Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(8): 3897–3925.
DOI:10.3934/jimo.2026139
Received: 13 May 2026
Revised: 04 July 2026
Accepted: 16 July 2026
Published: 22 July 2026

Research article

Ambiguity or transparency? A model for regulating native advertising, entertainment, and targeting decisions

Yafei Yu, Decheng Wen and Xiao Chen*

School of Management, Shandong University, 27 Shanda Nanlu, Jinan, Shandong 250100, China

* **Correspondence:** Email: x_chen@sdu.edu.cn.

We present the proofs of the main lemmas and propositions here.

A. Proof of Lemma 1

We show the case in which $q - ry < 0$, because the proof for $q - ry > 0$ is similar. From Equation 3.3, since π is concave in p on each interval, we can derive the vendor's optimal pricing decision. Case (1): $\frac{1}{2}(q + \frac{1-x}{x}e) < q - \hat{q} + ry - e \implies e < \frac{\frac{1}{2}q - \hat{q} + ry}{\frac{1}{2}\frac{1-x}{x}+1} \triangleq e_2$, $p^* = \frac{1}{2}(q + \frac{1-x}{x}e)$; Case (2): $\frac{1}{2}(q + \frac{1-x}{x}e) < q - e \implies e < \frac{\frac{1}{2}q}{\frac{1}{2}\frac{1-x}{x}+1} \triangleq e_1$, $p^* = q - e$; Case (3): $\frac{1}{2}q < q - \hat{q} - e + ry \implies e < \frac{q}{2} - \hat{q} + ry$, $p^* = q - \hat{q} - e + ry$; Case (4): $\frac{1}{2}q > q - \hat{q} - e + ry \implies e > \frac{q}{2} - \hat{q} + ry$, $p^* = \frac{1}{2}q$.

$$p^* = \begin{cases} q - e, & 0 \le e \le e_1, \\ \frac{1}{2}(q + \frac{1-x}{x}e), & e_1 \le e \le e_2, \\ (q - \hat{q} - e + ry), & e_2 < e \le \frac{q}{2} - \hat{q} + ry, \\ \frac{q}{2}, & \frac{q}{2} - \hat{q} + ry < e \le 1 \end{cases} \tag{A.1}$$

B. Proof of Lemma 2

We first prove that e_2^* cannot be the optimal entertainment level. When $e_1 < e \le e_2$, $\Pi = -\frac{e^2}{2} + \frac{1}{2}(1 - x)(e+t)\left(\frac{e(1-x)}{x} + q\right) + \frac{1}{4}x\left(\left(\frac{e(1-x)}{x} + q\right)\left(q - \frac{e(1-x)}{x}\right)\right) = \frac{q^2 x}{4} + \frac{1}{2}qt(1-x) + \left[\frac{1}{2}q(1-x) + \frac{t(1-x)^2}{2x}\right]e + \left[\frac{(1-x)^2}{4x} - \frac{1}{2}\right]e^2$.

(1) When $2 - \sqrt{3} < x \le 1$, $e_2^* = -\frac{(1-x)(qx+t(1-x))}{x^2-4x+1} > 0$. Recall that $e_2^* < e_1$ and $\frac{\partial^2 \Pi}{\partial e^2} < 0$. Π is concave and decreases on $(e_1, e_2]$. Thus, $\max\limits_{e_1<e\le e_2} \Pi(e) = \lim\limits_{e\to(e_1)^+} \Pi(e) = \Pi(e_1) \le \max\limits_{0<e\le e_1} \Pi(e)$. Thus, e_2^* is inferior to e_1.

(2) When $0 \le x < 2 - \sqrt{3}$, $e_2^* = -\frac{(1-x)(qx+t(1-x))}{x^2-4x+1} < 0$. Π is convex and increases on $(e_1, e_2]$. Thus, $\max\limits_{e_1<e\le e_2} \Pi(e) = \Pi(e_2)$. Thus, e_2^* is inferior to e_2. Therefore, e_2^* can never be the global optimizer.

Next, we identify the optimal entertainment level e^* within each interval of x.

When $0 \le x < \bar{x}_1$, $\frac{\partial \Pi}{\partial e} = -3e + q + t(x-1) > 0 (0 < e \le e_1)$, $\frac{\partial \Pi}{\partial e} = \frac{e(x-4)x+e+x(q(-x)+q+t(x-2))+t}{2x} > 0 (e_1 < e \le e_2)$. Since Π is concave and monotonically decreasing on the fourth interval, $\max\limits_{0<e\le e_2} \Pi(e) = \max\limits_{e_2<e\le 1} \Pi(e) = \Pi\left(e_3^*\right)$, where $\bar{x}_1 = \frac{1}{2}$.

When $\bar{x}_1 \le x < \bar{x}_2$, $\frac{\partial \Pi}{\partial e} = -3e + q + t(x-1) > 0 (0 < e \le e_1)$, $\frac{\partial \Pi}{\partial e} = \frac{e(x-4)x+e+x(q(-x)+q+t(x-2))+t}{2x} < 0 (e_1 < e \le e_2)$, $\max\limits_{0<e\le 1} \Pi(e) = \Pi(e_1)$, where $\bar{x}_2 = \frac{q-\sqrt{q^2-qt+t^2}}{t}$.

