



*Research article***Sharp joint Carathéodory compatibility and fifth-order lifting obstructions in bi-univalent function theory****Zainab H. Mahmood^{1,*}, Reem O. Rasheed², Bassim K. Mihsin³ and Waggas Galib Atshan⁴**¹ Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq² Department of Mathematics, College of Education Tuzkhurmatu, Tikrit University, Tikrit, Iraq³ General Directorate of Education in Karbala, Karbala, Iraq⁴ Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq* **Correspondence:** Email: zainab.mahmood@sc.uobaghdad.edu.iq.

Abstract: Let $\Psi(z) = 1 + L_1z + L_2z^2 + L_3z^3 + L_4z^4 + \dots$ be analytic in $\mathbb{D}$ with $L_1, \dots, L_4 \in \mathbb{R}$ and $L_1 > 0$, and let $\mathcal{M}$ consist of the bi-univalent functions f for which a prescribed linear operator with coefficients $\Phi_2, \dots, \Phi_5 > 0$, applied to f and to f^{-1} , yields two functions subordinate to Ψ . Put $\kappa = L_1\Phi_3/\Phi_2^2 - L_2/L_1$. Coupling the two subordinations through the second-order Carathéodory parametrization gives $(4-t^2)(x+y) = 2\kappa t^2$, and hence $|c_1|^2 \leq 4/(1+|\kappa|)$ on the second-order compatible coefficient body V_2 . This envelope is sharp on V_2 and it is not asserted to be sharp for $\mathcal{M}$. Its equality set is a single rotation orbit of an explicit finite Blaschke pair; for $\kappa \geq 0$, the envelope coincides with the bound produced by the usual summation argument; for $\kappa < 0$, it improves it. We then ask whether this equality orbit can be the low-order truncation of a member of $\mathcal{M}$. The inverse-compatibility residuals R_1 , R_2 , and R_3 vanish identically on it, and the residual R_4 , attached to the fifth coefficient, is $R_4 = -2P_\sigma[\Phi_4(L_1\Phi_2^2 + \sigma(L_1^2\Phi_3 - L_2\Phi_2^2))^2]^{-1}$, $\sigma = \text{sgn } \kappa$, where P_σ is an explicit polynomial that is affine in L_4 with positive leading coefficient. Thus $P_\sigma \neq 0$ means precisely that the equality orbit violates fifth-order inverse compatibility; it gives no information about other points of V_2 . Combining this with the rigidity of the equality orbit and normality of the Schwarz class, we prove that $P_\sigma \neq 0$ implies the strict separation $\sup_{f \in \mathcal{M}} |c_1|^2 < 4/(1+|\kappa|)$. The gap is not quantified, and the exact value of the supremum is left open. For the Janowski generator $(1 + Az)/(1 + Bz)$, one has $\kappa = A$ and $P_\sigma = A(A-B)^3(A+\sigma)(2A+\sigma-B)$, which is nonzero for $A \neq 0$; for $A = 0$, the corresponding polynomial vanishes and no separation follows. For $\Psi(z) = e^z$, one has $P_{+1} = 23/12$, whence $\sup_{f \in \mathcal{M}} |a_2| < \sqrt{2/3}$.

Keywords: bi-univalent functions; subordination; Schwarz functions; Carathéodory functions; inverse coefficients; coefficient body; joint compatibility; lifting obstruction

Mathematics Subject Classification: 30C45, 30C50, 30J10

Notation, standing hypotheses and equation labels are those of the main text; κ , V_2 , K , σ , a , u_0 , v_0 , A_j , B_j , R_j , P_σ and $M = L_1^2\Phi_3 - L_2\Phi_2^2$ retain their meanings there. Nothing in this supplement is used in the proof of any result of the main text. This file consists of the present document together with the Wolfram Mathematica notebook `WolframExactVerificationReport.nb`, the script `WolframFinalVerificationReport.wl` and the report `WolframExactVerificationReport.pdf` described in Section S1.4.

