[21]	Head-to-head	Yes	Lower reported error than FDM at the same degrees of freedom.	Adam optimization was expensive and computational complexity was not compared.
[24]	Head-to-head	Yes	The neural method and an identified spectral numerical method produced similar pointwise errors on the tested fractional-diffusion problem.	The comparison used only one spectral resolution, $M = 10$, and did not match runtime, memory, or computational budget; it is therefore evidence of same-problem accuracy comparison rather than cost superiority.

Continued on next page

Study	Comparison design	Eligible	Reported comparative outcome	Qualification
[26]	Head-to-head	Yes	Third-order HNS exceeded L1 and L2-1σ accuracy at coarse resolution.	The fifth-order variant became unstable on fine temporal intervals.
[33]	Head-to-head	Yes	DSC-PINN was generally more accurate for solutions with initial singularities and required less time and memory than the FDM configuration attaining comparable overall accuracy.	For $\alpha = 0.75$, finely resolved FDM produced a smaller L^∞ error, whereas DSC-PINN retained a slightly smaller relative L^2 error; the exception is therefore norm- and resolution-dependent.
[34]	Head-to-head	Yes	Lower reported errors than predictor-corrector, L1 predictor-corrector, Haar-wavelet, and PINN-L1 methods.	Transferable FODE scope-boundary study; not evidence of FPDE-wide competitiveness.

References

1. G. Pang, L. Lu, G. E. Karniadakis, fPINNs: Fractional physics-informed neural networks, *SIAM J. Sci. Comput.*, **41** (2019), A2603–A2626. https://doi.org/10.1137/18M1229845
2. S. Wang, H. Zhang, X. Jiang, Fractional physics-informed neural networks for time-fractional phase field models, *Nonlinear Dyn.*, **110** (2022), 2715–2739. https://doi.org/10.1007/s11071-022-07746-3
3. L. Guo, H. Wu, X. Yu, T. Zhou, Monte Carlo fPINNs: deep learning method for forward and inverse problems involving high dimensional fractional partial differential equations, *Comput. Methods Appl. Mech. Eng.*, **400** (2022), 115523. https://doi.org/10.1016/j.cma.2022.115523
4. L. Ma, R. Li, F. Zeng, L. Guo, G. E. Karniadakis, Bi-orthogonal fPINN: a physics-informed neural network method for solving time-dependent stochastic fractional PDEs, *Commun. Comput. Phys.*, **34** (2023), 1133–1176. https://doi.org/10.4208/cicp.OA-2023-0075
5. X. B. Yan, Z. Q. J. Xu, Z. Ma, Laplace-fPINNs: Laplace-based fractional physics-informed neural networks for solving forward and inverse problems of a time fractional equation, *East Asian J. Appl. Math.*, **14** (2024), 657–674. https://doi.org/10.4208/eajam.2023-197.171223
6. S. Wang, H. Zhang, X. Jiang, Physics-informed neural network algorithm for solving forward and inverse problems of variable-order space-fractional advection-diffusion equations, *Neurocomputing*, **535** (2023), 64–82. https://doi.org/10.1016/j.neucom.2023.03.032
7. H. Ren, X. Meng, R. Liu, J. Hou, Y. Yu, A class of improved fractional physics informed neural networks, *Neurocomputing*, **562** (2023), 126890. https://doi.org/10.1016/j.neucom.2023.126890
8. J. Zhang, Y. Zhao, Y. Tang, Adaptive loss weighting auxiliary output fPINNs for solving fractional partial integro-differential equations, *Physica D*, **460** (2024), 134066. https://doi.org/10.1016/j.physd.2024.134066
9. J. Shi, X. Yang, X. Liu, A novel fractional physics-informed neural networks method for solving the time-fractional Huxley equation, *Neural Comput. Appl.*, **36** (2024), 19097–19119. https://doi.org/10.1007/s00521-024-10177-3
10. J. Shi, X. Liu, X. Yang, Data-driven solutions and parameter estimation of the high-dimensional time-fractional reaction-diffusion equations using an improved fPINN method, *Nonlinear Dyn.*, **113** (2025), 9577–9604. https://doi.org/10.1007/s11071-024-10786-6

