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Research article

Systemic stockout risk in e-commerce fulfillment networks: A fractional stochastic control approach

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1. Proofs

Throughout the appendix, Assumption 1 (WIS convention) is in force; $\mathcal{E}_t := \exp(\nu t + \sigma B_t^H - \frac{1}{2}\sigma^2 t^{2H})$ denotes the geometric WIS factor associated with a given fBm B^H, and $\nu := T^{2H}$ where convenient.

1.1. Proof of Lemma 1

Averaging (3.3) over $i \in \mathcal{N}$ with $u \equiv 0$ gives $d\bar{Z}_t = \frac{\kappa}{N}\sum_i(\bar{Z}_t - Z_t^{(i)})dt + \nu\bar{Z}_t dt + \frac{\sigma}{N}\sum_i Z_t^{(i)} dB_t^{H,i}$, and $\sum_{i=1}^N(\bar{Z}_t - Z_t^{(i)}) = N\bar{Z}_t - N\bar{Z}_t = 0$ identically, which proves (3.5); no property of the noise is used, so the cancellation holds for every κ and every dependence structure. (i) $\bar{B}_t^H = N^{-1}\sum_i B_t^{H,i}$ is centered Gaussian with $\mathrm{Var}(\bar{B}_t^H) = N^{-2}\sum_i \mathbb{E}[(B_t^{H,i})^2] = t^{2H}/N$ by independence; being Gaussian, H-self-similar with stationary increments, it equals $N^{-1/2}\hat{B}_t^H$ in law for a standard fBm $\hat{B}^H$. (ii) Writing (3.5) in integral form and taking expectations, the WIS integral term has zero expectation for integrands in the domain of the divergence operator [28, Section 3]; [29, Chapter 2], which contains the solutions of linear Wick-type systems; hence, $\mathbb{E}[\bar{Z}_t] = \bar{Z}_0 + \nu\int_0^t \mathbb{E}[\bar{Z}_s]ds$, so $\mathbb{E}[\bar{Z}_t] = \bar{Z}_0 e^{\nu t}$. ■

1.2. Proof of Proposition 1

The closed form (3.7). By Lemma 1 and Assumption 1(b), $\hat{Z}_T = \bar{Z}_0 \exp(\nu T + \sigma_N \hat{B}_T^H - \frac{1}{2}\sigma_N^2 T^{2H})$, so $\ln\hat{Z}_T$ is Gaussian with mean $\ln\bar{Z}_0 + \nu T - \frac{1}{2}\sigma_N^2 T^{2H}$ and variance $\sigma_N^2 T^{2H}$ (using $\mathbb{E}[(\hat{B}_T^H)^2] = T^{2H}$). Standardizing $\{\hat{Z}_T \le Z_{\min}\} = \{\ln\hat{Z}_T \le \ln Z_{\min}\}$ yields (3.7). The common-shock statement is established in part (a) below. ■

(a) Exact solution under a common shock ($\rho = 1$), any $\kappa \ge 0$, any N. Let all warehouses be driven by the same fBm B^H and let $y_i(t) := \bar{\xi} + e^{-\kappa t}(\xi_i - \bar{\xi})$, where $\bar{\xi} := N^{-1}\sum_i \xi_i$. The deterministic vector $y(t)$ solves $\dot{y}_i = \kappa(\bar{y} - y_i)$ with $\bar{y}(t) = \bar{\xi}$, since averaging preserves the mean and the deviation $y_i - \bar{\xi}$ decays at rate κ. We claim that $Z_t^{(i)} := y_i(t)\,\mathcal{E}_t$ solves the coupled system. Because y_i is deterministic and C^1, the WIS product rule reduces to $dZ^{(i)} = \dot{y}_i\mathcal{E}_t dt + y_i d\mathcal{E}_t = \kappa(\bar{y} - y_i)\mathcal{E}_t dt + y_i(\nu\mathcal{E}_t dt + \sigma\mathcal{E}_t dB_t^H) = \kappa(\bar{Z}_t - Z_t^{(i)})dt + \nu Z_t^{(i)}dt + \sigma Z_t^{(i)} dB_t^H$, using $\bar{y}\mathcal{E} = \bar{Z}$; equivalently, in the pathwise representation of Assumption 1(c), the

same computation is the classical chain rule. Averaging gives $\bar{Z}_t = \bar{\xi}\,\mathcal{E}_t$ exactly, so $\ln \bar{Z}_T$ is Gaussian with mean $\ln \bar{Z}_0 + \nu T - \frac{1}{2}\sigma^2 T^{2H}$ and variance $\sigma^2 T^{2H}$, which yields the stockout exposure formula of Proposition 1 with σ in place of σ_N, independent of κ and N.