When $\bar{x}_2 \le x < 1$, $e_1(x) > \frac{1}{3}(q + tx - t)$, $\frac{\partial \Pi}{\partial e} = \frac{e(x-4)x+e+x(q(-x)+q+t(x-2))+t}{2x} < 0 (e_1 < e \le e_2)$, $\frac{\partial \Pi}{\partial e} = \frac{e(x-4)x+e+x(q(-x)+q+t(x-2))+t}{2x} < 0 (e_2 < e \le \frac{q}{2} - t)$, $\max\limits_{0<e\le 1} \Pi(e) = \Pi\left(e_1^*\right)$.

Thus, when $0 \le x < \bar{x}_1$, e_3^* is the global maximizer and independent of x. When $\bar{x}_1 \le x < \bar{x}_2$, e_1 is the global maximizer. When $\bar{x}_2 \le x < 1$, e_1^* is the global maximizer. By the envelope theorem, $\frac{\partial \Pi(e_1,x)}{\partial x} = -\frac{qt}{x+1} > 0$ for $t < 0$, and $\frac{\partial \Pi(e_1^*,x)}{\partial x} > 0$.

The proof of Lemma 2 is complete.

C. Proof of Lemma 3

We demonstrate that under a PSC, realized profit is a monotonic, non-decreasing function of ambiguity.

Case (1): When $0 \le x < \bar{x}_1$, for $-\frac{e_3^2}{2} + (e_3 + t)((1-x)(q-e_3) + e_3 x(q-e_3))$, we have $\frac{\partial \Pi(e_3,x)}{\partial x} = 0$. Hence, $\Pi(e_3, x)$ is independent of x. In the strict-regulation region, the platform may select any ambiguity level within the permissible regulatory range.

Case (2): When $\bar{x}_1 \le x < \bar{x}_2$, $\Pi(e_1, x) = -\frac{e_1^2}{2} + (1-x)(q-e_1)(e_1+t) + e_1 x(q-e_1)$, $\frac{\partial \Pi(e_1,x)}{\partial x} = \frac{q(-2x(q+t)+q-2t)}{(x+1)^3}$, the key issue is the sign of $q + 4qx - 2ry(1+x)$, which we denote as $f_1(x)$. If $q > 2ry$, $f_1(x) > f_1(0) > 0$. If $q < \frac{4ry}{5}$, $f_1(x) < f_1(1) < 0$. If $ry \ge q > \frac{4ry}{5}$, there exists x_0^1 such that $f_1(x) = 0$, where $x_0^1 = \frac{2ry-q}{4q-2ry}$. It follows that $x_0^1 - \bar{x}_2 > 0$, so $f_1(x) < 0$ if $q < ry$.

Case (3): When $\bar{x}_2 \le x < 1$, $\Pi(e_1^*, x) = -\frac{(e_1^*)^2}{2} + (1-x)(q-e_1^*)(e_1^*+t) + e_1^* x(q-e_1^*)$, $\frac{\partial \Pi(e_1^*,x)}{\partial x} = \frac{1}{3}t(t(x-1) - 2q)$, the key issue is the sign of $(q-ry)x - 3q + ry$, which we denote as $f_2(x)$. If $q > \frac{1}{3}ry$, $f_2(x) < f_2(0) < 0$. If $q < \frac{1}{3}ry$, there exists x_0^2 such that $f_2(x) = 0$, where $x_0^2 = \frac{3q-ry}{q-ry}$. It follows that $x_0^2 - \bar{x}_2 < 0$, so $f_2(x) < 0$.

Therefore, the economic profit function Π is non-decreasing in x, and the platform always adopts the maximum permitted ambiguity level $\bar{x}$ under a PSC.

D. Proof of Lemma 4

It is straightforward to verify that $CTRs(x) = xe^* + (1-x)(e^* + \hat{q} - ry)$ increases with x within each of the three intervals.

Case (1): When $0 \le x < \bar{x}_1$, $CTRs(e_3^*, x) = \frac{1}{3}(q - 2t) + t(1-x)$, $\frac{\partial CTRs(e_3^*,x)}{\partial x} = -t > 0$, $CTRs(e_3^*, x)$ increases with x.

Thus, $\max\limits_{x} CTRs(e_3^*, x) = CTRs(e_3, \bar{x}_1)$, $\lim\limits_{ry\to 2q} \bar{x}_1 = -1 (\bar{x}_1 > 0)$, $\lim\limits_{ry\to q} \bar{x}_1 = \frac{1}{2}$. Thus, in the first interval, a higher ambiguity level leads to a higher click-through rate. The maximum click-through rate is achieved at $\bar{x}_1$, with a value of $\frac{1}{3}q - \frac{t}{6}$.

