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*Research article*

# Dynamic learning for a DC pension plan with random risk aversion

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## A. Proof of Theorem 7

We prove the local existence of the solution $(g^{\gamma_1}, g^{\gamma_2})$ on a small time interval near $T$.

**Step 1: Cole–Hopf transformation and homogenization.** Let

$$s_i(t,\hat{\mu}) = \left(g^{\gamma_i}(t,\hat{\mu})\right)^{\frac{\gamma_i}{1-\gamma_i}}, \qquad i = 1, 2.$$

Then $s_i(t,\hat{\mu})$ satisfies the auxiliary parabolic equation

$$(s_i)_t + \frac{1}{2}\Sigma^2 (s_i)_{\hat{\mu}\hat{\mu}} + b_i (s_i)_{\hat{\mu}} - c_i s_i = 0, \qquad s_i(T,\hat{\mu}) = 1, \tag{1}$$

where

$$b_i = \kappa(\bar{\mu} - \hat{\mu}) + (1-\gamma_i)\frac{\Sigma\hat{\mu}}{\varepsilon} + (1-\gamma_i)\Sigma^2\varphi,$$

$$c_i = (1-\gamma_i)\left\{-\left(r + \sigma_{p1}\hat{\mu}(t) + \sigma_{p2}\lambda_p - \eta\right) - \frac{\left(\lambda_p - \sigma_{p2}\right)^2 + \left(\hat{\mu}(t) - \sigma_{p1}\right)^2}{\varepsilon}\right.$$
$$\left. + \frac{\gamma_i}{2}\frac{\left(\lambda_p - \sigma_{p2}\right)^2 + \left(\hat{\mu}(t) - \sigma_{p1}\right)^2}{\varepsilon^2} - \Sigma\varphi\hat{\mu}\left(1 - \frac{\gamma_i}{\varepsilon}\right) + \frac{\gamma_i}{2}\Sigma^2\varphi^2\right\},$$

and $\varepsilon, \varphi$ are given by

$$\varepsilon = \frac{p_1\gamma_1 s_1 + p_2\gamma_2 s_2}{p_1 s_1 + p_2 s_2}, \tag{2}$$

$$\varphi(t,\hat{\mu}) = \frac{p_1(1-\gamma_1)\partial_{\hat{\mu}} s_1 + p_2(1-\gamma_2)\partial_{\hat{\mu}} s_2}{p_1\gamma_1 s_1 + p_2\gamma_2 s_2}. \tag{3}$$

To work with homogeneous boundary conditions, we shift the unknown functions by their terminal value

$$v_i(t,\hat{\mu}) = s_i(t,\hat{\mu}) - 1, \qquad i = 1,2. \tag{4}$$

A direct computation shows that $v_i$ satisfies the inhomogeneous equation

$$\partial_t v_i + \frac{1}{2}\Sigma^2 \partial^2_{\hat{\mu}} v_i + b_i \partial_{\hat{\mu}} v_i - c_i v_i = c_i, \tag{5}$$

together with homogeneous terminal and Dirichlet boundary conditions

$$v_i(T,\hat{\mu}) = 0, \qquad v_i(t,\pm L) = 0. \tag{6}$$

The coefficients $b_i, c_i$ remain exactly as above, with $\varepsilon$ and $\varphi$ now expressed through $v_i$ by substituting $s_i = 1 + v_i$ into (2)–(3). The state variable $\hat{\mu}$ is restricted to a sufficiently large symmetric interval $[-L, L]$. From the economic viewpoint, this is without loss of generality. The expected growth rate $\hat{\mu}(t)$ follows a mean-reverting O–U process, and its dynamics ensure that it remains in a bounded range with overwhelming probability. Moreover, for any finite time horizon, the probability that $|\hat{\mu}(t)|$ exceeds a large threshold $L$ decays exponentially in $L$, so taking $L$ sufficiently large makes the truncation error negligible. Hence, we work on the bounded domain $[-L, L]$ and impose homogeneous Dirichlet boundary conditions $v_i(t,\pm L) = 0$ for the transformed problem. The existence of a unique local solution on this domain is sufficient for our purposes.