11. T. Zhang, D. Zhang, S. Shi, Z. Guo, Spectral-fPINNs: spectral method based fractional physics-informed neural networks for solving fractional partial differential equations, *Nonlinear Dyn.*, **113** (2025), 12565–12588. https://doi.org/10.1007/s11071-025-11067-6
12. M. Mohammadi, R. Mokhtari, M. Ramezani, RBF-fPINNs: radial basis function-enhanced fractional physics-informed neural networks, *Eng. Comput.*, **42** (2026), 42. https://doi.org/10.1007/s00366-025-02258-1
13. W. Liu, Y. Liu, H. Li, Y. Yang, Multi-output physics-informed neural network for one- and two-dimensional nonlinear time distributed-order models, *Netw. Heterog. Media*, **18** (2023), 1899–1918. https://doi.org/10.3934/nhm.2023080
14. M. Ramezani, M. Mohammadi, R. Mokhtari, dPINNs: a physics-informed framework for forward and inverse problems governed by distributed-order derivatives, *Eng. Anal. Bound. Elem.*, **179** (2025), 106418. https://doi.org/10.1016/j.enganabound.2025.106418
15. Z. Hu, K. Kawaguchi, Z. Zhang, G. E. Karniadakis, Tackling the curse of dimensionality in fractional and tempered fractional PDEs with physics-informed neural networks, *Comput. Methods Appl. Mech. Eng.*, **432** (2024), 117448. https://doi.org/10.1016/j.cma.2024.117448
16. S. M. Nuugulu, K. C. Patidar, D. T. Tarla, A physics informed neural network approach for solving time fractional Black-Scholes partial differential equations, *Optim. Eng.*, **26** (2025), 2419–2448. https://doi.org/10.1007/s11081-024-09910-7
17. S. M. Nuugulu, K. C. Patidar, D. T. Tarla, An efficient physics-informed neural network solution to the time-space fractional Black-Scholes equation, *Oper. Res. Forum*, **6** (2025), 172. https://doi.org/10.1007/s43069-025-00570-6
18. X. Kang, Y. Li, Y. Li, J. Hu, K. Zheng, Solving the fractional Allen-Cahn equation and the fractional Cahn-Hilliard equation with the fractional physics-informed neural networks, *Fractal Fract.*, **9** (2025), 773. https://doi.org/10.3390/fractalfract9120773
19. B. Lu, Z. Hao, C. Moya, G. Lin, FPINN-deeponet: a physics-informed operator learning framework for multi-term time-fractional mixed diffusion-wave equations, *J. Comput. Phys.*, **538** (2025), 114184. https://doi.org/10.1016/j.jcp.2025.114184
20. S. Jiang, X. Li, Solving the non-local Fokker-Planck equations by deep learning, *Chaos*, **33** (2023), 043107. https://doi.org/10.1063/5.0128935
21. Z. Hao, M. Park, Z. Cai, Neural network method for integral fractional Laplace equations, *East Asian J. Appl. Math.*, **13** (2023), 95–118. https://doi.org/10.4208/eajam.010122.210722
22. H. Qu, X. Liu, Z. She, Neural network method for fractional-order partial differential equations, *Neurocomputing*, **414** (2020), 225–237. https://doi.org/10.1016/j.neucom.2020.07.063
23. Y. Ye, H. Fan, Y. Li, X. Liu, H. Zhang, Deep neural network methods for solving forward and inverse problems of time fractional diffusion equations with conformable derivative, *Neurocomputing*, **509** (2022), 177–192. https://doi.org/10.1016/j.neucom.2022.08.030
24. X. Fang, L. Qiao, F. Zhang, F. Sun, Explore deep network for a class of fractional partial differential equations, *Chaos Solitons Fract.*, **172** (2023), 113528. https://doi.org/10.1016/j.chaos.2023.113528

25. Z. Ma, J. Hou, W. Zhu, Y. Peng, Y. Li, PMNN: physical model-driven neural network for solving time-fractional differential equations, *Chaos Solitons Fract.*, **177** (2023), 114238. https://doi.org/10.1016/j.chaos.2023.114238

26. J. Hou, Z. Ma, S. Ying, Y. Li, HNS: an efficient Hermite neural solver for solving time-fractional partial differential equations, *Chaos Solitons Fract.*, **181** (2024), 114637. https://doi.org/10.1016/j.chaos.2024.114637

27. X. Wang, X. Wang, H. Xu, H. Qi, Numerical analysis and comparison of fractional physics-informed neural networks in unsaturated flow process, *Commun. Nonlinear Sci. Numer. Simul.*, **147** (2025), 108833. https://doi.org/10.1016/j.cnsns.2025.108833

28. N. Xue, M. Chen, Multiprecision computing for multistage fractional physics-informed neural networks, *Chaos*, **35** (2025), 113109. https://doi.org/10.1063/5.0287809

