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Research article

Dynamic pricing, advertising and inventory optimization for perishable products under trade credit

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Appendix A. Proof of Proposition 1

For a fixed replenishment cycle $T < M$, the factor $1/T$ does not affect the optimal control decisions. By Pontryagin's Maximum Principle, the optimal controls maximize the Hamiltonian. The first-order condition with respect to the price decision, $p(t)$, is $\frac{\partial H_{cd}^1}{\partial p(t)} = 0$. Hence,

$$[1 + \eta_2(M - t)](\beta_0 - 2\beta_1 p(t) + \beta_2 G(t)) + \beta_1 \lambda_{cd}^{I1}(t) = 0.$$

Solving for $p(t)$ gives

$$p_{cd}^{1*}(t) = \frac{1}{2}\left[\frac{\beta_0 + \beta_2 G_{cd}^1(t)}{\beta_1} + \frac{\lambda_{cd}^{I1}(t)}{1 + \eta_2(M - t)}\right].$$

Moreover,

$$\frac{\partial^2 H_{cd}^1}{\partial p(t)^2} = -2\beta_1[1 + \eta_2(M - t)] < 0.$$

For the advertising decision, the first-order condition is

$$\frac{\partial H_{cd}^1}{\partial u(t)} = -au(t) + \lambda_{cd}^{G1}(t) = 0,$$

which gives

$$u(t) = \frac{\lambda_{cd}^{G1}(t)}{a}.$$

Moreover,

$$\frac{\partial^2 H_{cd}^1}{\partial u^2(t)} = -a < 0.$$

Considering the non-negativity constraint $u(t) \geq 0$, the optimal advertising policy becomes

$$u_{cd}^{1*}(t) = \max\left\{0, \frac{\lambda_{cd}^{G1}(t)}{a}\right\}.$$

Recall that the goodwill dynamics are given by

$$\frac{\partial G(t)}{\partial t} + \delta G(t) = u_{cd}^{1*}(t).$$

Using the integrating factor $e^{\delta t}$, we obtain

$$G_{cd}^{1*}(t) = e^{-\delta t}\left[G_0 + \int_0^t e^{\delta s}\, u_{cd}^{1*}(s)\, ds\right].$$

Since the integral representation of goodwill complicates the derivation of explicit analytical results, we adopt a local approximation to obtain tractable expressions. Assuming that $u(t)$ is continuously differentiable and evolves smoothly over time, we approximate the integral term as:

$$\int_0^t e^{\delta x} u(x)\, dx \approx t\varphi\, e^{\delta \varphi t} u(t).$$

Accordingly, the goodwill process is approximated by

$$G(t) \approx e^{-\delta t}\left(G_0 + t\varphi e^{\delta \varphi t} u(t)\right).$$

The approximation error satisfies:

$$\mid G(t) - e^{-\delta t}\left(G_0 + t\varphi e^{\delta \varphi t} u(t)\right) \mid \leq \frac{t^2}{2} e^{-\delta t} \sup_{\xi \in [0,t]} e^{\delta \varphi \xi} \mid \delta \varphi u(\xi) + u'(\xi) \mid.$$

Hence, the approximation becomes accurate when the advertising investment evolves smoothly over time. Moreover, this approximation captures the local effect of the current advertising effort on goodwill accumulation and allows the derivation of analytically tractable dynamic policies.

Now, using the goodwill approximation $G(t) \approx e^{-\delta t}\left[G_0 + \varphi t e^{\delta \varphi t} u(t)\right]$, the goodwill trajectory can be expressed as

$$G_{cd}^{1*}(t) = e^{-\delta t}\left[G_0 + \varphi t e^{\delta \varphi t} u_{cd}^{1*}(t)\right].$$

Next, the adjoint equation associated with inventory is

$$\frac{\partial \lambda_{cd}^{I1}(t)}{\partial \mathrm{t}} = -\frac{\partial H_{cd}^1}{\partial I(t)}.$$

Since

$$\frac{\partial H_{cd}^1}{\partial I(t)} = -h - \frac{t}{m}\lambda_{cd}^{I1}(t),$$

we have

$$\frac{\partial \lambda_{cd}^{I1}(t)}{\partial \mathrm{t}} = h + \frac{t}{m}\lambda_{cd}^{I1}(t).$$