**Step 2: Reformulation as a fixed-point problem.** Define the vector

$$\mathbf{v}(t,\hat{\mu}) = (v_1, v_2)',$$

where $v_i$ are the homogenized variables from Step 1. From (5)–(6), the system for $\mathbf{v}$ reads

$$\partial_t v_i + \mathcal{L}^{\gamma_i}(\mathbf{v}) v_i = c_i(\mathbf{v}), \qquad i = 1,2,$$

with the second-order linear operator

$$\mathcal{L}^{\gamma_i}(\mathbf{v}) v_i = \frac{1}{2}\Sigma^2 \partial^2_{\hat{\mu}} v_i + b_i(\mathbf{v}) \partial_{\hat{\mu}} v_i - c_i(\mathbf{v}) v_i,$$

where $b_i(\mathbf{v})$ and $c_i(\mathbf{v})$ are computed from $\mathbf{v}$ via the substitution $s_i = 1 + v_i$ in (2)–(3).

We now introduce the solution map $\mathfrak{F}$. Given a guess $\widetilde{\mathbf{v}} = (\widetilde{v}_1, \widetilde{v}_2)'$, we evaluate $\widetilde{\varepsilon}, \widetilde{\varphi}$ through the formulas above using $\widetilde{\mathbf{v}}$, and form the corresponding coefficients $\widetilde{b}_i = b_i(\widetilde{\mathbf{v}})$, $\widetilde{c}_i = c_i(\widetilde{\mathbf{v}})$. Then, for each $i = 1, 2$, solve the linear parabolic problem

$$\partial_t v_i + \frac{1}{2}\Sigma^2 \partial^2_{\hat{\mu}} v_i + \widetilde{b}_i \partial_{\hat{\mu}} v_i - \widetilde{c}_i v_i = \widetilde{c}_i, \quad v_i(T,\hat{\mu}) = 0,\ v_i(t,\pm L) = 0. \tag{7}$$

Equation (7) is decoupled; $v_1$ and $v_2$ are determined independently from their own equation, but both depend on the same guess $\widetilde{\mathbf{v}}$ through the coefficients. We define

$$\mathfrak{F}(\widetilde{\mathbf{v}}) = \mathbf{v} = (v_1, v_2)'.$$

A fixed point $\mathbf{v}^* = \mathfrak{F}(\mathbf{v}^*)$ solves (5)–(6), and therefore $s_i^* = 1 + v_i^*$ yields a solution of the original auxiliary system (1)–(3) with $s_i^*(T, \hat{\mu}) = 1$, $s_i^*(t, \pm L) = 1$. Reverting the Cole–Hopf transformation then gives the desired equilibrium strategies. We shall prove the existence of a unique fixed point in a weighted parabolic Sobolev space on a short time interval by using the Banach fixed-point theorem.

**Step 3: Function spaces with exponential weight.** We restrict the state variable $\hat{\mu}$ to a sufficiently large symmetric interval $[-L, L]$ and work on the bounded cylinder

$$Q = Q_\delta = [T - \delta, T] \times [-L, L].$$

Let $p > 3$. For a measurable function $f$ on $Q$, we define the $L^p$ norm

$$\|f\|_{L^p(Q)} = \left( \int_Q |f(t, \hat{\mu})|^p \mathrm{d}t\mathrm{d}\hat{\mu} \right)^{1/p}.$$

The standard (unweighted) parabolic Sobolev space $W_p^{1,2}(Q)$ is defined as the set of functions $f$ such that

$$\|f\|_{W_p^{1,2}(Q)} = \|f\|_{L^p(Q)} + \|\partial_t f\|_{L^p(Q)} + \|\partial_{\hat{\mu}} f\|_{L^p(Q)} + \|\partial_{\hat{\mu}}^2 f\|_{L^p(Q)} < \infty.$$

For a parameter $\lambda \geq 0$, we introduce the weighted parabolic Sobolev space $W_{p,\lambda}^{1,2}(Q)$ consisting of all functions $f$ for which

$$\|f\|_{W_{p,\lambda}^{1,2}(Q)} = \left\| e^{-\lambda(T-t)} f \right\|_{W_p^{1,2}(Q)} < \infty.$$

The standard (unweighted) space corresponds to $\lambda = 0$.