(b) Closure remainder. For general shocks, subtracting the benchmark diffusion term from the exact one in (3.5) leaves $\mathcal{R}_t = \sigma N^{-1} \sum_i \int_0^t (Z_s^{(i)} - \bar{Z}_s) dB_s^{H,i}$. Each summand is a WIS integral of an admissible integrand and has zero expectation [28, 29], so $\mathbb{E}[\mathcal{R}_t] = 0$. The closure is therefore unbiased, and its error is a pure fluctuation whose size is controlled by the cross-sectional dispersion of inventories, the quantity D_N of Section 3.3. Larger transshipment intensity κ compresses dispersion and hence the remainder; the two exact regimes are (a) above, where dispersion in the *ratio* $Z^{(i)}/\bar{Z}$ is deterministic and the remainder vanishes identically, and the large-N regime of Theorem 1.

(c) Exact aggregate-variance identities ($\kappa = 0$, factor structure (5.4)). With $Z_T^{(i)} = \xi_i e^{\nu T} \exp(\sigma B_T^{H,i} - \frac{1}{2}\sigma^2 v)$ and $\mathrm{Cov}(B_T^{H,i}, B_T^{H,j}) = \rho v + (1-\rho) v \delta_{ij}$, the Gaussian moment formula gives, for $i \neq j$, $\mathbb{E}[Z_T^{(i)} Z_T^{(j)}] = \xi_i \xi_j e^{2\nu T} e^{\rho \sigma^2 v}$ and, for $i = j$, $\mathbb{E}[(Z_T^{(i)})^2] = \xi_i^2 e^{2\nu T} e^{\sigma^2 v}$. Hence, exactly,

$$\mathrm{Var}(\bar{Z}_T) \;=\; e^{2\nu T}\left[\frac{(\sum_i \xi_i)^2 - \sum_i \xi_i^2}{N^2}\, e^{\rho\sigma^2 v} \;+\; \frac{\sum_i \xi_i^2}{N^2}\, e^{\sigma^2 v} \;-\; \bar{Z}_0^2\right], \tag{A.1}$$

while the benchmark's variance is $\bar{Z}_0^2 e^{2\nu T}(e^{\sigma_{\mathrm{agg}}^2 v} - 1)$. Two error factors follow from (A.1) by expansion at $\rho = 0$. For small $\sigma^2 v$, the ratio of true to benchmark variance is $1 + s_\xi^2/\bar{\xi}^2$, where $s_\xi^2 := N^{-1}\sum_i (\xi_i - \bar{\xi})^2$ (initial heterogeneity inflates true variance), while for balanced inventories, the ratio is $(e^{\sigma^2 v} - 1)/(N(e^{\sigma^2 v/N} - 1)) \approx (e^{\sigma^2 v} - 1)/(\sigma^2 v)$ for large N (heavy per-warehouse tails inflate it). These are exactly the two degradation channels documented in Table 1. ■

1.3. Proofs of Corollaries 1 and 2

Corollary 1. Write the argument of Φ in (3.7) as $g = (-a - \nu T)/x + x/2$ with $a = \ln(\bar{Z}_0/Z_{\min})$ and $x = \sigma_N T^H$, so $P = \Phi(g)$ and $\partial P/\partial\theta = \phi(g)\,\partial g/\partial\theta$ for any parameter θ. (i) $\partial g/\partial \bar{Z}_0 = -1/(\bar{Z}_0 x)$. (ii) Using $\partial x/\partial T = Hx T^{-1}$, direct differentiation and regrouping give $\partial g/\partial T = x^{-1}[HaT^{-1} + Hx^2/(2T) + \nu(H-1)]$; for $a \geq 0$, $\nu \leq 0$, $H < 1$, every bracket term is nonnegative and the middle term is strictly positive. (iii) Using $\partial x/\partial H = x \ln T$, $\partial g/\partial H = [(a + \nu T)x^{-2} + \frac{1}{2}] x \ln T$, whose bracket is positive when $a + \nu T \geq 0$ and whose sign is then that of $\ln T$. (iv) $\partial g/\partial \sigma_N = [(a + \nu T)x^{-2} + \frac{1}{2}] T^H$. All expressions were verified symbolically, and the $T < 1$ reversal in (iii) was verified numerically.