Because the initial inventory $I(0)$ is a free variable and appears in the objective function as $-cI_{cd}(0)$, the corresponding transversality condition is $\lambda_{cd}^{I1}(0) = c$. Solving the above linear differential equation gives

$$\lambda_{cd}^{I1*}(t) = e^{t^2/(2m)}\left[c + h\int_0^t e^{-s^2/(2m)}\, ds\right].$$

By the substitution $x = \frac{s}{\sqrt{2m}}$, the integral becomes

$$\int_0^t e^{-s^2/(2m)}\, ds = \sqrt{2m} \int_0^{t/\sqrt{2m}} e^{-x^2}\, dx.$$

Therefore,

$$\lambda_{cd}^{I1*}(t) = \left[c + h\sqrt{2m} \int_0^{t/\sqrt{2m}} e^{-x^2}\, dx\right] e^{t^2/(2m)}.$$

Finally, the goodwill adjoint equation is obtained as

$$\frac{\partial \lambda_{cd}^{G1}(t)}{\partial t} = -\frac{\partial H_{cd}^{1}}{\partial G(t)},$$

which gives

$$\frac{\partial \lambda_{cd}^{G1}(t)}{\partial t} - \delta \lambda_{cd}^{G1}(t) = \beta_2[\lambda_{cd}^{I1}(t) - (1 + \eta_2(M - t))p_{cd}^{1*}(t)].$$

Since the terminal goodwill level is free, $\lambda_{cd}^{G1}(T) = 0$.

Solving this differential equation backward from T yields

$$\lambda_{cd}^{G1*}(t) = e^{\delta t} \int_t^T e^{-\delta x}\, \beta_2[(1 + \eta_2(M - x))p_{cd}^{1*}(x) - \lambda_{cd}^{I1*}(x)]dx.$$

Thus, the optimal pricing, advertising, goodwill, and adjoint variables are characterized by the above semi-closed-form expressions. This completes the proof.

Appendix B. Proof of Proposition 2

For a fixed replenishment cycle $M \le T \le m$, the factor $1/T$ does not affect the optimal control decisions. Therefore, Pontryagin's Maximum Principle can be applied to the two intervals separately, namely, $0 \le t \le M$ and $M \le t \le T$.

For $0 \le t \le M$, $\frac{\partial H_{cd}^{21}}{\partial p(t)} = 0$ yields

$$[1 + \eta_2(M - t)](\beta_0 - 2\beta_1 p(t) + \beta_2 G(t)) + \beta_1 \lambda_{cd}^{I2}(t) = 0.$$

Solving for $p(t)$ gives

$$p_{cd}^{21*}(t) = \frac{1}{2}\left[\frac{\beta_0 + \beta_2 G_{cd}^{21*}(t)}{\beta_1} + \frac{\lambda_{cd}^{I2*}(t)}{1 + \eta_2(M - t)}\right], 0 \le t \le M.$$

Moreover,

$$\frac{\partial^2 H_{cd}^{21}}{\partial p(t)^2} = -2\beta_1[1 + \eta_2(M - t)] < 0.$$

For the second interval, $M \le t \le T$, the first-order condition $\frac{\partial H_{cd}^{22}}{\partial p(t)} = 0$ yields

$$\beta_0 - 2\beta_1 p(t) + \beta_2 G(t) + \beta_1 \lambda_{cd}^{I2}(t) = 0.$$

Therefore,

$$p_{cd}^{22*}(t) = \frac{\beta_0 + \beta_2 G_{cd}^{22*}(t) + \beta_1 \lambda_{cd}^{I2*}(t)}{2\beta_1}, M \le t \le T,$$

$$\frac{\partial^2 H_{cd}^{22}}{\partial p^2} = -2\beta_1 < 0,$$

the solution is optimal.