The product space we work with is

$$\mathbf{X}_{\delta,\lambda} = \{\mathbf{v} = (v_1, v_2) \in W_{p,\lambda}^{1,2}(Q) \times W_{p,\lambda}^{1,2}(Q) : v_i(T, \hat{\mu}) = 0,\ v_i(t, \pm L) = 0\},$$

endowed with the norm $\|\mathbf{v}\|_{\mathbf{X}_{\delta,\lambda}} = \|v_1\|_{W_{p,\lambda}^{1,2}(Q)} + \|v_2\|_{W_{p,\lambda}^{1,2}(Q)}$.

Because the spatial domain is one-dimensional and $p > 3$, the parabolic Sobolev embedding (which is unaffected by the smooth weight) gives

$$W_{p,\lambda}^{1,2}(Q) \hookrightarrow C^{1+\alpha/2, 2+\alpha}(Q) \quad \text{for some } \alpha \in (0, 1). \tag{8}$$

In particular, for any $f \in W_{p,\lambda}^{1,2}(Q)$,

$$\|f\|_{L^\infty(Q)} + \|\partial_{\hat{\mu}} f\|_{L^\infty(Q)} \leq C_0 \|f\|_{W_{p,\lambda}^{1,2}(Q)},$$

where the constant $C_0$ can be chosen independent of $\delta$ ([1], Lemma 2.5, Proposition 2.10).

Now we fix $\lambda > 0$ (e.g., $\lambda = 1$) and define the radius

$$R = \frac{1}{2} \min\Big(1, \frac{1}{C_0}\Big).$$

Then $C_0 R \leq \frac{1}{2}$. For a given $R$, we define the closed ball

$$B_R = \{\mathbf{v} \in \mathbf{X}_{\delta,\lambda} : \|\mathbf{v}\|_{\mathbf{X}_{\delta,\lambda}} \leq R\}.$$

For any $\mathbf{v} \in B_R$, we have

$$\|v_i\|_{L^\infty(Q)} \le C_0 R \le \frac{1}{2}, \qquad \|\partial_{\hat{\mu}} v_i\|_{L^\infty(Q)} \le C_0 R \le \frac{1}{2}.$$

Consequently, $s_i = 1 + v_i$ satisfies $s_i \ge \frac{1}{2}$ and $s_i \le \frac{3}{2}$, so the expressions (2)–(3) are well-defined, strictly positive, and uniformly bounded away from zero and infinity. Moreover, the coefficients $b_i, c_i$ belong to $L^\infty(Q)$ and the maps $\mathbf{v} \mapsto b_i, \mathbf{v} \mapsto c_i$ are Lipschitz from $(B_R, \|\cdot\|_{\mathbf{X}_{\delta,\lambda}})$ to $L^\infty(Q)$,

$$\|b_i(\mathbf{v}^{(1)}) - b_i(\mathbf{v}^{(2)})\|_{L^\infty(Q)} + \|c_i(\mathbf{v}^{(1)}) - c_i(\mathbf{v}^{(2)})\|_{L^\infty(Q)} \le L_{\text{coeff}} \|\mathbf{v}^{(1)} - \mathbf{v}^{(2)}\|_{\mathbf{X}_{\delta,\lambda}}, \tag{9}$$

where $L_{\text{coeff}}$ is a constant.