Case (2): When $\bar{x}_1 \leq x < \bar{x}_2$, $CTRs(e_1, x) = \frac{q}{2\left(\frac{1-x}{2x}+1\right)} + t(1-x)$, $\frac{\partial CTRs(e_3,x)}{\partial x} = \frac{q}{(x+1)^2} - t > 0$, $\bar{x}_2$ represents a hyperbola in terms of t, where $t = -q$ is an infinite discontinuity point. The relevant range of t is $\left[-\frac{q}{4}, 0\right]$. $\max_x CTRs(e_1, \frac{q}{4}) = \frac{q}{2}$. The maximum click-through rate is achieved at $\bar{x}_2$.

Case (3): When $\bar{x}_2 \leq x < 1$, $CTRs(e_1^*, x) = \frac{1}{3}(q + tx - t) + t(1-x)$. $\frac{\partial CTRs(e_1^*,x)}{\partial x} = -\frac{1}{3}t > 0$, so the click-through rate increases with x in the third interval. $\max_x CTRs(e_1^*, x) = CTRs(e_1^*, 1) = \frac{1}{3}q$. The maximum click-through rate is achieved at 1, with a value of $\frac{q}{3}$.

Recall that $\bar{x}_2$ increases monotonically as t increases. Because the click-through rate increases monotonically with x, for all t that ensure $\bar{x}_1$ and $\bar{x}_2$ belong to $[0,1]$, $\bar{x}_2$ is always greater than $\bar{x}_1$, and the click-through rate in the second interval is always greater than that in the first interval, where $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{q-2(\hat{q}-\mathrm{ry})}{2(q+(\hat{q}-\mathrm{ry}))}$. Thus, under a CPC contract, the platform engages in self-regulation. We therefore need only compare the maximum CTRs in the first and second intervals. Define $f(t) = CTRs(e_1, \bar{x}_2) - CTRs(e_3^*, \bar{x}_1) = -1 - (-2+q)t + \frac{q^2}{q+t} + \frac{1}{1+q+t} - \frac{1}{3}q + \frac{t}{6}$, $f(t)$ is well-defined and continuous in $(-q, 0)$, $f'(t) = (2-q) - \frac{q^2}{(q+t)^2} - \frac{1}{(1+q+t)^2} + \frac{1}{6}$, $f''(t) = \frac{2q^2}{(q+t)^3} + \frac{2}{(1+q+t)^3} > 0$. The function $f(t)$ is strictly convex and quasi-concave in $(-q, 0]$ because its upper contour set is convex ($f(t) = \alpha$ has one real root and a pair of complex conjugate roots).

We examine the limit as t approaches the boundary. When $t \to -q^+$, $q + t \to 0^+$, $\frac{q^2}{q+t} \to +\infty$, while the other terms remain finite, then $f(t) \to +\infty$. $f(0) = -1 + (2-q) \cdot 0 + \frac{q^2}{q} + \frac{1}{1+q} - \frac{1}{3}q + \frac{0}{6} = -1 + q + \frac{1}{1+q} - \frac{1}{3}q = \frac{2}{3}q + \frac{1}{1+q} - 1$.

When $0 < q < \frac{1}{2}$, $f(0) < 0$, the equation $f(t) = 0$ has exactly one root, and there exists a one-to-one correspondence between t and ry. Thus, we denote this unique root as ry_{Small}. When $\frac{1}{2} < q$, $f(0) > 0$, the equation $f(t) > 0$ holds when $ry \in [q, 2q]$.

Finally, we prove that ry_{Small} is increasing in $q \in (0, \frac{1}{2}]$. By the implicit function theorem and the constraint $t = q - ry$, $\frac{\partial ry_{Small}}{\partial q} = \frac{\partial f'/\partial q + \partial f'/\partial t}{\partial f'/\partial t} = \frac{\frac{2q(q-t)}{(q+t)^3} + \frac{4}{(q+t+1)^3} - 1}{2\left(\frac{q^2}{(q+t)^3} + \frac{1}{(q+t+1)^3}\right)}$. Similarly, the denominator is positive, so we only need to prove that the numerator is also positive. We denote the numerator by $n_1(t, q)$. It is straightforward to check that $n_1(t, q)$ is decreasing in (t, q) under the constraints $q \in (0, \frac{1}{2}]$ and $t \in (-q, 0)$. $\min_t n(t; q) = n(0; q) = -1 + \frac{2}{q} + \frac{4}{(1+q)^3}$ and $n(0; q)$ is decreasing in $q \in (0, \frac{1}{2}]$. $\min_q n(0; q) = n\left(0; \frac{1}{2}\right) = 3 + \frac{4}{(1+\frac{1}{2})^3} > 0$. Therefore, ry_{Small} is increasing in q.

E. Proof of Proposition 1

This result follows directly from Lemma 3 and Lemma 4.

F. Proof of Proposition 2

This proof is similar to the proof of Lemma 3.

G. Proof of Proposition 3

Because the government cannot control the level of advertising aversion for each consumer, we take the derivative of $ry(xe + (1-x)[(1 - (e + \hat{q} - ry)])$. $\frac{\partial PH(x,e)}{\partial x} = \left[(2e - 1 + t) + (2x-1)\frac{de}{dx}\right] ry$.

