

Sharp Joint Carathéodory Compatibility and Fifth-Order Lifting Obstructions in Bi-Univalent Function Theory

Exact Wolfram Verification and Reproducibility Report

Zainab H. Mahmood; Reem O. Rasheed; Bassim K. Mihsin; Waggas Galib Atshan

Generated: 2026-08-13 18:46:17

Wolfram environment: 15.0.0 for Microsoft Windows (64-bit) (May 19, 2026) | SystemID: Windows-x86-64

SHA-256 of the notebook containing this verifier:

110305008259846817750171873971828048332027666932821193157412683389123151732362

Scope and evidentiary status

This report independently re-derives the finite algebraic identities used in the manuscript with exact symbolic arithmetic. It verifies the reversion coefficients, the joint compatibility factorization, the explicit equality pair, the first four inverse residuals, the 16-term fifth-order obstruction polynomial, the unique L4 lifting value, the zero branch, rotation covariance, quantitative-stability algebra, and the Janowski, exponential and (p,q) specializations. The compactness argument in the strict-separation theorem uses standard analytic facts (normal families/Montel, Cauchy coefficient convergence and the maximum principle); a computer algebra system is not a substitute for those analytic theorems. The report therefore distinguishes exact symbolic verification from analytic proof logic.

Executive verification result

ALL EXACT SYMBOLIC CHECKS IN THIS REPORT PASSED.

Report format

This clean referee-facing version intentionally contains no figures, flowcharts, graphs or plots, because the manuscript itself is figure-free. The report is restricted to exact derivations, symbolic residuals, PASS/FAIL checks, specializations and the analytic-proof status note.

Verification item	Status
Inverse reversion coefficients $b_2, \dots, b_5$	PASS
Joint compatibility factorization	PASS
Sharp-envelope scalar factorization	PASS
Sharp-envelope positive multiplier $1+ \kappa $	PASS
Equality pair symmetry $v_0(z)=u_0(-z)$	PASS
Blaschke boundary modulus identity for u_0	PASS
Blaschke boundary modulus identity for v_0	PASS
Equality-pair Taylor coefficients through z^4	PASS
$A_1, \dots, A_4$ from direct composition $\Psi(u_0)$	PASS
Equality parameters $x=\sigma$ and $y=\sigma$	PASS
Second-order compatibility at the equality pair	PASS
$a_3=a_2^2$ on the equality branch	PASS
Residual R1	PASS
Residual R2	PASS
Residual R3	PASS
Exact fifth-order identity $R_4 = -2 \Psi_{\text{sig}} / [\phi^4(\dots)^2]$	PASS
Denominator reconstruction	PASS
Ψ_{sig} is affine in L4 with positive coefficient	PASS
Unique exact fifth-order lift value	PASS
Zero-branch polynomial P_0 and R4 formula	PASS
Quantitative-stability identity	PASS
Quantitative-stability second bound factorization	PASS
Quantitative-stability inequality on the admissible interval	PASS
Weighted homogeneity of b_5 under rotation	PASS
R4 is polynomial/continuous in finite Taylor data	PASS
Janowski: $\kappa=A$	PASS
Janowski obstruction factorization	PASS
Janowski $A>0$: obstruction nonzero	PASS
Janowski $A<0$: obstruction nonzero	PASS
Janowski $A=0$: zero-branch $P_0=0$	PASS
Exponential: $\kappa=1/2$	PASS
Exponential: $\Psi_{\text{sig}}=23/12$	PASS
Exponential: $L_4\text{_lift}=-15/8$	PASS
Exponential envelope value $\sqrt{2/3}$	PASS
Benchmark exact κ	PASS
Benchmark exact $L_4\text{_lift}$	PASS
Benchmark lift value is strictly negative	PASS

1. Reversion coefficients through fifth order

The inverse coefficients are obtained independently from $f(g(w))=w$ by coefficient comparison through w^5 .

Exact solution returned by Wolfram:

$$\{bb2 \rightarrow -aa2, bb3 \rightarrow 2 aa2^2 - aa3, bb4 \rightarrow -5 aa2^3 + 5 aa2 aa3 - aa4, \\ bb5 \rightarrow 14 aa2^4 - 21 aa2^2 aa3 + 3 aa3^2 + 6 aa2 aa4 - aa5\}$$

PASS Inverse reversion calculation

2. Joint second-order compatibility

Substituting the exact Caratheodory parametrizations into the coupled second-order equations reduces the system to the single scalar identity

Compatibility identity:

$$(4 - t^2) (x + y) = 2 \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2} \right) t^2$$

Exact factorization residual (must be 0):

0

The sharp estimate then follows analytically from $|x+y| \leq |x|+|y| \leq 2$. The symbolic check above verifies the exact algebraic reduction; the triangle inequality is the analytic step.