**Step 4: Linear estimate via maximal regularity.** We fix a guess $\widetilde{\mathbf{v}} \in B_R$ and let $\widetilde{b_i}, \widetilde{c_i}$ be the associated coefficients. Equation (7) for $v_i$ reads

$$\partial_t v_i + \frac{1}{2}\Sigma^2 \partial^2_{\hat{\mu}} v_i + \widetilde{b_i} \partial_{\hat{\mu}} v_i - \widetilde{c_i} v_i = \widetilde{c_i}, \qquad v_i(T, \hat{\mu}) = 0, \quad v_i(t, \pm L) = 0. \tag{10}$$

We introduce the reversed time variable

$$\tau = T - t, \qquad \tau \in [0, \delta],$$

and set

$$\tilde{v}_i(\tau, \hat{\mu}) = v_i(T - \tau, \hat{\mu}).$$

Since $\partial_t v_i = -\partial_\tau \tilde{v}_i$ and $\tilde{v}_i(0, \hat{\mu}) = v_i(T, \hat{\mu}) = 0$, Equation (10) transforms into the forward parabolic problem

$$\partial_\tau \tilde{v}_i = \frac{1}{2}\Sigma^2 \partial^2_{\hat{\mu}} \tilde{v}_i + \widetilde{b_i} \partial_{\hat{\mu}} \tilde{v}_i - \widetilde{c_i} \tilde{v}_i - \widetilde{c_i}, \qquad \tilde{v}_i(0, \hat{\mu}) = 0, \quad \tilde{v}_i(\tau, \pm L) = 0. \tag{11}$$

Now we introduce the exponential weight in reversed time,

$$w_i(\tau, \hat{\mu}) = e^{-\lambda \tau} \tilde{v}_i(\tau, \hat{\mu}).$$

Differentiating and using (11), we obtain

$$\partial_\tau w_i - \frac{1}{2}\Sigma^2 \partial^2_{\hat{\mu}} w_i - \widetilde{b_i} \partial_{\hat{\mu}} w_i + (\widetilde{c_i} + \lambda) w_i = -\, e^{-\lambda \tau} \widetilde{c_i}, \tag{12}$$

with initial condition $w_i(0, \hat{\mu}) = 0$ and boundary conditions $w_i(\tau, \pm L) = 0$.

Equation (12) is a uniformly parabolic equation with a positive zero-order coefficient $\widetilde{c_i} + \lambda$. By the weighted maximal regularity theory for parabolic equations ([2], Theorem 3.4; [3]), problem (12) admits a unique solution $w_i \in W_p^{1,2}(Q_\tau)$, and there exists a constant $C_p$ such that

$$\|w_i\|_{W_p^{1,2}(Q_\tau)} \le C_p \left\| e^{-\lambda \tau} \widetilde{c_i} \right\|_{L^p(Q_\tau)}. \tag{13}$$

It remains to translate this estimate back to the original variables. By definition,

$$v_i(t, \hat{\mu}) = \tilde{v}_i(T - t, \hat{\mu}) = e^{\lambda(T-t)} w_i(T - t, \hat{\mu}),$$

and the weighted norm on $Q = [T-\delta, T] \times [-L, L]$ is

$$\|v_i\|_{W^{1,2}_{p,\lambda}(Q)} = \left\|e^{-\lambda(T-t)} v_i\right\|_{W^{1,2}_p(Q)}.$$

Changing the time coordinate $\tau = T - t$ gives exactly

$$\|v_i\|_{W^{1,2}_{p,\lambda}(Q)} = \|w_i\|_{W^{1,2}_p(Q_\tau)}, \qquad \left\|e^{-\lambda(T-t)}\widetilde{c}_i\right\|_{L^p(Q)} = \left\|e^{-\lambda\tau}\widetilde{c}_i\right\|_{L^p(Q_\tau)}.$$

Thus (13) becomes the desired weighted estimate

$$\|v_i\|_{W^{1,2}_{p,\lambda}(Q)} \le C_p \|\widetilde{c}_i\|_{L^p_\lambda(Q)}, \tag{14}$$

where $\|\cdot\|_{L^p_\lambda(Q)} = \|e^{-\lambda(T-t)} \cdot\|_{L^p(Q)}$.

