Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(8): 3668–3698.
DOI:10.3934/jimo.2026132
Received: 19 May 2026
Revised: 03 July 2026
Accepted: 06 July 2026
Published: 13 July 2026

Research article

Service rivalry reshapes strategic interplay: Manufacturer encroachment and extended warranty provision under retailer-owned extended warranty

Yue Qi[1], Taofeng Ye[2,*] and Guoqing Zhang[3]

[1] Business School, Jiangsu Vocational College of Information Technology, Wuxi, China
[2] School of Economics and Management, Jiangsu University of Science and Technology, Zhenjiang, China
[3] Department of Industrial and Manufacturing Systems Engineering, University of Windsor, Windsor, Canada

* **Correspondence:** Email: ytf_jj@163.com.

A. Equilibrium results

A.1. Results of Model NNS

In the Model NNS, products are sold exclusively through the retail channel, and only retailers offer EW in the market. The inverse product demand function is given by $p_p = v(1-\alpha) - p_s - \alpha(1-\alpha)B_r q_{pr} - r\alpha\beta_r$, and the demand function for the retailer's EW is $q_s = \frac{(A-B_r)\alpha(1-\alpha)q_{pr}+r\alpha(1-\beta_r)-p_s}{(1-\alpha)\alpha(A-B_r)}$.

In the retailer's response stage, the observed wholesale price pins down the retailer's joint choice of product quantity and EW price. Its profit function is given by

$$\begin{aligned}\pi_r(q_{pr}, p_s) &= (p_p - w_p)q_{pr} + (p_s - \alpha(1-\beta_r)r)q_s \\ &= \left[v(1-\alpha) - p_s - \alpha(1-\alpha)B_r q_{pr} - r\alpha\beta_r - w_p\right] q_{pr} \\ &\quad + (p_s - \alpha(1-\beta_r)r)\,\frac{(A-B_r)\alpha(1-\alpha)q_{pr} + r\alpha(1-\beta_r) - p_s}{(1-\alpha)\alpha(A-B_r)}\end{aligned}.$$

By taking the first-order derivatives of the retailer's profit function with respect to q_{pr} and p_s, and setting both equations to zero simultaneously. By solving the system of equations, we obtain $p_s = r\alpha(1-\beta_r)$ and $q_{pr}\left(w_p\right) = \frac{v(1-\alpha)-r\alpha-w_p}{2(1-\alpha)\alpha B_r}$. To verify that this stationary point is a maximum, we examine the Hessian matrix of $\pi_r(q_{pr}, p_s)$ with respect to (q_{pr}, p_s):

$$H_r^{NNS} = \begin{pmatrix} -2(1-\alpha)\alpha B_r & 0 \\ 0 & -\dfrac{2}{(1-\alpha)\alpha(A-B_r)} \end{pmatrix}.$$

Since $0 < \alpha < 1$ and $A > B_r > 0$, the leading principal minor is negative and $\det(H_r^{NNS}) = \dfrac{4B_r}{A-B_r} > 0$. Thus, H_r^{NNS} is negative definite, and the retailer's profit is jointly concave in (q_{pr}, p_s).

In the first stage, anticipating the retailer's optimal response, the manufacturer sets the optimal wholesale price to maximize its own profit $\pi_m\left(w_p\right) = \left(w_p - (1-\alpha)\,c\right)\frac{v(1-\alpha)-r\alpha-w_p}{2(1-\alpha)\alpha B_r}$, by considering the constraint $w_p \leqslant v\,(1-\alpha) - r\alpha$, where the constraint ensures non-negative demand for the product in the retail channel. The second-order condition is satisfied because $\frac{\partial^2\pi_m}{\partial w_p^2} = -\frac{1}{(1-\alpha)\alpha B_r} < 0$, under $0 < \alpha < 1$ and $B_r > 0$. Therefore, the manufacturer's profit is concave in w_p, and the first-order condition yields the unique optimal wholesale price. Solving the manufacturer's optimization problem gives $w_p^{NNS*} = \frac{(v+c)(1-\alpha)-r\alpha}{2}$. Substituting it into the constraint confirms that the constraint holds identically. We obtain the equilibrium outcomes and profits for both the manufacturer and the retailer as follows, $p_s^{NNS*} = r\alpha\,(1-\beta_r)$, $q_{pr}^{NNS*} = \frac{\Phi}{4(1-\alpha)\alpha B_r}$, $\pi_m^{NNS*} = \frac{\Phi^2}{8(1-\alpha)\alpha B_r}$, and $\pi_r^{NNS*} = \frac{\Phi^2}{16(1-\alpha)\alpha B_r}$,
where $\Phi = (1-\alpha)\,(v-c) - \alpha r$.