For both intervals, $\frac{\partial H_{cd}^2}{\partial u(t)} = -au(t) + \lambda_{cd}^{G2}(t) = 0$. Thus,

$$u(t) = \frac{\lambda_{cd}^{G2}(t)}{a}.$$

Considering the constraint $u(t) \geq 0$, the optimal advertising effort is

$$u_{cd}^{21*}(t) = \max\left\{0, \frac{\lambda_{cd}^{G21*}(t)}{a}\right\}, 0 \leq t \leq M,$$

and

$$u_{cd}^{22*}(t) = \max\left\{0, \frac{\lambda_{cd}^{G22*}(t)}{a}\right\}, M \leq t \leq T.$$

Again, the goodwill state equation gives

$$G(t) = e^{-\delta t}\left[G_0 + \int_0^t e^{\delta s}\, u(s)ds\right],$$

and using the same approximation as above,

$$\int_0^t e^{\delta s}\, u(s)ds \simeq \varphi t e^{\delta \varphi t} u(t),$$

we obtain

$$G_{cd}^{21*}(t) = e^{-\delta t}\left[G_0 + \varphi t e^{\delta \varphi t} u_{cd}^{21*}(t)\right], 0 \leq t \leq M,$$

and

$$G_{cd}^{22*}(t) = e^{-\delta t}\left[G_0 + \varphi t e^{\delta \varphi t} u_{cd}^{22*}(t)\right], M \leq t \leq T.$$

The continuity condition at the switching point is $G_{cd}^{21*}(M^+) = G_{cd}^{22)*}(M^-)$.

The inventory costate satisfies $\frac{\partial \lambda_{cd}^{I2}(t)}{\partial t} = -\frac{\partial H_{cd}^2}{\partial I(t)}$.

For $0 \leq t \leq M$,

$$\frac{\partial \lambda_{cd}^{I2}(t)}{\partial t} = h + \frac{t}{m}\lambda_{cd}^{I2}(t).$$

Since $I(0)$ is free and the objective contains the term $-cI_{cd}(0)$, $\lambda_{cd}^{I2}(0) = c$. The solution is:

$$\lambda_{cd}^{I2*}(t) = \left[c + h\sqrt{2m}\int_0^{\frac{t}{\sqrt{2m}}} e^{-x^2}\, dx\right] e^{\frac{t^2}{2m}}, 0 \leq t \leq M.$$

For $M \leq t \leq T$,

$$\frac{\partial \lambda_{cd}^{I2}(t)}{\partial t} = h + \eta_1 c + \frac{t}{m}\lambda_{cd}^{I2}(t).$$

Using the continuity condition $\lambda_{cd}^{I2}(M^+) = \lambda_{cd}^{I2}(M^-)$, the solution becomes

$$\lambda_{cd}^{I2*}(t) = e^{t^2/(2m)}\left[\lambda_{cd}^{I2*}(M)e^{-\frac{M^2}{2m}} + (h + \eta_1 c)\sqrt{2m}\int_{\frac{M}{\sqrt{2m}}}^{\frac{t}{\sqrt{2m}}} e^{-x^2}\, dx, M \leq t \leq T\right].$$

The goodwill costate satisfies

$$\frac{\partial \lambda_{cd}^{G21}(t)}{\partial t} = -\frac{\partial H_{cd}^2}{\partial G}.$$

For $0 \leq t \leq M$,

$$\frac{\partial \lambda_{cd}^{G21}(t)}{\partial t} - \delta \lambda_{cd}^{G21}(t) = \beta_2 \left[\lambda_{cd}^{I2}(t) - (1 + \eta_2(M - t)) p_{cd}^{(21)*}(t) \right].$$

For $M \le t \le T$,

$$\frac{\partial \lambda_{cd}^{G22}(t)}{\partial t}) - \delta \lambda_{cd}^{G22}(t) = \beta_2 [\lambda_{cd}^{I2}(t) - p_{cd}^{22*}(t)].$$

Because $G(T)$is free, $\lambda_{cd}^{G22}(T) = 0$. Solving the last equation gives

$$\lambda_{cd}^{G22*}(t) = e^{\delta t} \int_t^T e^{-\delta x} \beta_2 \left[p_{cd}^{(22)*}(x) - \lambda_{cd}^{(I2)*}(x) \right] dx, M \le t \le T.$$

At $t = M$, the costate continuity condition requires $\lambda_{cd}^{G21}(M^-) = \lambda_{cd}^{G22}(M^+)$. Hence,

$$\lambda_{cd}^{G21*}(t) = \quad e^{\delta t} \int_t^M e^{-\delta x} \beta_2 [\lambda_{cd}^{I2*}(x) - (1 + \eta_2(M - x)) p_{cd}^{21*}(x)] dx + e^{-\delta(M-t)} \lambda_{cd}^{G22*}(M), \qquad 0 \le t \le M.$$

This completes the proof.