Corollary 2. With $\ln \hat{Z}_T \sim N(m, v_*^2)$, $m = \ln \bar{Z}_0 + \nu T - \frac{1}{2}v_*^2$, $v_* = \sigma_N T^H$, the lognormal partial expectation is $\mathbb{E}[\hat{Z}_T \mathbf{1}\{\hat{Z}_T \leq K\}] = e^{m + v_*^2/2}\Phi(\frac{\ln K - m - v_*^2}{v_*}) = \bar{Z}_0 e^{\nu T}\Phi(d - v_*)$, where $d = (\ln K - m)/v_*$ is the argument of (3.7) at $K = Z_{\min}$. Then, $\mathbb{E}[(Z_{\min} - \hat{Z}_T)^+] = Z_{\min}\mathbb{P}\{\hat{Z}_T \leq Z_{\min}\} - \mathbb{E}[\hat{Z}_T \mathbf{1}\{\hat{Z}_T \leq Z_{\min}\}]$ gives (3.8). ■

1.4. Proof of Proposition 2

(i) Expanding, $\sum_{i,j}(z_j - z_i)^2 = \sum_{i,j}(z_j^2 - 2z_i z_j + z_i^2) = 2N\sum_i z_i^2 - 2(\sum_i z_i)^2 = 2N\sum_i (z_i - \bar{z})^2$. Multiplying by $(\kappa/N)^2$, taking expectations, and taking the square root gives $\|F_t^{(N)}\| = \sqrt{2}\,\kappa D_N(t)$.

(ii) The matrix $\Delta_{ij} := z_j - z_i$ is antisymmetric, so $\sum_{i,j}\Delta_{ij}^+ = \frac{1}{2}\sum_{i,j}(\Delta_{ij}^+ + \Delta_{ji}^+) = \frac{1}{2}\sum_{i,j}|\Delta_{ij}|$; hence, $V_t = \frac{\kappa}{N}\cdot\frac{1}{2}\sum_{i,j}|Z_t^{(j)} - Z_t^{(i)}| = \frac{\kappa N}{2} G_N(t)$.

(iii) On $\mathbb{R}^{N^2}$, $\|\Delta\|_2 \le \|\Delta\|_1 \le N\|\Delta\|_2$, and by (i) $\|\Delta\|_2 = (2N\sum_i (z_i - \bar{z})^2)^{1/2} = \sqrt{2}\,N d_N$. Since $V_t = \frac{\kappa}{2N}\|\Delta\|_1$, the bounds follow. For the Gaussian cross-section approximation, suppose that the z_i are approximately i.i.d. with standard deviation s. Then $G_N \to \mathbb{E}|z - z'| = 2s/\sqrt{\pi}$ and $d_N \to s$, so $V_t \approx \kappa N d_N/\sqrt{\pi}$; the approximation was verified numerically to within 1% for Gaussian cross-sections with $N \ge 25$. ■

1.5. Proof of Lemma 2

Let $\beta(\tau) := b(T-\tau)$, so (4.6) becomes the forward equation $\beta'(\tau) = 2a(T-\tau)\beta + h(T-\tau) - r^{-1}(T-\tau)\beta^2$, $\beta(0) = k \ge 0$, with $a(t) := \sigma^2 H t^{2H-1}$ continuous and bounded on $[0, T]$ (here, $2H - 1 > 0$). The right side is continuous in τ and locally Lipschitz in β, so a unique local solution exists. Nonnegativity follows because the field equals $h \ge 0$ on $\{\beta = 0\}$, so the region $\beta \ge 0$ is forward invariant. An a priori bound follows from $\beta' \le 2\bar{a}\beta + \bar{h}$ for $\beta \ge 0$, where $\bar{a} = \sigma^2 H T^{2H-1}$ and $\bar{h} = \sup h$. Grönwall's inequality then gives $\beta(\tau) \le (k + \bar{h}T)e^{2\bar{a}\tau}$. The solution therefore remains in a compact set on $[0, T]$ and extends globally; uniqueness follows from the Lipschitz property on that compact set. ■