When $0 < x < \bar{x}_1$, it is straightforward to see that $x_{PH} = 0$ minimizes $PH(x, e_3^*)$ and min $\frac{PH(x,e_3^*)}{ry} = 1/3(3 - 2q + ry)$.

When $\bar{x}_1 < x < \bar{x}_2$, $\frac{\partial^2 PH(x,e_1)}{\partial x^2} = \frac{6q}{(x+1)^3} > 0$, so $PH(x, e_1)$ is convex in this interval. $x_{PH} = x_M$ minimizes $PH(x, e)$ and min $\frac{PH(x,e_1)}{ry} = 2\left(\sqrt{3}\sqrt{q(3q - ry - 1)} + ry + 1\right) - 7q$, where $x_M = \frac{\sqrt{3}q}{\sqrt{q(2q+t-1)}} - 1$.

When $\bar{x}_2 < x < 1$, $\frac{\partial^2 PH(x,e_1^*)}{\partial x^2} = \frac{4t}{3} < 0$, so $PH(x, e_1^*)$ is concave in this interval. $x_{PH} = 1$ minimizes $PH(x, e)$ and min $\frac{PH(x,e_1^*)}{ry} = \frac{q}{3}$.

We compare the minimum values of platform harm under the three scenarios and then determine the parameter range for the following inequality.

If $ry > 3q - 1$ (i.e., the consumer aversion level is extremely high), the night-watchman platform always sets $x_{PH} = 1$. This is because x_M does not exist and $\frac{PH(0,e_3^*(0))}{ry} > \frac{PH(1,e_1^*(1))}{ry}$ (since $ry > 3q - 3$).

If $3q - 1 \le ry$ and $ry \notin \mathcal{S}$ (i.e., the consumer aversion level is relatively high), the night-watchman platform sets $x_{PH} = x_M$. This is because $\frac{PH(x_M,e_1(x_M))}{ry}$ is smaller than $\frac{PH(1,e_1^*(1))}{ry}$ and $\frac{PH(0,e_3^*(0))}{ry}$.

If $ry \in \mathcal{S}$ (i.e., the consumer aversion level is low), the night-watchman platform sets $x_{PH} = 0$. This is because $\frac{PH(0,e_3^*(0))}{ry}$ is smaller than $\frac{PH(x_M,e_1(x_M))}{ry}$ and $\frac{PH(1,e_1^*(1))}{ry}$. We prove the last case here, because the second case can be established similarly. $f(q,z) = PH(x = x_M) - PH(x = 0) = -\frac{5q}{3} - \frac{4}{3} - \frac{4}{3}z + 2\sqrt{3}\sqrt{qz} > 0$ and $z = 3q - 1 - ry$ is equivalent to $z^2 - \left(\frac{17q}{4} - 2\right)z + \left(\frac{25q^2}{16} + \frac{5q}{2} + 1\right) < 0$. When $q > \frac{16}{7}$, $\delta = \frac{189q^2}{16} - 27q > 0$. This yields two positive roots $z_1, z_2 = \frac{\frac{17q}{4} - 2 \pm \sqrt{\delta}}{2}$ (since $\left(\frac{25q^2}{16} + \frac{5q}{2} + 1\right) > 0$, both roots are positive). Denote $z_1 > z_2 > 0$. Therefore, when $q > \frac{16}{7}$ and $3q - 1 - z_2 \le ry < 3q - 1 - z_1$, the night-watchman platform chooses $x_{PH}^* = 0$.

H. Proof of Proposition 4

We first discuss the case in which $q < ry < 3q - 3$. If this condition does not hold, it follows from Lemma 3 and Proposition 3 that $x = 1$ simultaneously minimizes platform harm and maximizes profit, so the behavior of the balanced platform is identical to that of the night-watchman platform.

We next examine how the balanced platform chooses x by weighing platform harm against profit.

If the platform implements strict regulation, $0 < x < \bar{x}_1$, $NP_1(e, x) = \Pi(e_3^*) - PH(e_3^*, x) = (e_3^* + t)(q - e_3^* - t) - 1/2 * (e_3^*)^2 - (xrye_3^* + (1 - x)ry[(1 - (e_3^* + \hat{q} - ry)])$, $\frac{\partial NP_1(e,x)}{\partial x} = xrye_3^* + ry[(1 - (e_3^* + \hat{q} - ry)] > 0$.

If the platform implements moderate regulation, $\bar{x}_1 < x < \bar{x}_2$, $NP_2(e, x) = \Pi(e_1) - PH(e_1, x) = (1 - x)(q - e_1)(e_1 + t) + e_1 x(q - e_1) - 1/2(e_1)^2 - [xrye_1 + (1 - x)ry[(1 - (e_1 + \hat{q} - ry)]$, $\frac{\partial^2 NP_2(e,x)}{\partial x^2} = \frac{3q^2(2x-1)+2qry(1+x)}{(x+1)^4} > 0$ because $x > \bar{x}_1$; therefore, NP is convex in this interval.