29. D. Dabiri, J. DaRosa, M. Saadat, D. Mangal, S. Jamali, A detailed and comprehensive account of fractional physics-informed neural networks: from implementation to efficiency, *Artif. Intell. Eng.*, **2** (2026), 251–264. https://doi.org/10.1049/aie2.70024

30. E. Abdo, L. Chai, R. Hu, X. Yang, Convergence analysis of PINNs for fractional diffusion equations in bounded domains, *arXiv*, 2026. https://doi.org/10.48550/arXiv.2601.01462

31. H. Momeni, A. Yazdani Cherati, A. Valinejad, AFP-PINN: adaptive fractional polynomial-based physics-informed neural network for solving distributed-order time-fractional diffusion equations and its error analysis, *Phys. Scr.*, **100** (2025), 076026. https://doi.org/10.1088/1402-4896/adeaf8

32. Q. Wu, F. Ma, W. Wo, FS-PINNs: fractional spectrally adapted physics-informed neural networks for fractional partial differential equations, *Math. Methods Appl. Sci.*, **48** (2025), 13882–13889. https://doi.org/10.1002/mma.11149

33. H. Li, Z. Tan, DSC-PINN: a decomposed singularity-capturing PINN for time-fractional PDEs involving initial singularities, *J. Comput. Phys.*, **561** (2026), 114965. https://doi.org/10.1016/j.jcp.2026.114965

34. S. M. Sivalingam, V. Govindaraj, Solution representation and a generalised physics-informed neural network for ψ-Caputo-type fractional time-varying systems, *Nonlinear Dyn.*, **113** (2025), 19125–19157. https://doi.org/10.1016/j.jcp.2026.114965

35. T. Ahmad, M. Sulaiman, D. Bassir, F. S. Alshammari, G. Laouini, Enhanced numerical solutions for fractional PDEs using Monte Carlo PINNs coupled with cuckoo search optimization, *Fractal Fract.*, **9** (2025), 225. https://doi.org/10.3390/fractalfract9040225

36. J. Zhang, X. Tian, C. Zhou, FFNN: fractional order basis function multi-step neural network method for fractional partial differential equations, *Appl. Soft Comput.*, **185** (2025), 113907. https://doi.org/10.1016/j.asoc.2025.113907

37. M. Saadat, D. Mangal, S. Jamali, UniFIDES: universal fractional integro-differential equations solver, *APL Mach. Learn.*, **3** (2025), 016116. https://doi.org/10.1063/5.0258122

38. R. Zhang, F. Li, J. Liu, LT-PINNs: physics-informed neural networks based on Laplace transform for solving Caputo-type fractional partial differential equations, *Chin. Phys. B*, **35** (2026), 030201. https://doi.org/10.1088/1674-1056/adf827