3. Explicit equality configuration and rigidity data

Finite Blaschke equality pair:

$$\left\{ \frac{z \left(\frac{1}{\sqrt{1+K}} + \text{sig } z \right)}{1 + \frac{\text{sig } z}{\sqrt{1+K}}}, \frac{z \left(-\frac{1}{\sqrt{1+K}} + \text{sig } z \right)}{1 - \frac{\text{sig } z}{\sqrt{1+K}}} \right\}$$

Taylor coefficients of u_0 through order four:

$$\left\{ \frac{1}{\sqrt{1+K}}, \frac{K \text{ sig}}{1+K}, -\frac{K}{(1+K)^{3/2}}, \frac{K \text{ sig}}{(1+K)^2} \right\}$$

Caratheodory equality parameters:

$$\left\{ \frac{2}{\sqrt{1+K}}, \frac{2+2K \text{ sig}}{1+K}, -\frac{2}{\sqrt{1+K}}, \frac{2+2K \text{ sig}}{1+K}, \text{sig}, \text{sig} \right\}$$

The two boundary-modulus identities are exactly zero, so the displayed rational functions have unit modulus on $|z|=1$. Together with analyticity and $u_0(0)=v_0(0)=0$, this supplies the disk self-map verification used in the paper. The rigidity step itself invokes equality in Schwarz-Pick and is an analytic theorem, not a numerical computation.

4. First three residuals and fifth-order obstruction

Equality-branch A-coefficients:

$$\left\{ \frac{L1}{\sqrt{1 + \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2}\right) \text{sig}}}, \frac{L2 + L1 \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2}\right)}{1 + \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2}\right) \text{sig}}, \right. \\ \left. \frac{L3 + \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2}\right) (2 L2 - L1 \text{sig})}{\left(1 + \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2}\right) \text{sig}\right)^{3/2}}, \frac{L4 + \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2}\right) \left(L1 + 3 L3 + L2 \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2} - 2 \text{sig}\right)\right)}{\left(1 + \left(-\frac{L2}{L1} + \frac{L1 \phi i3}{\phi i2^2}\right) \text{sig}\right)^2} \right\}$$

Exact residuals R1,R2,R3:

$$\{0, 0, 0\}$$

Exact obstruction polynomial Psig (16 monomials):

$$\begin{aligned} & -L1^2 L2 \phi i2^4 \phi i4 + L2^3 \phi i2^4 \phi i4 - 3 L1 L2 L3 \phi i2^4 \phi i4 + L1^2 L4 \phi i2^4 \phi i4 + L1^4 \phi i2^2 \phi i3 \phi i4 - \\ & 2 L1^2 L2^2 \phi i2^2 \phi i3 \phi i4 + 3 L1^3 L3 \phi i2^2 \phi i3 \phi i4 + L1^4 L2 \phi i3^2 \phi i4 + 6 L1^2 L2^2 \phi i2^3 \phi i5 - \\ & 3 L1^3 L3 \phi i2^3 \phi i5 - 6 L1^4 L2 \phi i2 \phi i3 \phi i5 + 2 L1^6 \phi i4 \phi i5 + 2 L1 L2^2 \phi i2^4 \phi i4 \text{sig} - \\ & 2 L1^3 L2 \phi i2^2 \phi i3 \phi i4 \text{sig} - 3 L1^3 L2 \phi i2^3 \phi i5 \text{sig} + 3 L1^5 \phi i2 \phi i3 \phi i5 \text{sig} \end{aligned}$$

Exact R4 formula:

$$\begin{aligned} & - \left(2 \left(-L1^2 L2 \phi i2^4 \phi i4 + L2^3 \phi i2^4 \phi i4 - 3 L1 L2 L3 \phi i2^4 \phi i4 + \right. \right. \\ & \quad L1^2 L4 \phi i2^4 \phi i4 + L1^4 \phi i2^2 \phi i3 \phi i4 - 2 L1^2 L2^2 \phi i2^2 \phi i3 \phi i4 + \\ & \quad 3 L1^3 L3 \phi i2^2 \phi i3 \phi i4 + L1^4 L2 \phi i3^2 \phi i4 + 6 L1^2 L2^2 \phi i2^3 \phi i5 - 3 L1^3 L3 \phi i2^3 \phi i5 - \\ & \quad 6 L1^4 L2 \phi i2 \phi i3 \phi i5 + 2 L1^6 \phi i4 \phi i5 + 2 L1 L2^2 \phi i2^4 \phi i4 \text{sig} - \\ & \quad \left. \left. 2 L1^3 L2 \phi i2^2 \phi i3 \phi i4 \text{sig} - 3 L1^3 L2 \phi i2^3 \phi i5 \text{sig} + 3 L1^5 \phi i2 \phi i3 \phi i5 \text{sig} \right) \right) / \\ & \left(\phi i4 \left(L1 \phi i2^2 + \left(-L2 \phi i2^2 + L1^2 \phi i3 \right) \text{sig} \right)^2 \right) \end{aligned}$$