If the platform implements lenient regulation, $\bar{x}_2 < x < 1$, $NP_3(e, x) = \Pi(e_1^*) - PH(e_1^*, x) = (1 - x)(q - e_1^*)(e_1^* + t) + e_1^* x(q - e_1^*) - 1/2 * (e_1^*)^2 - [xrye_1^* + (1 - x)ry[(1 - (e_1^* + \hat{q} - ry)]$, $\frac{\partial^2 NP_3(e,x)}{\partial x^2} = \frac{1}{3}(q - ry)(q + 3ry) < 0$, and setting $\frac{\partial NP_3(e,x)}{\partial x}$ equal to zero yields $x_M' = \frac{ry^2+3(1-2q)ry+3q^2}{(3ry+q)(q-ry)}$. It is easy to see that the denominator of x_M' is negative. Denote the numerator by $n(ry)$, which is a parabolic function of ry with symmetry axis $ry = 3q - \frac{3}{2} > 3q - 3$. Therefore, $n(ry)$ always decreases in the interval under consideration. The smaller root of $n(ry) = 0$ is $ry_{root1} = \frac{1}{2}\left(-\sqrt{3}\sqrt{8q^2 - 12q + 3} + 6q - 3\right)$, which is always meaningful and smaller than q when $q > \frac{3}{2}$. Thus, x_M' is positive in the interval under consideration.

Checking whether $x_M' - 1 > 0$ is equivalent to checking whether $2ry - \frac{\sqrt{4ry^2 - 3ry}}{\sqrt{2}} < q$. Denote

$2ry - \frac{\sqrt{4ry^2-3ry}}{\sqrt{2}}$ by $h(ry)$. Then $\frac{\partial^2 h(ry)}{\partial ry^2} = \frac{9}{4\sqrt{2}(ry(4ry-3))^{3/2}} > 0$ given $ry > q > \frac{3}{2}$. $h(\frac{3}{2}) = \frac{3}{2}$ and $h(3q-3) = \frac{3}{2}$ when $q = \frac{3}{2}$. Recall that $\frac{\partial h(ry)}{\partial ry} > 0$ when $ry = \min_q\{3q-3\}$, and $h(ry)$ is concave. $\lim_{ry\to\infty}\left(2ry - \frac{\sqrt{4ry^2-3ry}}{\sqrt{2}}\right) = \infty$. Thus, $h(ry) - q = 0$ has a unique root when $q > \frac{3}{2}$, which we denote by ry'.

Therefore, (a) if $q < ry < \min\{ry', 3q-3\}$, $x'_M - 1 > 0$, and the balanced platform sets the ambiguity level $x^*_{NP} = \bar{x}_1$; (b) if $\max\{q, ry'\} < ry$, $x'_M - 1 < 0$, and the balanced platform sets the ambiguity level $x^*_{NP} = x'_M$.

This completes the proof of Proposition 4.

I. Proof of Proposition 5

Case (i): $r < r'$.

The balanced platform maximizes $NP(x = 1/2) = \frac{1}{6}(q^2 + 5q\,ry - 3ry - 5ry^2) = \frac{1}{6}q^2 + \frac{5}{6}q\,ry - \frac{1}{2}ry - \frac{5}{6}ry^2$ when $r < r'$, under the constraint $h(ry) < q \le ry$, where $h(ry) = 2ry - \frac{\sqrt{4ry^2-3ry}}{\sqrt{2}}$. For $h(ry)$ to be real and positive, $2ry - 3 \ge 0$ and the feasible ry for a non-empty region is $[\frac{3}{2}, \infty)$.

To find the maximum, we check the necessary condition. $\frac{\partial NP(x=1/2)}{\partial q} = \frac{1}{3}q + \frac{5}{6}ry = 0$; $\frac{\partial NP(x=1/2)}{\partial ry} = \frac{5}{6}q - \frac{1}{2} - \frac{5}{3}ry = 0$. Solving these equations simultaneously gives the stationary point $ry = -\frac{2}{15}$, $q = \frac{1}{3}$, which lies outside the feasible region. Since $NP(x = 1/2)(q, q) \to \infty$ and $NP(x = 1/2)(q, h^{-1}(q)) \to -\infty$ as $q \to \infty$, and since the feasible region lies outside $q = -\frac{5}{2}ry$, we have $\frac{\partial NP(x=1/2)}{\partial q} > 0$. Thus, the optimal path of quality improvement is $y^* = \frac{q}{r}$.

Case (ii): $r' \le r$.