Supplementary File S1

S1.1. Secondary coefficient consequences

Lemma S1.1. *Let $h \in \mathcal{P}$, $t = |c_1| \in [0, 2]$ and $v \in \mathbb{C}$. Then*

$$\left| c_2 - \frac{1+v}{2}c_1^2 \right| \leq \frac{4 + (|v| - 1)t^2}{2}. \quad (\text{S1.1})$$

Proof. Replacing h by $h(\rho \cdot)$ with $|\rho| = 1$ multiplies c_2 and c_1^2 by ρ^2 and leaves both sides of (S1.1) unchanged, so we may assume $c_1 = t \geq 0$. For $t < 2$ write $2c_2 = t^2 + (4 - t^2)x$, $|x| \leq 1$. Then

$$c_2 - \frac{1+v}{2}t^2 = \frac{t^2}{2} + \frac{(4 - t^2)x}{2} - \frac{t^2}{2} - \frac{vt^2}{2} = -\frac{vt^2}{2} + \frac{(4 - t^2)x}{2},$$

so the modulus is at most $|v|t^2/2 + (4 - t^2)/2$. For $t = 2$ one has $c_2 = 2$ and $|2 - 2(1 + v)| = 2|v|$, which equals the right side of (S1.1). $\square$

In Propositions S1.1 and S1.4 we assume $0 \leq L_2 \leq L_1$. For the family $L_1 = \sin \theta$, $L_2 = \frac{1}{2} \sin \theta \cos \theta$ with $\theta \in (0, \pi/2)$, whose value at $\theta = \pi/4$ is the data (6.16) of the main text, one has $L_2/L_1 = \frac{1}{2} \cos \theta \in (0, \frac{1}{2})$.

Proposition S1.1. *For every $f \in \mathcal{M}$,*

$$|a_3| \leq \frac{L_1}{\Phi_3}, \quad (\text{S1.2})$$

attained on V_2 by $u = \rho z^2$, $v = -\rho z^2$.

Proof. From the coefficient identity (2.14) for a_3 ,

$$a_3 = \frac{L_1}{2\Phi_3} \left[c_2 - \frac{L_1 - L_2}{2L_1} c_1^2 \right], \quad \frac{L_1 - L_2}{2L_1} = \frac{1+v}{2}, \quad v = -\frac{L_2}{L_1},$$

and $L_2 \geq 0$, $L_1 > 0$ give $|v| = L_2/L_1$. Lemma S1.1 yields

$$|a_3| \leq \frac{L_1}{4\Phi_3} \left[4 - \left(1 - \frac{L_2}{L_1} \right) t^2 \right] \leq \frac{L_1}{\Phi_3},$$

the last step by $L_2 \leq L_1$. At $u = \rho z^2$ one has $a_3 = L_1\rho/\Phi_3$. $\square$

Proposition S1.2. *For $\mu \in \mathbb{C}$ put $v_\mu = \mu L_1\Phi_3/\Phi_2^2 - L_2/L_1 = \mu\kappa - (1 - \mu)L_2/L_1$. Then*

$$a_3 - \mu a_2^2 = \frac{L_1}{2\Phi_3} \left[c_2 - \frac{1+v_\mu}{2} c_1^2 \right], \quad (\text{S1.3})$$

and for every $f \in \mathcal{M}$,

$$|a_3 - \mu a_2^2| \leq \frac{L_1}{\Phi_3} \left[1 + \frac{\max\{0, |\nu_\mu| - 1\}}{1 + |\kappa|} \right]. \quad (\text{S1.4})$$

Proof. From the identities (2.13) and (2.14) for a_2, a_3 ,

$$a_3 - \mu a_2^2 = \frac{L_1}{2\Phi_3} c_2 - \frac{L_1 - L_2}{4\Phi_3} c_1^2 - \frac{\mu L_1^2}{4\Phi_2^2} c_1^2 = \frac{L_1}{2\Phi_3} \left[c_2 - \frac{1}{2} \left(\frac{L_1 - L_2}{L_1} + \frac{\mu L_1 \Phi_3}{\Phi_2^2} \right) c_1^2 \right],$$

and $\frac{L_1 - L_2}{L_1} + \frac{\mu L_1 \Phi_3}{\Phi_2^2} = 1 + \nu_\mu$, which is (S1.3). Lemma S1.1 gives

$$|a_3 - \mu a_2^2| \leq \frac{L_1}{4\Phi_3} \left[4 + (|\nu_\mu| - 1)t^2 \right].$$

If $|\nu_\mu| \leq 1$ the bracket is at most 4. If $|\nu_\mu| > 1$ it increases in t^2 , and $t^2 \leq 4/(1 + |\kappa|)$ by Theorem 3.5 of the main text gives (S1.4). $\square$

For $\mu = 1$ one has $\nu_1 = \kappa$, and when $|\kappa| > 1$ the bound $|a_2^2 - a_3| \leq L_1/\Phi_3$ of Proposition 3.9 of the main text is stronger than (S1.4), since it uses the identity $(4 - t^2)(x + y) = 2\kappa t^2$ rather than only the envelope.