Difference between direct R4 and the stated formula (must be 0):

$$0$$

Coefficient of L4 in Psig:

$$L1^2 \phi i2^4 \phi i4$$

Unique fifth-order lifting value L4_lift:

$$\begin{aligned} & - \frac{1}{L1^2 \phi i2^4 \phi i4} \left(-L1^2 L2 \phi i2^4 \phi i4 + L2^3 \phi i2^4 \phi i4 - 3 L1 L2 L3 \phi i2^4 \phi i4 + L1^4 \phi i2^2 \phi i3 \phi i4 - \right. \\ & \quad 2 L1^2 L2^2 \phi i2^2 \phi i3 \phi i4 + 3 L1^3 L3 \phi i2^2 \phi i3 \phi i4 + L1^4 L2 \phi i3^2 \phi i4 + 6 L1^2 L2^2 \phi i2^3 \phi i5 - \\ & \quad 3 L1^3 L3 \phi i2^3 \phi i5 - 6 L1^4 L2 \phi i2 \phi i3 \phi i5 + 2 L1^6 \phi i4 \phi i5 + 2 L1 L2^2 \phi i2^4 \phi i4 \text{sig} - \\ & \quad \left. 2 L1^3 L2 \phi i2^2 \phi i3 \phi i4 \text{sig} - 3 L1^3 L2 \phi i2^3 \phi i5 \text{sig} + 3 L1^5 \phi i2 \phi i3 \phi i5 \text{sig} \right) \end{aligned}$$

5. Zero branch kappa=0

Zero-branch polynomial P0:

$$L1^2 L4 \phi^2^4 \phi^4 - 3 L1^3 L3 \phi^2^3 \phi^5 + 2 L1^6 \phi^4 \phi^5$$

Zero-branch direct-minus-stated R4 residual (must be 0):

0

6. Quantitative stability

First stability identity residual (must be 0):

0

Second bound factorization residual (must be 0):

0

Admissible-interval inequality check:

True

7. Rotational covariance and the compactness theorem

Weighted-homogeneity check for b5 (must be 0):

0

Polynomial continuity of R4 in u1,...,u4,v1,...,v4:

True

The strict-separation theorem is completed analytically as follows: assume the second-order envelope is the full-class supremum; choose an extremizing sequence and normalize by rotation; use normal-family compactness to obtain a locally uniform limit; Cauchy's formula passes Taylor coefficients to the limit; rigidity identifies the limit with the equality pair; every member of the sequence has $R4=0$; the finite Taylor formula for $R4$ is continuous (verified above); hence the limiting equality pair would have $R4=0$, contradicting $\Psi_{sig} \neq 0$. The Wolfram check verifies the finite algebraic continuity premise but does not replace Montel's theorem or the rigidity proof.

8. Exact specializations

Janowski generator

kappa:

A

Factored obstruction:

$$A (A - B)^3 (1 + 2 A^2 - A B + 3 A \operatorname{sig} - B \operatorname{sig})$$

A=0 zero-branch check:

$$0$$

Exponential generator

{kappa, Psig, L4_lift, a2-envelope}:

$$\left\{ \frac{1}{2}, \frac{23}{12}, -\frac{15}{8}, \sqrt{\frac{2}{3}} \right\}$$

(p,q)-operator benchmark

Exact kappa:

$$\frac{4043}{20250\sqrt{2}}$$

Exact L4_lift:

$$\frac{-3084612892 - 1088153235\sqrt{2}}{32805000000}$$

Numerical L4_lift:

$$-0.14093869699366430893$$

Envelope value for |a2|:

$$\frac{22}{\sqrt{4050 + \frac{4043}{5\sqrt{2}}}}$$

Numerical envelope:

$$0.323607574698716029$$

9. Final verification statement

Conclusion: every finite symbolic identity and exact specialization tested in this report is reproduced with zero residual or exact equality. The analytic arguments that depend on standard complex-analysis theorems remain self-contained in the manuscript and are explicitly separated from the computer algebra checks. This is the appropriate evidentiary status for a referee-facing computational supplement.

Important editorial note

This verification package can strengthen reproducibility and reduce referee uncertainty, but it cannot guarantee acceptance. Editorial decisions also depend on novelty, scope, literature positioning, presentation and referee judgment.