The balanced platform maximizes $NP(x = x'_M) = \frac{-3q^4-18q^3ry+169q^2ry^2-28qry^2(3+4ry)+ry^2(3+4ry)^2}{6(q+3ry)^2}$ under the constraint $3q - 3 < q \le h(ry)$. The constraint region is $\{(q, ry) \mid q \ge \frac{3}{2},\ h^{-1}(q) \le ry \le 3q-3\}$. Since $NP(x = 1/2)(q, 3q-3) \to \infty$ and $NP(x = 1/2)(q, h^{-1}(q)) \to \infty$ as $q \to \infty$, and because quality is fixed during the sales phase, $q = \bar{q}$. $\frac{\partial f}{\partial ry} = \frac{(-3+14q-4ry)ry((-3+14q)q-8qry-12ry^2)}{3(q+3ry)^3}$. The numerator consists of three factors. Because $ry = 9q(2q+1) > 0$ holds for $q > 0$, the curve $(-3+14q)q - 8qry - 12ry^2 = 0$ lies below the 45° line, whereas $-3 + 14q - 4ry = 0$ lies above the 45° line. Thus, $\frac{\partial f}{\partial ry} < 0$ in the feasible region and the optimal path of quality improvement is $y^* = \frac{h^{-1}(q)}{r}$.

J. Proof of Proposition 6

The government's goal is to maximize the total revenue of all members of society. We denote the platform's commission, the vendor's profit, and consumer surplus by Π_p, Π_v, and CS, respectively.

$SW = \Pi_v + \Pi_p + CS + \frac{1}{2}e^2 = \alpha\left[(q-p)\cdot p - \frac{1}{2}e^2\right] + (1-\alpha)\left[(q-p)p - \frac{1}{2}e^2\right] + (q-p)q - (q-p)p + \frac{1}{2}e^2 = (q-p)q$. By Lemma 1, SW is given by the targeting and entertainment decision of the vendor.

$$SW = (q-p)q = \begin{cases} e\cdot q, & 0 \le e < e_1, \\ \left(\frac{1}{2}q - \frac{1-x}{x}e\right)q, & e_1 \le e < e_2, \\ \frac{q^2}{2}, & e_2 \le e \le 1 \end{cases}$$

Note that $e_2 = ry - \frac{q}{2}$ when consumers know the quality, so the third targeting interval collapses to a single point. The government never chooses $x_g^* \in (\bar{x}_1, \bar{x}_2)$. This is because $SW(0 < x \leq \bar{x}_1) - SW(\bar{x}_1 < x \leq \bar{x}_2) = \frac{q^2}{2} - e_1 \cdot q = \frac{q^2}{2}\left(1 - \frac{1}{\frac{1+x}{2x}}\right) = \frac{q^2}{2}\frac{1-x}{1+x} > 0$.

$SW(0 < x \leq \bar{x}_1) - SW(\bar{x}_2 < x \leq 1) = SW_1 - SW_3 = \frac{1}{2}q^2 - \left[\frac{1}{3}q^2 + (1-x)q(q-ry)\right] = \frac{1}{6}q^2 + (1-x)q(q-ry)$.

This difference equals $\frac{1}{6}q^2$ on the 45° line of the $ry - q$ Cartesian coordinate system. This means that, in the neighborhood of the 45° line, the government chooses $x_g^* = \bar{x}_1$, whereas otherwise, $x_g^* = \bar{x}_2$, because SW is continuous in (q, ry), and $\frac{\partial SW(\bar{x}_2 < x \leq 1)}{\partial x} = \frac{1}{3}qt < 0 \implies$ the government chooses x as close as possible to $\bar{x}_2$. Taking the smallest boundary and denoting it by $ry = q + \tilde{q}$, recall that the CTR term $e_1^* + q - ry$ must be positive. This yields $ry \leq \frac{6}{5}q$.

K. Proof of Proposition 7

To achieve higher purchase volumes in the purchase stage, a high-quality vendor does not engage in adverse selection. However, a low-quality vendor has an incentive to mimic the behavior of a high-quality vendor to secure greater purchase volumes. Therefore, in each ambiguity-regulation region, the condition $\Pi_{LH}(e_{SEP}^{H*}, x) \leq \Pi_{LL}^*(e_{SEP}^{L*}, x)$ must be satisfied. Here, e_{SEP}^{H*} and e_{SEP}^{L*} denote the equilibrium entertainment levels chosen by the high-quality and low-quality vendors in the separating equilibrium, respectively; the superscript H or L denotes the vendor's quality type; while the numerical superscripts $1, 2, 3$ indicate the three corresponding ambiguity-regulation regions in which the separating-equilibrium entertainment level is derived.

When $x \in [0, \bar{x}_1)$, $\Pi_{LH}(e_{sep}^{H1*}, x) \leq \Pi_{LL}^*(e_{sep}^{L1*}, x)$ is equivalent to $-\frac{e^2}{2} + (e + q_H - ry)(-e - q_H + q_L + ry) \geq \frac{1}{6}\left(1 - 2q_L^2 - 2ry(1+ry) + q_L(2+4ry)\right)$. The solution of this inequality is $e \geq \frac{1}{3}(-2q_H + q_L + 2ry + \beta_1)$ or $e \leq \frac{1}{3}(-2q_H + q_L + 2ry - \beta_1)$.

$\beta_1 = \sqrt{-2q_H^2 + 2q_H(q_L + 2ry) + q_L(3q_L - 6ry - 2) + 2ry - 1}$. Comparing the profit of a high-quality vendor at the two endpoints gives $e_{sep}^{H1*} = \frac{1}{3}(-2q_H + q_L + 2ry + \beta_1)$.