A.2. Results of Model ENS

In the Model ENS, products are sold through both direct and retail channels, and only the retailer offers an EW in the market. Therefore, the total supply of products in the market is given by $q_p = q_{pr} + q_{pm}$. The inverse demand function for the product is $p_p = v\,(1-\alpha) - p_s - \alpha\,(1-\alpha)\,B_r\left(q_{pr} + q_{pm}\right) - r\alpha\beta_r$, and the demand function for the retailer's extended warranty service is $q_s = \frac{(A-B_r)\alpha(1-\alpha)\left(q_{pr}+q_{pm}\right)+r\alpha(1-\beta_r)-p_s}{(1-\alpha)\alpha(A-B_r)}$.

In the third stage, after observing the retailer's EW price and product order quantity, the manufacturer determines the sales volume of the product in the direct channel. The manufacturer's profit function is

$$\begin{aligned}\pi_m\left(q_{pm}\right) &= \left(w_p - (1-\alpha)\,c\right)q_{pr} \\ &\quad + \left(v\,(1-\alpha) - p_s - \alpha\,(1-\alpha)\,B_r\left(q_{pr}+q_{pm}\right) - r\alpha\beta_r - (1-\alpha)\,c - S\right)q_{pm}\end{aligned}.$$

The manufacturer's profit function is differentiated with respect to q_{pm}, and the derivative is set to zero. Solving this yields $q_{pm}\left(q_{pr}\right) = \frac{(v-c)(1-\alpha)-S-p_s-\alpha(1-\alpha)B_r q_{pr}-r\alpha\beta_r}{2(1-\alpha)\alpha B_r}$.

In the second stage, anticipating the manufacturer's product sales volume in the direct channel and observing the wholesale price set by the manufacturer, the retailer determines the product order quantity for the retail channel. At this stage, the retailer's profit function is

$$\begin{aligned}\pi_r(q_{pr}, p_s) &= \left(\frac{(v+c)\,(1-\alpha) + S - p_s - (1-\alpha)\,\alpha B_r q_{pr} - r\alpha\beta_r}{2} - w_p\right)q_{pr} \\ &\quad + (p_s - \alpha\,(1-\beta_r)\,r)\times \\ &\quad \frac{(A-B_r)\left((v-c)\,(1-\alpha) + (1-\alpha)\,\alpha B_r q_{pr} - S\right) - (A+B_r)\,(p_s + r\alpha\beta_r) + 2r\alpha B_r}{2\,(1-\alpha)\,\alpha\,(A-B_r)\,B_r}\end{aligned}.$$

Differentiating $\pi_r\left(q_{pr}\right)$ with regard to q_{pr}, and making it equal to zero gives $q_{pr}\left(w_p\right)=\frac{(v+c)(1-\alpha)-r\alpha+S-2w_p}{2(1-\alpha)\alpha B_r}$ and $p_s=\frac{((v-c)(1-\alpha)-S)(A-B_r)+r\alpha(B_r(3-2\beta_r)+A(1-2\beta_r))}{2(A+B_r)}$. For this two-variable maximization problem, the Hessian matrix of the retailer's profit with respect to (q_{pr}, p_s) is

$$H_r^{ENS}=\begin{pmatrix}-(1-\alpha)\alpha B_r & 0\\ 0 & -\dfrac{A+B_r}{(1-\alpha)\alpha(A-B_r)B_r}\end{pmatrix}.$$

Because $0<\alpha<1$ and $A>B_r>0$, we have $H_{11}<0$ and $\det(H_r^{ENS})=\frac{A+B_r}{A-B_r}>0$. Therefore, H_r^{ENS} is negative definite, and the retailer's first-order conditions characterize the unique joint maximum.