Since $\widetilde{\mathbf{v}} \in B_R$ guarantees a uniform bound on $\widetilde{c}_i$, its weighted $L^p$ norm is controlled by

$$\|\widetilde{c}_i\|_{L^p_\lambda(Q)} \le \left(\int_Q |\widetilde{c}_i(t,\hat{\mu})|^p \, dt \, d\hat{\mu}\right)^{1/p} = \|\widetilde{c}_i\|_{L^p(Q)} \le \|\widetilde{c}_i\|_{L^\infty} |Q|^{1/p} \le C_\delta \|\widetilde{c}_i\|_{L^\infty},$$

with $C_\delta = |Q|^{1/p} = (2L\delta)^{1/p}$. Thus the map $\mathfrak{F}$ is well-defined from $B_R$ into $\mathbf{X}_{\delta,\lambda}$.

**Step 5: Contraction and invariance for small $\delta$.** From Step 4, for any $\tilde{\mathbf{v}} \in B_R$, we have

$$\|\mathfrak{F}(\tilde{\mathbf{v}})\|_{\mathbf{X}_{\delta,\lambda}} \le 2C_p \|\widetilde{c}_i\|_{L^\infty} (2L)^{1/p} \, \delta^{1/p}. \tag{15}$$

We denote

$$C = 2C_p \|\widetilde{c}_i\|_{L^\infty} (2L)^{1/p}, \tag{16}$$

which is a finite constant depending only on $R$ and the model parameters.

Now take two elements $\widetilde{\mathbf{v}}^{(1)}, \widetilde{\mathbf{v}}^{(2)} \in B_R$ and set $\mathbf{v}^{(k)} = \mathfrak{F}(\widetilde{\mathbf{v}}^{(k)})$ $(k = 1, 2)$. Their difference $\Delta\mathbf{v} = \mathbf{v}^{(1)} - \mathbf{v}^{(2)}$ has components $\Delta v_i = v_i^{(1)} - v_i^{(2)}$. For each $i = 1, 2$, $\Delta v_i$ satisfies the linear problem

$$\partial_t \Delta v_i + \tfrac{1}{2}\Sigma^2 \partial^2_{\hat{\mu}} \Delta v_i + \widetilde{b}_i^{(2)} \partial_{\hat{\mu}} \Delta v_i - \widetilde{c}_i^{(2)} \Delta v_i = \mathcal{S}_i, \tag{17}$$

with $\Delta v_i(T, \hat{\mu}) = 0$, $\Delta v_i(t, \pm L) = 0$, and

$$\mathcal{S}_i = (\widetilde{c}_i^{(1)} - \widetilde{c}_i^{(2)}) + (\widetilde{b}_i^{(2)} - \widetilde{b}_i^{(1)}) \partial_{\hat{\mu}} v_i^{(1)} + (\widetilde{c}_i^{(2)} - \widetilde{c}_i^{(1)}) v_i^{(1)}. \tag{18}$$

Similar to (13), applying the maximal regularity estimate to (17)–(18) yields

$$\|\Delta v_i\|_{W^{1,2}_{p,\lambda}(Q)} \le C_p \|\mathcal{S}_i\|_{L^p_\lambda(Q)}. \tag{19}$$