Proposition S1.3. For $\gamma_1 = a_2/2$ and $\gamma_2 = a_3/2 - a_2^2/4$,

$$\gamma_2 = \frac{L_1}{4\Phi_3} \left[c_2 - \frac{1 + \nu'}{2} c_1^2 \right], \quad \nu' = \frac{L_1 \Phi_3}{2\Phi_2^2} - \frac{L_2}{L_1} = \frac{\kappa - L_2/L_1}{2}, \quad (\text{S1.5})$$

and

$$|\gamma_1| \leq \frac{L_1}{2\Phi_2 \sqrt{1 + |\kappa|}}, \quad |\gamma_2| \leq \frac{L_1}{2\Phi_3} \left[1 + \frac{\max\{0, |\nu'| - 1\}}{1 + |\kappa|} \right]. \quad (\text{S1.6})$$

Proof. $\gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2)$, so (S1.5) is (S1.3) at $\mu = \frac{1}{2}$ divided by 2, and $\nu' = \nu_{1/2}$. The bounds follow from $|\gamma_1| = |a_2|/2$ with Corollary 3.6 of the main text, and from (S1.4) at $\mu = \frac{1}{2}$. $\square$

Proposition S1.4. Assume $0 \leq L_2 \leq L_1$. For $g = f^{-1}$ with $f \in \mathcal{M}$,

$$|b_2| = |a_2| \leq \frac{L_1}{\Phi_2 \sqrt{1 + |\kappa|}}, \quad |b_3| \leq \frac{L_1}{\Phi_3}, \quad (\text{S1.7})$$

the second attained on V_2 by $u = \rho z^2$, $v = -\rho z^2$.

Proof. $b_2 = -a_2$ gives the first bound, by Corollary 3.6 of the main text. From $\Phi_3 b_3 = B_2$,

$$b_3 = \frac{L_1}{2\Phi_3} \left[d_2 - \frac{L_1 - L_2}{2L_1} d_1^2 \right].$$

Apply Lemma S1.1 to k , whose first coefficient satisfies $|d_1| = t$, with $\nu = -L_2/L_1$ and $|\nu| = L_2/L_1$:

$$|b_3| \leq \frac{L_1}{4\Phi_3} \left[4 - \left(1 - \frac{L_2}{L_1} \right) t^2 \right] \leq \frac{L_1}{\Phi_3}.$$

At $u = \rho z^2$, $v = -\rho z^2$ one has $t = 0$, $d_2 = -2\rho$ and $b_3 = -L_1\rho/\Phi_3$. $\square$

S1.2. Lifting of the equality configuration of Proposition 3.9

The configuration $u_\star(z) = \rho z^2$, $v_\star(z) = -\rho z^2$ attains the bound of Proposition 3.9 of the main text and those of Propositions S1.1 and S1.4 on V_2 . (The equality configurations of Theorem 3.5, including the branch $\kappa = 0$ with lifting polynomial P_0 , are treated in Section 4 and Remark 4.8 of the main text.) Its behavior under the inverse-compatibility conditions is as follows.

Proposition S1.5. *For $u_\star(z) = \rho z^2$, $v_\star(z) = -\rho z^2$ with $|\rho| = 1$,*

$$R_1 = R_2 = R_3 = 0, \quad R_4 = \frac{\rho^2(3L_1^2\Phi_5 - 2L_2\Phi_3^2)}{\Phi_3^2}, \quad (\text{S1.8})$$

so this configuration lifts through order five if and only if $3L_1^2\Phi_5 = 2L_2\Phi_3^2$. For $L_1 = \sin \theta$, $L_2 = \frac{1}{2} \sin \theta \cos \theta$, $\theta \in (0, \pi/2)$, this is equivalent to

$$\tan \theta = \frac{\Phi_3^2}{3\Phi_5}. \quad (\text{S1.9})$$

Proof. Here $A_1 = A_3 = 0$, $A_2 = L_1\rho$, $A_4 = L_2\rho^2$, and $\Psi(v_\star(z)) = 1 - L_1\rho z^2 + L_2\rho^2 z^4 + O(z^6)$ gives $B_1 = B_3 = 0$, $B_2 = -L_1\rho$, $B_4 = L_2\rho^2$. Thus $a_2 = 0$, $a_3 = L_1\rho/\Phi_3$, $a_4 = 0$, $a_5 = L_2\rho^2/\Phi_5$, so $b_2 = b_4 = 0$, $b_3 = -a_3$ and $b_5 = 3a_3^2 - a_5$. Then $R_1 = R_3 = 0$, $R_2 = -L_1\rho + L_1\rho = 0$, and

$$R_4 = \Phi_5(3a_3^2 - a_5) - L_2\rho^2 = \frac{3L_1^2\rho^2\Phi_5}{\Phi_3^2} - 2L_2\rho^2.$$