In the first stage, by anticipating the retailer's optimal responses regarding the product order quantity and EW price, the manufacturer sets the wholesale price to maximize its profit. Thus, the manufacturer's objective is to maxmize

$$\begin{aligned}\pi_m(w_p)=&\left(w_p-(1-\alpha)c\right)\cdot\frac{(v+c)(1-\alpha)-r\alpha+S-2w_p}{2(1-\alpha)\alpha B_r}\\&+\left(\frac{(Ac+vB_r)(1-\alpha)+AS+(A+B_r)w_p-r\alpha B_r}{2(A+B_r)}-(1-\alpha)c-S\right)\\&\cdot\frac{(A+B_r)\,w_p-r\alpha B_r-S\,(A+2B_r)-(1-\alpha)\,(Ac+(2c-v)B_r)}{2(1-\alpha)\alpha B_r\,(A+B_r)}\end{aligned},$$

by considering the constraint $q_{pr}\geqslant 0$ and $q_{pm}\geqslant 0$, i.e., $w_p\leqslant\frac{(v+c)(1-\alpha)-r\alpha+S}{2}$ and $\frac{(1-\alpha)(Ac+(2c-v)B_r)+r\alpha B_r+S(A+2B_r)}{A+B_r}\leqslant w_p$. The two constraints above ensure non-negative demand in the retail and direct channels, respectively. The condition $\frac{(v+c)(1-\alpha)-r\alpha+S}{2}>\frac{(1-\alpha)(Ac+(2c-v)B_r)+r\alpha B_r+S(A+2B_r)}{A+B_r}$ must be satisfied, leading to the basic constraint $S<\Phi$. After substituting the retailer's optimal response functions into the manufacturer's profit function, the manufacturer's objective is concave in w_p. The second-order condition is satisfied, since $\frac{\partial^2\pi_m}{\partial w_p^2}<0$ under the feasibility conditions stated before. Therefore, the first-order condition, together with the constraints above, yields the constrained optimal wholesale price. The resulting equilibrium is stated below. Substituting the equilibrium solutions into the constraints, we find that the first inequality constraint always holds, while the second constraint requires $S\leqslant\frac{\Phi(A+5B_r)}{3A+7B_r}$.

In the Model ENN, the equilibrium outcomes and optimal profits for the manufacturer and retailer are as follows:

(1) When $S\leqslant\dfrac{\Phi(A+5B_r)}{3A+7B_r}$ is satisfied, we have $q_{pr,1}^{ENS*}=\dfrac{\Phi(A-B_r)+S(3A+5B_r)}{6(1-\alpha)\alpha B_r(A+B_r)}$, $w_{p,1}^{ENS*}=\dfrac{(1-\alpha)(A(2c+v)+(c+2v)B_r)-r\alpha(A+2B_r)-SB_r}{3(A+B_r)}$, $q_{pm,1}^{ENS}=\dfrac{\Phi(A+5B_r)-S(3A+7B_r)}{6(1-\alpha)\alpha B_r(A+B_r)}$

$p_{s,1}^{ENS*}=\dfrac{((v-c)(1-\alpha)-S)(A-B_r)+r\alpha(B_r(3-2\beta_r)+A(1-2\beta_r))}{2(A+B_r)}$,

$\pi_{m,1}^{ENS*}=\dfrac{S^2(3A^2+12AB_r+13B_r^2)-8\Phi B_r(A+2B_r)S+\Phi^2(A^2+4AB_r+7B_r^2)}{12(1-\alpha)\alpha B_r(A+B_r)^2}$, and

$\pi_{r,1}^{ENS*}=\dfrac{S^2(9A^2+15AB_r+8B_r^2)-2\Phi(A-B_r)(3A+2B_r)S+\Phi^2(A-B_r)(5A+4B_r)}{36(1-\alpha)\alpha B_r(A+B_r)^2}$.

Let $\Delta_m^{ENS}=S^2\left(3A^2+12AB_r+13B_r^2\right)-8\Phi B_r\,(A+2B_r)\,S+\Phi^2\left(A^2+4AB_r+7B_r^2\right)$ and

$\Delta_r^{ENS} = S^2\left(9A^2+15AB_r+8B_r^2\right)-2\Phi\,(A-B_r)\,(3A+2B_r)\,S+\Phi^2\,(A-B_r)\,(5A+4B_r)$.