Following the same procedure, $e_{sep}^{H2*} = \frac{1}{3}(q_H(x-1) + q_L - ryx + ry + \beta_2)$, $e_{sep}^{H3*} = \frac{1}{3}(q_H(x-1) + q_L - ryx + ry + \beta_3)$,

$$\beta_2 = -\Bigg(q_H^2(x-1)^2 - 2q_H(x-1)\left[2q_L + ry(x-1)\right] + \frac{q_L^2\left[2x(5x-2) - 5\right] + 2q_L(ry)(x-1)(x+1)(2x-1) + (ry)^2(x^2-1)^2}{(x+1)^2}\Bigg)^{1/2},$$

and

$$\beta_3 = \Big((q_H - q_L)(x-1) \times \left[-q_H - 5q_L + 2ry + (q_H + q_L - 2ry)x\right]\Big)^{1/2}.$$

Figure 6 illustrates the results of Proposition 7.

L. Proof of Proposition 8

It is straightforward to see that when $x \in (0, \overline{x}_1]$, e^{H*}_{sep} and e^{L*}_{sep} are independent of x. We discuss the following two intervals. When $x \in [\overline{x}_1, \overline{x}_2]$, e^*_{sep} is defined by $\Pi_{LH}(e^*_{\text{sep}}, x) = \Pi_{LL}(x)$, and the objective is to find $\frac{de^{H*}_{\text{sep}}}{dx} = \frac{\frac{\partial \Pi^*_{LL}}{\partial x} - \frac{\partial \Pi_{LH}}{\partial x}}{\frac{\partial \Pi_{LH}}{\partial e^{H*}_{\text{sep}}}}$. If (a) $q_L < \frac{\beta_2-(1-x)t_H}{2}$, $\frac{\partial \Pi_{LH}}{\partial x} = (e^*_{\text{sep}} - q_L)(q_H - ry) < 0$; (b) $q_L(1+4x) - 2ry(1+x) < 0$, $\frac{\partial \Pi^*_{LL}}{\partial x} > 0$; (c) $(1-x)(ry - q_H) + q_L - 3e^{H*}_{\text{sep}} < 0$, $\frac{\partial \Pi_{LH}}{\partial e} < 0$. Condition (c) holds because $e^{H*}_{\text{sep}} > e^{H*}_1 > e^{LH*}_1$, where $e^{H*}_1 = \frac{q_H-(1-x)(q_H-ry)}{3}$ and $e^{LH*}_1 = \frac{q_L-(1-x)(q_H-ry)}{3}$. Therefore, the only condition is to ensure that $q_L < \min\left\{\frac{\beta_2-(1-x)t_H}{2}, \frac{2ry(1+x)}{1+4x}\right\}$, where $t_H = (q_H - ry)$.

When $x \in [\bar{x}_2, 1]$, $\frac{\partial \Pi^*_{LL}}{\partial x} = -\frac{1}{6}\underbrace{(q_H - ry)}_{<0}(4q_L + 2(1-x)(q_L - ry))$, $q_L > \frac{(1-x)ry}{3-x}$ yields $\frac{\partial \Pi^*_{LL}}{\partial x} > 0$. $\frac{\partial \Pi_{LH}}{\partial x} = e(q_L - e) - (q_L - e)(e + q_H - ry) = (q_L - e)[\underbrace{e - (e + q_H - ry)}_{>0}]$, where $q_L - e = \frac{1}{3}[q_H(1-x) - ry(1-x) + 2q_L - \beta_3]$ $= \frac{1}{3}[\underbrace{(1-x)(q_H - ry)}_{<0} + 2q_L - \beta_3] < 0$ if $q_L < \frac{\beta_3-(1-x)(q_H-ry)}{2}$.

$$\left.\frac{\partial \Pi_{LH}}{\partial e}\right|_{e=e^*_{sep}} = -3e^{H*}_{sep} + q_L + (1-x)ry - (1-x)q_H = \underbrace{(1-x)(ry - q_H)}_{>0} + q_L - 3e^{H*}_{sep} < 0$$ because $e^{H*}_{sep} > e^{LH*}_1 = \frac{1}{3}(q_L - (1-x)t_H)$. Thus, when $x \in [\bar{x}_2, 1]$ and $\frac{(1-x)ry}{3-x} < q_L < \frac{-(1-x)t_H+\beta_3}{2}$, the H-type vendor reduces its e^*_{sep} when regulation becomes more lenient. Obtaining an explicit solution for the declining region of e is difficult, so we present the simulation results of Proposition 8 in the figure below. This is reasonable because this region satisfies $q_H > q_L$ and q_L is neither too high nor too low.