To estimate $\|\mathcal{S}_i\|_{L^p_\lambda(Q)}$, we use the Lipschitz property (9) and the fact that $e^{-\lambda(T-t)} \le 1$,

$$\begin{aligned} \|\mathcal{S}_i\|_{L^p_\lambda(Q)} &\le \left\|\widetilde{c}_i^{(1)} - \widetilde{c}_i^{(2)}\right\|_{L^\infty} \left(1 + \|v_i^{(1)}\|_{L^\infty}\right) \|e^{-\lambda(T-t)}\|_{L^p(Q)} \\ &\quad + \left\|\widetilde{b}_i^{(2)} - \widetilde{b}_i^{(1)}\right\|_{L^\infty} \|\partial_{\hat{\mu}} v_i^{(1)}\|_{L^\infty} \|e^{-\lambda(T-t)}\|_{L^p(Q)} \\ &\le L_{\text{coeff}} \left(1 + \|v_i^{(1)}\|_{L^\infty} + \|\partial_{\hat{\mu}} v_i^{(1)}\|_{L^\infty}\right) (2L)^{1/p} \delta^{1/p} \, \|\widetilde{\mathbf{v}}^{(1)} - \widetilde{\mathbf{v}}^{(2)}\|_{\mathbf{X}_{\delta,\lambda}}. \end{aligned} \tag{20}$$

From (15), we have the bound estimate $\|\mathbf{v}^{(1)}\|_{\mathbf{X}_{\delta,\lambda}} \le C\delta^{1/p}$ and the Sobolev embedding (8), and we obtain

$$\|v_i^{(1)}\|_{L^\infty} \le C_0\|\mathbf{v}^{(1)}\|_{\mathbf{X}_{\delta,\lambda}} \le C_0 C\delta^{1/p}, \qquad \|\partial_{\hat{\mu}} v_i^{(1)}\|_{L^\infty} \le C_0\|\mathbf{v}^{(1)}\|_{\mathbf{X}_{\delta,\lambda}} \le C_0 C\delta^{1/p}.$$

Inserting these bounds into the estimate for $\|\mathcal{S}_i\|_{L^p_\lambda(Q)}$ yields

$$\|\mathcal{S}_i\|_{L^p_\lambda(Q)} \le L_{\text{coeff}}(2L)^{1/p}\delta^{1/p}(1 + 2C_0 C\delta^{1/p})\|\widetilde{\mathbf{v}}^{(1)} - \widetilde{\mathbf{v}}^{(2)}\|_{\mathbf{X}_{\delta,\lambda}}.$$

Now we plug this into (19) and sum over $i = 1, 2$:

$$\|\Delta\mathbf{v}\|_{\mathbf{X}_{\delta,\lambda}} \le 2C_p L_{\text{coeff}}(2L)^{1/p}\delta^{1/p}(1 + 2C_0 C\delta^{1/p})\|\widetilde{\mathbf{v}}^{(1)} - \widetilde{\mathbf{v}}^{(2)}\|_{\mathbf{X}_{\delta,\lambda}}. \tag{21}$$

Since $\delta^{1/p} \to 0$ as $\delta \to 0^+$, we can choose $\delta$ so small that

$$C\delta^{1/p} \le \frac{R}{2} \qquad \text{and} \qquad 2C_p L_{\text{coeff}}(2L)^{1/p}\delta^{1/p}(1 + 2C_0 C\delta^{1/p}) \le \frac{1}{2}.$$

Such a $\delta$ exists because the left-hand sides are continuous and vanish at $\delta = 0$. For this $\delta$, we have

- Invariance: $\|\mathfrak{F}(\tilde{\mathbf{v}})\|_{\mathbf{X}_{\delta,\lambda}} \le C\delta^{1/p} \le R/2 \le R$, hence $\mathfrak{F}(B_R) \subseteq B_R$.
- Contraction: From (21), $\|\mathfrak{F}(\tilde{\mathbf{v}}^{(1)}) - \mathfrak{F}(\tilde{\mathbf{v}}^{(2)})\|_{\mathbf{X}_{\delta,\lambda}} \le \frac{1}{2}\|\tilde{\mathbf{v}}^{(1)} - \tilde{\mathbf{v}}^{(2)}\|_{\mathbf{X}_{\delta,\lambda}}$.