(2) When $\dfrac{\Phi(A+5B_r)}{3A+7B_r} < S < \Phi$ is satisfied, we have $q_{pr,2}^{ENS*} = \dfrac{(\Phi-S)(A+3B_r)}{2(1-\alpha)\alpha B_r(A+B_r)}$, $q_{pm,2}^{ENS*} = 0$,

$w_{p,2}^{ENS*} = \dfrac{(1-\alpha)(Ac+(2c-v)B_r)+r\alpha B_r+S(A+2B_r)}{A+B_r}$,

$p_{s,2}^{ENS*} = \dfrac{((v-c)(1-\alpha)-S)(A-B_r)+r\alpha(A(1-2\beta_r)+B_r(3-2\beta_r))}{2(A+B_r)}$,

$\pi_{m,2}^{ENS*} = \dfrac{(\Phi-S)(A+3B_r)(S(A+2B_r)-\Phi B_r)}{2(1-\alpha)\alpha B_r(A+B_r)^2}$ and $\pi_{r,2}^{ENS*} = \frac{(\Phi-S)^2\left(A^2+3AB_r+4B_r^2\right)}{4(1-\alpha)\alpha B_r(A+B_r)^2}$.

A.3. Results of Model NMS

In the Model NMS, products are sold exclusively through the retail channel. In addition, both the manufacturer and the retailer offer their own EWs via their respective channels. In this model, the inverse demand function for the product is given by $p_p = v\,(1-\alpha)-p_e-\alpha\,(1-\alpha)\,B_m\left(q_{pr}+q_{pm}\right)-r\alpha\beta_m$, while the demand functions for the EW offered by the manufacturer and the retailer are given by $q_e = \frac{p_e-p_s-\alpha(1-\alpha)(B_r-B_m)\left(q_{pr}+q_{pm}\right)-r\alpha(\beta_r-\beta_m)}{(1-\alpha)\alpha(B_m-B_r)}$ and $q_s = \frac{r\alpha(B_r-B_m)-(A-B_m)(p_s+r\alpha\beta_r)+(A-B_r)(p_e+r\alpha\beta_m)}{(1-\alpha)\alpha(A-B_r)(B_r-B_m)}$, respectively.

In the third stage, after observing the retailer's product order quantity and the selling price of its EW, the manufacturer determines p_e to maximize its profit

$$\pi_m\,(p_e) = \left(w_p-(1-\alpha)\,c\right)q_{pr}+(p_e-\alpha\,(1-\beta_m)\,r-s)\,\frac{p_e-p_s-\alpha\,(1-\alpha)\,(B_r-B_m)\,q_{pr}-r\alpha\,(\beta_r-\beta_m)}{(1-\alpha)\,\alpha\,(B_m-B_r)}.$$

Differentiating $\pi_m\,(p_e)$ with respect to p_e and making the derivative to zero gives $p_e\left(p_s,q_{pr}\right) = (s+r\alpha\,(1-2\beta_m+\beta_r)+p_s+\alpha\,(1-\alpha)\,(B_r-B_m)\,q_{pr})/2$.

In the second stage, anticipating the manufacturer's pricing for its EW and observing the manufacturer's product wholesale price, the retailer determines the optimal product order quantity for the retail channel and the selling price of its self-operated EW to maximize its own profit. The retailer's objective is to maximize

$$\begin{aligned}\pi_r\left(q_{pr},p_s\right) = {} & \left(\frac{2v\,(1-\alpha)-r\alpha\,(1+\beta_r)-s-p_s-(1-\alpha)\,\alpha\,(B_m+B_r)\,q_{pr}}{2}-w_p\right)q_{pr} \\ & +(p_s-\alpha\,(1-\beta_r)\,r)\,\frac{\begin{pmatrix}(1-\alpha)\,\alpha\,(A-B_r)\,(B_r-B_m)\,q_{pr} \\ -(A-2B_m+B_r)\,(p_s-r\alpha\,(1-\beta_r))+(A-B_r)\,s\end{pmatrix}}{2\,(1-\alpha)\,\alpha\,(A-B_r)\,(B_r-B_m)}.\end{aligned}$$

Differentiating $\pi_r(q_{pr},p_s)$ with respect to q_{pr} and p_s, setting them to zero yields $q_{pr} = \frac{2(v(1-\alpha)-r\alpha-w_p)-s}{2(1-\alpha)\alpha(B_m+B_r)}$ and $p_s = \frac{(A-B_r)s+2r\alpha(A-2B_m+B_r)(1-\beta_r)}{2(A-2B_m+B_r)}$. The Hessian matrix of $\pi_r(q_{pr},p_s)$ with respect to (q_{pr},p_s) is

$$H_r^{NMS} = \begin{pmatrix}-(1-\alpha)\alpha(B_m+B_r) & 0 \\ 0 & -\dfrac{A-2B_m+B_r}{(1-\alpha)\alpha(A-B_r)(B_r-B_m)}\end{pmatrix}.$$