1.6. Proof of Proposition 3

By (4.4)–(4.5) and Fubini's theorem, the cost of $u = LX$ is the deterministic functional $C(u) = \int_0^T (h_t + r_t L_t^2) m_t\,dt + k m_T$. Using (4.5) and the Riccati equation (4.6),

$$\frac{d}{dt}(b_t m_t) \;=\; \dot{b}_t m_t + b_t \dot{m}_t \;=\; m_t\Big(-h_t + r_t^{-1} b_t^2 + 2 b_t L_t\Big),$$

so that, pointwise in t,

$$(h_t + r_t L_t^2) m_t + \frac{d}{dt}(b_t m_t) \;=\; m_t\, r_t \Big(L_t + r_t^{-1} b_t\Big)^2,$$

by completing the square. Integrating over $[0, T]$ and using $b_T = k$, $m_0 = x_0^2$ gives (4.7). Since $m_t > 0$ and $r_t \ge \delta > 0$, the integral in (4.7) is nonnegative and vanishes precisely when $L_t = -r_t^{-1} b_t$ a.e., which is bounded by Lemma 2 and hence admissible. ■

1.7. Proof of Proposition 4

(i) Averaging (5.1) with $u \equiv 0$ cancels transshipment (Lemma 1's algebra), and averaging the independent noise families gives, in law, $d\hat{Z} = \nu \hat{Z} dt + \sigma_{1,N} \hat{Z} dW + \sigma_{2,N} \hat{Z} dB^H$ for the benchmark, with W being a standard Brownian motion and B^H a standard fBm, independent. Itô calculus in W and WIS calculus in B^H give the unique solution $\hat{Z}_T = \bar{Z}_0 \exp(\nu T + \sigma_{1,N} W_T + \sigma_{2,N} B_T^H - \frac{1}{2}\sigma_{1,N}^2 T - \frac{1}{2}\sigma_{2,N}^2 T^{2H})$; the mixed exponent $M_T := \sigma_{1,N} W_T + \sigma_{2,N} B_T^H$ is Gaussian, centered, with variance $v^2(T) = \sigma_{1,N}^2 T + \sigma_{2,N}^2 T^{2H}$ by independence. Standardizing $\{\hat{Z}_T \le Z_{\min}\}$ yields (5.2).

(ii) Proposition 2 is a family of pathwise algebraic identities in the array $(Z_t^{(i)})_i$ at a fixed time. No distributional property of the noise enters its proof (Appendix A.4), so it holds under (5.1) verbatim.

(iii) Under $u = LX \in \mathcal{U}_{\text{lin}}$, the controlled deviation solves $X_t = x_0 \exp(\int_0^t L + \sigma_1 W_t + \sigma_2 B_t^H - \frac{1}{2}\sigma_1^2 t - \frac{1}{2}\sigma_2^2 t^{2H})$, and by independence of W and B^H, $m_t := \mathbb{E}[X_t^2] = x_0^2 \exp(2\int_0^t L + \sigma_1^2 t + \sigma_2^2 t^{2H})$, so $\dot{m} = m(2L_t + \sigma_1^2 + 2\sigma_2^2 H t^{2H-1})$. The structure is asymmetric. The Brownian factor contributes $\mathbb{E}[e^{2\sigma_1 W_t}] e^{-\sigma_1^2 t} = e^{\sigma_1^2 t}$ (constant-rate correction), whereas

the fractional factor contributes $e^{\sigma_2^2 t^{2H}}$ (memory-growing correction). With b solving (5.3), $\frac{d}{dt}(b_t m_t) = m_t(-h_t + r_t^{-1}b_t^2 + 2b_t L_t)$, and completing the square exactly as in the proof of Proposition 3 gives $C(u) = b_0 x_0^2 + \int_0^T m_t r_t (L_t + r_t^{-1}b_t)^2 dt$. Existence, uniqueness, and nonnegativity of b follow by the argument of Lemma 2 with $a(t)$ replaced by $\frac{1}{2}\sigma_1^2 + \sigma_2^2 H t^{2H-1}$. The minimizer is $L_t = -r_t^{-1}b_t$, with minimal cost $b_0 x_0^2$. ■