Thus $\mathfrak{F}$ is a strict contraction on the closed ball $B_R$.

**Step 6: Banach fixed-point theorem.** $B_R$ is a non-empty complete metric space (as a closed subset of $\mathbf{X}_{\delta,\lambda}$). By the Banach fixed-point theorem, $\mathfrak{F}$ possesses a unique fixed point $\mathbf{v}^* \in B_R$ satisfying $\mathbf{v}^* = \mathfrak{F}(\mathbf{v}^*)$. This fixed point is exactly the solution of the nonlinear PDE system (5)–(6) on $Q = [T - \delta, T] \times [-L, L]$ when $\delta$ is small enough.

**Step 7: Recovery of the original functions and regularity.** Set $s_i^* = 1 + v_i^*$ for $i = 1, 2$. From the definition of $B_R$, we have $\|v_i^*\|_{W^{1,2}_{p,\lambda}(Q)} \le R$, so $s_i^* \ge 1 - C_0 R \ge \frac{1}{2}$ and $s_i^* \in W^{1,2}_{p,\lambda}(Q)$. Because $p > 3$ and the spatial dimension is 1, the parabolic Sobolev embedding (8) implies that each $s_i^*$ is Hölder continuous. The original functions are recovered via the Cole–Hopf transformation. This completes the proof.

## B. Proof for Proposition 13

The core of the proof is to establish the convexity of the mapping $\gamma \to g^\gamma(t, \hat{\mu})^{\frac{\gamma}{1-\gamma}}$. To clearly isolate the effect of random risk aversion from that of learning, this Appendix assumes that $\hat{\mu}(t) \equiv \bar{\mu}$. As will be evident from the subsequent proof, incorporating learning behavior only increases the complexity of the proof without altering the result.

We define

$$\Delta(t) = p_1 g^{\gamma_1}(t)^{\frac{\gamma_1}{1-\gamma_1}} + p_2 g^{\gamma_2}(t)^{\frac{\gamma_2}{1-\gamma_2}} - g^{\mathbb{E}[\gamma]}(t)^{\frac{\bar{\gamma}}{1-\bar{\gamma}}},$$

where $\Delta(t)$ satisfies

$$\Delta_t(t) = -\left(\left(r + \sigma_{p1}\bar{\mu} + \sigma_{p2}\lambda_p - \eta\right) + \frac{\left(\lambda_p - \sigma_{p2}\right)^2 + \left(\bar{\mu} - \sigma_{p1}\right)^2}{2\varepsilon(t)}\right)\Delta(t)$$

$$+\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon^2(t)}\left(\varepsilon(t)-\bar{\gamma}\right)g^{\bar{\gamma}}(t)^{\frac{\bar{\gamma}}{1-\bar{\gamma}}}.$$

A key observation is that $\varepsilon(t)$ is a wealth-weighted average of $\gamma_1$ and $\gamma_2$, with weights of

$$\frac{p_1 g^{\gamma_1}(t)^{\frac{\gamma_1}{1-\gamma_1}}}{p_1 g^{\gamma_1}(t)^{\frac{\gamma_1}{1-\gamma_1}}+p_2 g^{\gamma_2}(t)^{\frac{\gamma_2}{1-\gamma_2}}} \quad \text{and} \quad \frac{p_2 g^{\gamma_2}(t)^{\frac{\gamma_2}{1-\gamma_2}}}{p_1 g^{\gamma_1}(t)^{\frac{\gamma_1}{1-\gamma_1}}+p_2 g^{\gamma_2}(t)^{\frac{\gamma_2}{1-\gamma_2}}}.$$

Its time derivative satisfies

$$\varepsilon_t(t)=\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon^2(t)}\left[\mathbb{E}[\gamma^2]-\left(\mathbb{E}[\gamma]^2\right)\right]>0,$$

which can be found in [4], Section 4.3. Given the terminal condition $\varepsilon(T)=\bar{\gamma}$, it follows that $\varepsilon(t)<\bar{\gamma}$.