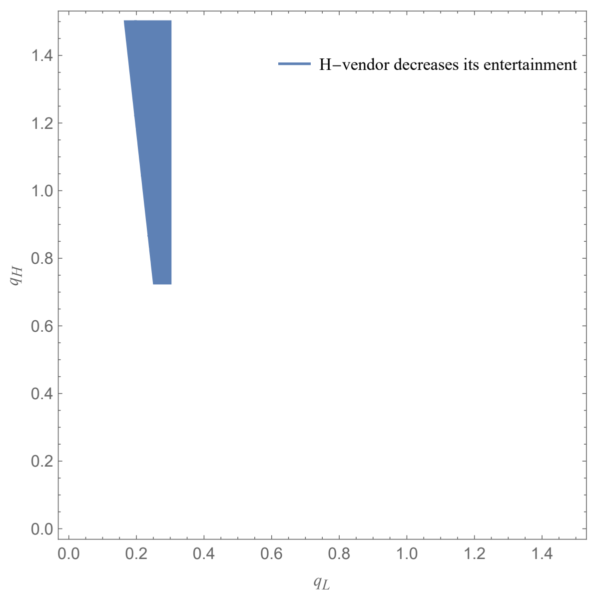

Figure 8. $\underline{q}_L \leq q_L \leq \bar{q}_L$.

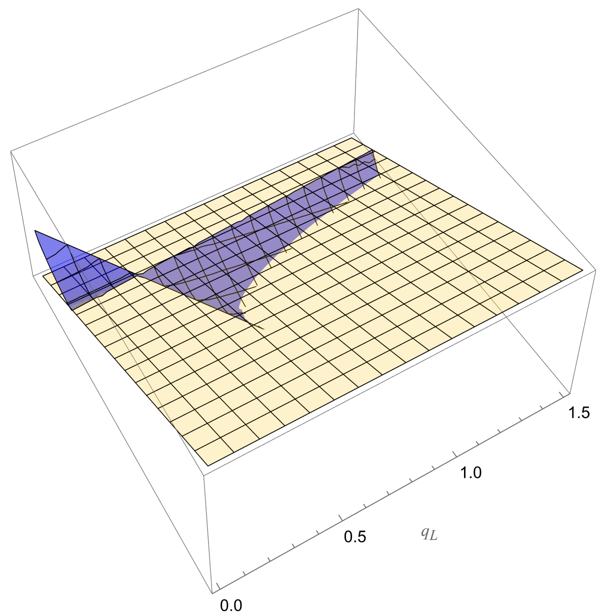

Figure 9. $\min\left\{\frac{\beta_2-(1-x)t_H}{2}, \frac{2ry(1+x)}{1+4x}, \frac{\beta_3-(1-x)t_H}{2}\right\} - q_L$, $q_L - \frac{(1-x)ry}{3-x}$.

M. Proof of Proposition 9

We focus solely on the case in which $q_H - q_L < \Delta$, meaning that the quality difference between the two vendor types is not excessively large. This assumption streamlines the comparison between pooling and separating equilibria (LMSE). It also ensures that the high-quality vendor achieves the highest profit relative to any other pooling equilibrium. When $x \in [0, \overline{x}_1]$, the high-quality vendor does not deviate from the equilibrium for a given effort e. To ensure that the low-quality vendor does not deviate, we require

$$\Pi_{LP}(e, x) \geq \Pi^*_{L,\text{dev}}(x)$$

This implies

$$e \in \left(-\frac{1}{3}q_H + \frac{2}{3}ry - \frac{1}{6}\sqrt{\Delta_1}, -\frac{1}{3}q_H + \frac{2}{3}ry + \frac{1}{6}\sqrt{\Delta_1}\right)$$

Taking the derivative with respect to e gives

$$\frac{\partial \Pi_{LP}(e, x)}{\partial e} = 0 \quad \Rightarrow \quad e^*_H = \frac{1}{3}(ry - q_L) > 0$$

We then compute the difference:

$$e_{HP} - e_C = \frac{1}{3}(ry - q_L) + \frac{1}{3}q_H - \frac{2}{3}ry = \frac{1}{3}(q_H - q_L) > 0$$

Thus, $e_{HP} > e_C > e_C - \frac{1}{6}\sqrt{\Delta_1}$.

This simplifies to

$$q_H - q_L < 2\sqrt{\Delta_1}$$

$$\Delta_1 = -2\left(2 + q_H^2 + (4 - 7q_L)q_L\right) + 8\,(1 + q_H - 2q_L)\,ry \tag{M.1}$$

$$\begin{aligned}\Delta_2 = \frac{1}{(1+x)^2}\Big(&2q_H\,(q_L(-5+x) - 2ry(-1+x))\,(-1+x)(1+x)^2 + q_H^2(-1+x^2)^2 + 4ry^2(-1+x^2)^2 \\ &- 4q_L ry(-1+4x-4x^3+x^4) + q_L^2\,(-11 + x\,(-8 + x\,(30 + (-8+x)x)))\Big)\end{aligned} \tag{M.2}$$

$$\Delta_3 = (q_H - q_L)(-1+x)\,(q_H(-1+x) - 4ry(-1+x) + q_L(-11+3x)) \tag{M.3}$$

The proofs for the remaining two intervals are similar.