For the last assertion, $3L_1^2\Phi_5 - 2L_2\Phi_3^2 = \sin \theta (3 \sin \theta \Phi_5 - \cos \theta \Phi_3^2)$ and $\sin \theta > 0$ on $(0, \pi/2)$. $\square$

For the data of Section 6.3 of the main text, $3L_1^2\Phi_5 - 2L_2\Phi_3^2 = \frac{134829}{22000} - \frac{25921}{6050} = \frac{446279}{242000} \neq 0$, and $\Phi_3^2/(3\Phi_5) = 1036840/1483119$, whose arctangent is $0.6101178 \dots \neq \pi/4$. No full-class conclusion is drawn here from (S1.8).

S1.3. The sign of $L_{4,\text{lift}}^{(1)}$ along the family of Section 6.3

Proposition S1.6. *Keep $p, q, \delta, \lambda, \gamma$ as in Section 6.3 of the main text, so that $\Phi_2 = \frac{45}{22}$, $\Phi_3 = \frac{161}{55}$, $\Phi_4 = \frac{18}{5}$, $\Phi_5 = \frac{44943}{11000}$, and let $L_1 = \sin \theta$, $L_2 = \frac{1}{2} \sin \theta \cos \theta$, $L_3 = \frac{1}{6} \sin^3 \theta$. Then for every $\theta \in [\pi/4, \pi/2)$ one has $\kappa > 0$, $\sigma = 1$ and*

$$L_{4,\text{lift}}^{(1)} < 0. \quad (\text{S1.10})$$

Proof. Write $s = \sin \theta \in [\frac{\sqrt{2}}{2}, 1)$ and $c = \cos \theta \in (0, \frac{\sqrt{2}}{2}]$. Since $\Phi_3/\Phi_2^2 = 7084/10125$,

$$\kappa(\theta) = \frac{7084}{10125} \sin \theta - \frac{\cos \theta}{2}, \quad \kappa'(\theta) = \frac{7084}{10125} \cos \theta + \frac{\sin \theta}{2} > 0 \text{ on } (0, \frac{\pi}{2}),$$

and $\kappa(\pi/4) = 4043\sqrt{2}/40500 > 0$, so $\kappa > 0$ on $[\pi/4, \pi/2)$.

By Corollary 4.5 of the main text, $L_{4,\text{lift}}^{(1)} = -P_{+1}|_{L_4=0}/(L_1^2\Phi_2^4\Phi_4)$, so (S1.10) is equivalent to $P_{+1}|_{L_4=0} > 0$. Substituting $L_1 = s$, $L_2 = sc/2$, $L_3 = s^3/6$, $\sigma = 1$ and the four rational Φ_j into P_σ with $L_4 = 0$,

$$P_{+1}|_{L_4=0} = \frac{9s^3}{1171280000} G(s, c), \quad (\text{S1.11})$$

$$G(s, c) = 1025156250 c^3 + 3956698125 c^2 s + 4100625000 c^2 - 9594276620 c s^2 - 12563758125 c s - 4100625000 c + 4422205137 s^3 + 9551286360 s^2 + 5738040000 s.$$

Using $c^2 = 1 - s^2$ and $c^3 = c(1 - s^2)$ and collecting the terms with and without a factor c ,

$$G(s, c) = N(s) - c D(s), \quad (\text{S1.12})$$

$$\begin{aligned} N(s) &= 4422205137 s^3 + 9551286360 s^2 + 5738040000 s + (1 - s^2)(3956698125 s + 4100625000) \\ &= 465507012 s^3 + 5450661360 s^2 + 9694738125 s + 4100625000, \end{aligned} \quad (\text{S1.13})$$

$$\begin{aligned} D(s) &= 9594276620 s^2 + 12563758125 s + 4100625000 - 1025156250 (1 - s^2) \\ &= 10619432870 s^2 + 12563758125 s + 3075468750. \end{aligned} \quad (\text{S1.14})$$