Now, we consider the time-reversed variable $\tau=T-t$ and let $\widetilde{\Delta}(\tau)=\Delta(T-\tau)$. Then $\widetilde{\Delta}(\tau)$ satisfies the following linear ODE:

$$\begin{aligned}\widetilde{\Delta_\tau}(\tau)=&\left(\left(r+\sigma_{p1}\bar{\mu}+\sigma_{p2}\lambda_p-\eta\right)+\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon(T-\tau)}\right)\widetilde{\Delta}(\tau)\\&-\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon^2(T-\tau)}\left(\varepsilon(T-\tau)-\bar{\gamma}\right)g^{\bar{\gamma}}(T-\tau))^{\frac{\bar{\gamma}}{1-\bar{\gamma}}},\end{aligned}$$

where $-\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon^2(T-\tau)}\left(\varepsilon(T-\tau)-\bar{\gamma}\right)$ is positive. Applying the variation-of-constants formula, the solution is

$$\begin{aligned}\widetilde{\Delta}(\tau)=&\exp\left\{\int_0^\tau\left(\left(r+\sigma_{p1}\bar{\mu}+\sigma_{p2}\lambda_p-\eta\right)+\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon(T-s)}\right)\mathrm{d}s\right\}\\&\times\int_0^\tau\left\{\left[-\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon^2(T-s)}\left(\varepsilon(T-s)-\bar{\gamma}\right)g^{\bar{\gamma}}(T-s))^{\frac{\bar{\gamma}}{1-\bar{\gamma}}}\right]\right.\\&\left.\times\exp\left\{\int_0^s\left(\left(r+\sigma_{p1}\bar{\mu}+\sigma_{p2}\lambda_p-\eta\right)+\frac{\left(\lambda_p-\sigma_{p2}\right)^2+\left(\bar{\mu}-\sigma_{p1}\right)^2}{2\varepsilon(T-u)}\right)\mathrm{d}u\right\}\right\}\mathrm{d}s.\end{aligned}$$

Therefore, we obtain $\widetilde{\Delta}(\tau)>0$ for all $\tau>0$, which is equivalent to $\Delta(t)>0$ for all $t<T$. By using Jensen's inequality, the proof is complete.

## C. Proof for Proposition 15

We concentrate on the case of a degenerate distribution for the risk aversion parameter $\gamma$. The extension to random risk aversion is analogous and does not change the conclusion.

Let $\phi^{\text{learning}}$ be the equilibrium strategy with learning, and $\phi^{\text{no-learning}}$ be the equilibrium strategy when the member ignores learning (i.e., fixed $\hat{\mu}(t) \equiv \bar{\mu}$). Under a degenerate distribution, $\varepsilon(t, \Gamma) \equiv \gamma$, and from (43)–(44), we obtain

$$\phi_I^{\text{no-learning}}(t) = \left(\frac{\lambda_p}{\sigma_{p2}} - 1\right)\frac{1}{\gamma} + 1, \qquad \phi_S^{\text{no-learning}}(t) = \frac{\bar{\mu} - \lambda_p \sigma_{p1}/\sigma_{p2}}{\sigma_S\, \gamma}.$$

Thus $\phi^{\text{no-learning}}$ depends only on model parameters and does not involve $Y^\phi$ or $P$. Hence it satisfies Definition 1 and belongs to the admissible set $\Pi$. Consequently,

$$V_{\bar{\gamma},\hat{\mu}}(t, y, p) = \sup_{\phi\in\Pi} J^\phi(t, y, p, \hat{\mu}) \geq J^{\phi^{\text{no-learning}}}(t, y, p, \hat{\mu}) = V_{\bar{\gamma},\bar{\mu}}(t, y, p).$$

Equality holds if and only if $\phi^{\text{no-learning}}$ is also optimal for the learning problem.

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