1.8. Proof of Proposition 5 and Theorem 1

Proposition 5(i). By Lemma 1's cancellation, $d\bar{Z} = \nu\bar{Z}dt + \sigma N^{-1}\sum_i Z^{(i)} dB^{H,i}$. Under (5.4), the aggregate driver $\bar{B}_t^H = N^{-1}\sum_i B_t^{H,i} = \sqrt{\rho}B_t^{H,\mathrm{sys}} + \sqrt{1-\rho}\,N^{-1}\sum_i B_t^{H,\mathrm{ind},i}$ is centered Gaussian with $\mathrm{Var}(\bar{B}_T^H) = \rho T^{2H} + \frac{1-\rho}{N}T^{2H} = (\rho + \frac{1-\rho}{N})T^{2H}$, i.e., a scaled fBm with scale $\sigma_{\mathrm{agg}}/\sigma$. The Gaussian-aggregate benchmark $d\hat{Z} = \nu\hat{Z}dt + \sigma_{\mathrm{agg}}\hat{Z}d\hat{B}^H$ then has WIS solution $\hat{Z}_T = \bar{Z}_0 \exp(\nu T + \sigma_{\mathrm{agg}}\hat{B}_T^H - \frac{1}{2}\sigma_{\mathrm{agg}}^2 T^{2H})$, and standardization yields (5.5). At $\rho = 0$, $\sigma_{\mathrm{agg}} = \sigma/\sqrt{N} = \sigma_N$; at $\rho = 1$, the benchmark coincides with the exact solution of Appendix A.2(a) for every N, κ.

Proposition 5(ii). As in A.7(ii), the result follows from pathwise algebra and is independent of the noise law.

Proposition 5(iii). In the decentralized problem, warehouse i's admissible controls, dynamics (4.1), and cost (4.3) involve the noise only through its own driver $B^{H,i}$. Under (5.4), $B^{H,i} = \sqrt{\rho}B^{H,\mathrm{sys}} + \sqrt{1-\rho}B^{H,\mathrm{ind},i}$ is a sum of independent fBms and is centered Gaussian with covariance $\rho\, c_H(s,t) + (1-\rho)c_H(s,t) = c_H(s,t)$, where c_H is the standard fBm covariance, so the marginal law of $B^{H,i}$ is that of a standard fBm for every ρ. Both the state solution (4.4) and the cost functional $C(u) = \int (h + rL^2)m + km_T$ are functionals of that marginal law alone, so the Riccati equation (4.6), the optimal feedback, and the value $b_0 x_0^2$ are unchanged. Correlation across warehouses affects only joint, systemic outcomes. ■

Theorem 1. Condition on the systemic path $B^{H,\mathrm{sys}}$. Based on it, the terminal inventories $Z_T^{(i)} = \xi_i \exp(\nu T + \sigma\sqrt{\rho}B_T^{H,\mathrm{sys}} - \frac{1}{2}\sigma^2 T^{2H}) \exp(\sigma\sqrt{1-\rho}B_T^{H,\mathrm{ind},i})$ are independent across i, with conditional means $\mathbb{E}[Z_T^{(i)} \mid B^{H,\mathrm{sys}}] = \xi_i G_T$, where $G_T := \exp(\nu T + \sigma\sqrt{\rho}B_T^{H,\mathrm{sys}} - \frac{1}{2}\rho\sigma^2 T^{2H})$ (the idiosyncratic factor contributes $e^{\frac{1}{2}(1-\rho)\sigma^2 T^{2H}}$, which combines with the Wick correction), and conditional variances $\xi_i^2 G_T^2 (e^{(1-\rho)\sigma^2 T^{2H}} - 1)$, uniformly bounded in i given the conditioning because $\sup_i \xi_i < \infty$. By Kolmogorov's strong law for independent, uniformly bounded variance summands (applied conditionally), $\bar{Z}_T - N^{-1}\sum_i \xi_i G_T \to 0$ almost surely, and since $N^{-1}\sum_i \xi_i = \bar{Z}_0^{(N)} \to \bar{z}_0$, the limit (5.6) follows; integrating over the law of $B^{H,\mathrm{sys}}$ preserves the a.s. statement. For (5.7), the limit variable is lognormal with continuous distribution function, so a.s. convergence implies convergence of the probabilities at the fixed threshold $Z_{\min}$, giving the stated limit, which is strictly positive and below one for $\rho > 0$. For $\rho = 0$, $G_T = e^{\nu T}$ is deterministic and $\bar{Z}_T \to \bar{z}_0 e^{\nu T}$, so the limiting stockout exposure is the indicator stated. The rate follows conditionally on $B^{H,\mathrm{sys}}$ from $\mathrm{sd}(\bar{Z}_T)/\mathbb{E}[\bar{Z}_T] = N^{-1/2}(e^{(1-\rho)\sigma^2 T^{2H}} - 1)^{1/2}(\frac{N^{-1}\sum \xi_i^2}{(N^{-1}\sum \xi_i)^2})^{1/2}$, which is the $O(N^{-1/2})$ rate with the stated constant; the Monte Carlo experiments of Section 6.1 reproduce both the rate and the constant. ■

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