All coefficients of N and D are positive, so $N, D > 0$ for $s > 0$. On $[\pi/4, \pi/2)$ one has $0 < c \leq \sqrt{2}/2$, hence

$$G(s, c) \geq N(s) - \frac{\sqrt{2}}{2} D(s) =: W(s), \quad (\text{S1.15})$$

$$\begin{aligned} W(s) &= 465507012 s^3 + (5450661360 - 5309716435 \sqrt{2}) s^2 \\ &\quad + (9694738125 - \frac{12563758125}{2} \sqrt{2}) s + (4100625000 - 1537734375 \sqrt{2}). \end{aligned}$$

Differentiating,

$$W'(s) = 1396521036 s^2 + (10901322720 - 10619432870 \sqrt{2}) s + 9694738125 - \frac{12563758125}{2} \sqrt{2},$$

a convex quadratic, so its maximum on $[\sqrt{2}/2, 1]$ is attained at an endpoint. Both endpoint values are negative:

$$W'\left(\frac{\sqrt{2}}{2}\right) = -226434227 - \frac{1662435405}{2} \sqrt{2} < 0, \quad W'(1) = 21992581881 - \frac{33802623865}{2} \sqrt{2} < 0,$$

the second because the first quantity below is smaller than the second:

$$\begin{aligned} (2 \cdot 21992581881)^2 &= 1\,934\,694\,631\,169\,957\,992\,644, \\ 2 \cdot 33802623865^2 &= 2\,285\,234\,760\,317\,335\,076\,450. \end{aligned}$$

Hence W is strictly decreasing on $[\sqrt{2}/2, 1]$ and $W(s) \geq W(1)$, where

$$W(1) = 19711531497 - \frac{26258659745}{2} \sqrt{2} > 0,$$

because the first quantity below exceeds the second:

$$\begin{aligned} (2 \cdot 19711531497)^2 &= 1\,554\,177\,895\,828\,892\,244\,036, \\ 2 \cdot 26258659745^2 &= 1\,379\,034\,423\,207\,366\,930\,050. \end{aligned}$$

By (S1.15), $G(s, c) \geq W(1) > 0$, so $P_{+1}|_{L_4=0} > 0$ by (S1.11). $\square$

Consequently, for every $\theta \in [\pi/4, \pi/2)$ and every prescribed $L_4 \geq 0$ one has $L_4 \neq L_{4,\text{lift}}^{(1)}$, which is the situation of Corollary 6.3 and Remark 6.4 of the main text; as there, this is a conditional statement, since the family does not determine L_4 . Outside $[\pi/4, \pi/2)$ the sign is not constant: at $\theta = \pi/12$ and $\theta = \pi/6$ one has $\kappa < 0$, $\sigma = -1$ and $L_{4,\text{lift}}^{(-1)} = +0.0778\dots, +0.00488\dots$ respectively, so the hypothesis $L_4 > 0$ does not exclude the lifting condition there.

S1.4. Exact symbolic verification

The accompanying Wolfram Mathematica notebook and .wl verifier perform independent exact symbolic checks of the displayed identities of the main text using exact arithmetic. They reproduce the inverse coefficients $b_2, \dots, b_5$; the expansion of u_0 and $v_0(z) = u_0(-z)$; the coefficients $A_1, \dots, A_4$ and the corresponding inverse data; the joint identity $(4 - t^2)(x + y) = 2\kappa t^2$; the cancellations $R_1 = R_2 = R_3 = 0$; and

$$R_4 = -\frac{2P_\sigma}{\Phi_4(L_1\Phi_2^2 + \sigma M)^2}.$$

They also verify that P_σ is affine in L_4 with coefficient $L_1^2\Phi_2^4\Phi_4$, the unique fifth-order lifting value, the branch $\kappa = 0$, rotation covariance, the quantitative-stability algebra, and the exact Janowski, exponential and (p, q) -example specializations. The accompanying Wolfram report records every check with PASS/FAIL status; all exact symbolic checks passed. The same identities are checked independently in exact arithmetic by the SymPy script of Supplementary File S2. These computations verify the displayed algebra and replace no proof: Schwarz–Pick rigidity, the maximum principle, normal families, Montel’s theorem and Cauchy coefficient convergence remain analytic arguments in the main text.



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)