39. M. Raissi, P. Perdikaris, G. E. Karniadakis, Physics-informed neural networks: a deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations, *J. Comput. Phys.*, **378** (2019), 686–707. https://doi.org/10.1016/j.jcp.2018.10.045
40. L. Lu, P. Jin, G. Pang, Z. Zhang, G. E. Karniadakis, Learning nonlinear operators via DeepONet based on the universal approximation theorem of operators, *Nat. Mach. Intell.*, **3** (2021), 218–229. https://doi.org/10.1038/s42256-021-00302-5
41. D. Kumar, S. Chaudhary, K. V. Srinivas, Fractional Crank-Nicolson-Galerkin finite element scheme for the time-fractional nonlinear diffusion equation, *Numer. Methods Partial Differ. Equ.*, **35** (2019), 2056–2075. https://doi.org/10.1002/num.22399
42. S. Wang, Y. Teng, P. Perdikaris, Understanding and mitigating gradient flow pathologies in physics-informed neural networks, *SIAM J. Sci. Comput.*, **43** (2021), A3055–A3081. https://doi.org/10.1137/20M1318043
43. R. Bischof, M. A. Kraus, Multi-objective loss balancing for physics-informed deep learning, *Preprint*, 2021. https://doi.org/10.13140/RG.2.2.20057.24169
44. A. D. Jagtap, K. Kawaguchi, G. E. Karniadakis, Adaptive activation functions accelerate convergence in deep and physics-informed neural networks, *J. Comput. Phys.*, **404** (2020), 109136. https://doi.org/10.1016/j.jcp.2019.109136
45. S. Mishra, R. Molinaro, Estimates on the generalization error of physics-informed neural networks for approximating PDEs, *IMA J. Numer. Anal.*, **43** (2023), 1–43. https://doi.org/10.1093/imanum/drab093
46. K. Shukla, A. D. Jagtap, G. E. Karniadakis, Parallel physics-informed neural networks via domain decomposition, *Preprint*, 2021. https://doi.org/10.48550/arXiv.2104.10013
47. B. Moseley, A. Markham, T. Nissen-Meyer, Finite basis physics-informed neural networks (FBPINNs): a scalable domain decomposition approach for solving differential equations, *Adv. Comput. Math.*, **49** (2023), 62. https://doi.org/10.1007/s10444-023-10065-9
48. L. Lu, X. Meng, Z. Mao, G. E. Karniadakis, DeepXDE: a deep learning library for solving differential equations, *SIAM Rev.*, **63** (2021), 208–228. https://doi.org/10.1137/19M1274067
49. G. E. Karniadakis, I. G. Kevrekidis, L. Lu, P. Perdikaris, S. Wang, L. Yang, Physics-informed machine learning, *Nat. Rev. Phys.*, **3** (2021), 422–440. https://doi.org/10.1038/s42254-021-00314-5
50. S. Cuomo, V. Schiano Di Cola, F. Giampaolo, G. Rozza, M. Raissi, F. Piccialli, Scientific machine learning through physics-informed neural networks: where we are and what's next, *J. Sci. Comput.*, **92** (2022), 88. https://doi.org/10.1007/s10915-022-01939-z
51. C. Meng, S. Griesemer, D. Cao, S. Seo, Y. Liu, When physics meets machine learning: a survey of physics-informed machine learning, *Mach. Learn. Comput. Sci. Eng.*, **1** (2025), 20. https://doi.org/10.1007/s44379-025-00016-0
52. A. A. Sophiya, A. K. Nair, S. Maleki, S. K. Krishnababu, A comprehensive analysis of PINNs: variants, applications, and challenges, *arXiv*, 2025. https://doi.org/10.48550/arXiv.2505.22761

53. F. Chen, D. Sondak, P. Protopapas, M. Mattheakis, S. Liu, D. Agarwal, et al., NeuroDiffEq: a Python package for solving differential equations with neural networks, *J. Open Source Softw.*, **5** (2020), 1931. https://doi.org/10.21105/joss.01931

54. W. E, B. Yu, The Deep Ritz Method: a deep learning-based numerical algorithm for solving variational problems, *Commun. Math. Stat.*, **6** (2018), 1–12. https://doi.org/10.1007/s40304-018-0127-z

55. J. Sirignano, K. Spiliopoulos, DGM: a deep learning algorithm for solving partial differential equations, *J. Comput. Phys.*, **375** (2018), 1339–1364. https://doi.org/10.1016/j.jcp.2018.08.029

56. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

57. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.

58. K. Diethelm, *The analysis of fractional differential equations*, Berlin: Springer-Verlag, 2010. https://doi.org/10.1007/978-3-642-14574-2

59. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Berlin: Springer-Verlag, 2014. https://doi.org/10.1007/978-3-662-61550-8

60. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3

61. F. Mainardi, *Fractional calculus and waves in linear viscoelasticity*, London: Imperial College Press, 2010.

62. C. Lubich, Discretized fractional calculus, *SIAM J. Math. Anal.*, **17** (1986), 704–719. https://doi.org/10.1137/0517050

63. C. Lubich, Convolution quadrature and discretized operational calculus. I, *Numer. Math.*, **52** (1988), 129–145. https://doi.org/10.1007/BF01398686

64. M. Stynes, E. O'Riordan, J. L. Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation, *SIAM J. Numer. Anal.*, **55** (2017), 1057–1079. https://doi.org/10.1137/16M1082329

65. B. Jin, R. Lazarov, Z. Zhou, Error estimates for a semidiscrete finite element method for fractional order parabolic equations, *SIAM J. Numer. Anal.*, **51** (2013), 445–466. https://doi.org/10.1137/120873984

66. G. Acosta, J. P. Borthagaray, A fractional Laplace equation: regularity of solutions and finite element approximations, *SIAM J. Numer. Anal.*, **55** (2017), 472–495. https://doi.org/10.1137/15M1033952

67. A. Bonito, J. E. Pasciak, Numerical approximation of fractional powers of elliptic operators, *Math. Comp.*, **84** (2015), 2083–2110. https://doi.org/10.1090/S0025-5718-2015-02937-8



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)