Appendix C. Proof of Proposition 3

Again the Pontryagin's Maximum Principle is applied. The first-order condition $\frac{\partial H_{nd}}{\partial p(t)} = 0$ gives

$$\beta_0 - 2\beta_1 p(t) + \beta_2 G(t) + \beta_1 \lambda_{nd}^I(t) = 0.$$

Therefore,

$$p_{nd}^*(t) = \frac{1}{2} \left[\frac{\beta_0 + \beta_2 G_{nd}(t)}{\beta_1} + \lambda_{nd}^I(t) \right].$$

Furthermore,

$$\frac{\partial^2 H_{nd}}{\partial p^2} = -2\beta_1 < 0.$$

The first-order condition with respect to $u(t)$, $\frac{\partial H_{nd}}{\partial u} = 0$, gives

$$-au(t) + \lambda_{nd}^G(t) = 0.$$

Hence, the unconstrained optimal advertising effort is

$$u(t) = \frac{\lambda_{nd}^G(t)}{a}.$$

Considering the constraint $u(t) \ge 0$, the optimal advertising policy becomes

$$u_{nd}^*(t) = \max \left\{ 0, \frac{\lambda_{nd}^G(t)}{a} \right\}.$$

The goodwill dynamics satisfy

$$\frac{\partial G(t)}{\partial \mathrm{t}} + \delta G(t) = u_{nd}^*(t).$$

Solving the last equation gives

$$G_{nd}^*(t) = e^{-\delta t} \left[G_0 + \int_0^t e^{\delta s} u_{nd}^*(s) ds \right].$$

Using again the approximation $\int_0^t e^{\delta s} u_{nd}^*(s) ds \simeq \varphi t e^{\delta \varphi t} u_{nd}^*(t)$, we obtain

$$G_{nd}^{*}(t) = e^{-\delta t}\left[G_0 + \varphi t e^{\delta \varphi t} u_{nd}^{*}(t)\right].$$

The inventory costate satisfies

$$\frac{\partial \lambda_{nd}^{I}}{\partial t} = -\frac{\partial H_{nd}}{\partial I}.$$

Since

$$\frac{\partial H_{nd}}{\partial I(t)} = -h - \frac{t}{m}\lambda_{nd}^{I}(t),$$

we obtain

$$\frac{\partial \lambda_{nd}^{I}}{\partial t} = h + \frac{t}{m}\lambda_{nd}^{I}(t).$$

Because $I(0)$ is a free variable and the objective contains the term $-cI_{nd}(0)$, the transversality condition is $\lambda_{nd}^{I}(0) = c$. Solving the linear differential equation gives

$$\lambda_{nd}^{I*}(t) = e^{t^2/(2m)}\left[c + h\int_0^t e^{-s^2/(2m)}\, ds\right]$$

or equivalently

$$\lambda_{nd}^{I*}(t) = \left[c + h\sqrt{2m}\int_0^{t/\sqrt{2m}} e^{-x^2}\, dx\right] e^{t^2/(2m)}.$$

The goodwill costate equation is

$$\frac{\partial \lambda_{nd}^{G}}{\partial t} = -\frac{\partial H_{nd}}{\partial G(t)}.$$

Since

$$\frac{\partial H_{nd}}{\partial G} = \beta_2 p(t) - \beta_2 \lambda_{nd}^{I}(t) - \delta \lambda_{nd}^{G}(t),$$

we obtain

$$\frac{\partial \lambda_{nd}^{G}}{\partial t} - \delta \lambda_{nd}^{G}(t) = \beta_2[\lambda_{nd}^{I}(t) - p_{nd}^{*}(t)].$$

Because $G(T)$ is free and no terminal goodwill value exists, $\lambda_{nd}^{G}(T) = 0$. Multiplying the last equation by $e^{-\delta t}$, we get

$$\frac{d}{dt}\left(e^{-\delta t}\lambda_{nd}^{G}(t)\right) = e^{-\delta t}\beta_2[\lambda_{nd}^{I}(t) - p_{nd}^{*}(t)].$$

Integrating backward from t to T, we obtain

$$\lambda_{nd}^{G*}(t) = e^{\delta t}\int_t^T e^{-\delta x}\,\beta_2[p_{nd}^{*}(x) - \lambda_{nd}^{I*}(x)]dx.$$

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