Under the feasibility conditions $0 < B_m < B_r < A$ and $A-2B_m+B_r > 0$, we have $H_{11} < 0$ and $\det(H_r^{NMS}) = \frac{(B_m+B_r)(A-2B_m+B_r)}{(A-B_r)(B_r-B_m)} > 0$. Hence, H_r^{NMS} is negative definite, and the retailer's profit is jointly concave in (q_{pr},p_s).

In the first stage, by anticipating the retailer's optimal product order quantity and the selling price of its EW, the manufacturer's objective is to maximize

$$\pi_m\left(w_p\right) = \left(w_p - (1-\alpha)\,c\right) q_{pr} + \left(p_e - \alpha\,(1-\beta_m)\,r - s\right) q_e$$

over w_p by considering the constraint $q_{pr} \geqslant 0$, i.e., $w_p \leqslant \frac{2v(1-\alpha)-2r\alpha-s}{2}$, which is used to ensure non-negative sales in the retail channel. After substituting the retailer's optimal response functions into the manufacturer's profit function, the manufacturer's objective is concave in w_p. The second-order condition is satisfied, since $\frac{\partial^2 \pi_m}{\partial w_p^2} < 0$ under the feasibility conditions stated before. The optimal wholesale price is $w_p = \frac{(A-B)s+2((A+3B)c+4Bv)(1-\alpha)-8Br\alpha}{2(A+7B)}$. Substituting this into the constraint shows that the inequality always holds. Therefore, in the Model NMN, the equilibrium outcomes and optimal profits for the manufacturer and the retailer can be written as

$$w_p^{NMS*} = \frac{\begin{pmatrix} (1-\alpha)(A-2B_m+B_r)\left(2c\,(B_m+B_r)+(3B_m+B_r)\,v\right) \\ -(A-2B_m+B_r)(3B_m+B_r)\,r\alpha + s\left(B_m^2+B_r^2-B_m\,(A+B_r)\right) \end{pmatrix}}{(A-2B_m+B_r)(5B_m+3B_r)},$$

$$p_s^{NMS*} = \frac{(A-B_r)s+2r\alpha(A-2B_m+B_r)(1-\beta_r)}{2(A-2B_m+B_r)},\; q_{pr}^{NMS*} = \frac{4\Phi(A-2B_m+B_r)-s(3A-8B_m+5B_r)}{2(1-\alpha)\alpha(A-2B_m+B_r)(5B_m+3B_r)},$$

$$p_e^{NMS*} = \frac{\begin{pmatrix} s\left((3A-B_r)\,B_r+3B_m\,(3A+B_r)-14B_m^2\right) \\ -2\,(A-2B_m+B_r)\begin{pmatrix} (v-c)(1-\alpha)(B_m-B_r) \\ -r\alpha\left(B_r\,(2-3\beta_m)+B_m\,(6-5\beta_m)\right) \end{pmatrix} \end{pmatrix}}{2(A-2B_m+B_r)(5B_m+3B_r)},$$

$$\pi_m^{NMS*} = \frac{\begin{pmatrix} s^2\left(4B_m^3+2B_m^2\,(A+9B_r)+B_r\left(3A^2+12AB_r+13B_r^2\right)-B_m\left(A^2+18AB_r+33B_r^2\right)\right) \\ +2\Phi\,(B_r-B_m)(A-2B_m+B_r)\left(2\Phi\,(A-2B_m+B_r)-(3A-8B_m+5B_r)\,s\right) \end{pmatrix}}{4(1-\alpha)\alpha(B_r-B_m)(A-2B_m+B_r)^2(5B_m+3B_r)},$$

$$\pi_r^{NMS*} = \frac{\begin{pmatrix} 2\begin{pmatrix} -32B_m^4-B_m^3\,(A-65B_r)+B_mB_r\left(15A^2-33AB_r-46B_r^2\right) \\ +B_r^2\left(9A^2+15AB_r+8B_r^2\right)+B_m^2\left(8A^2-45AB_r+37B_r^2\right) \end{pmatrix} s^2 \\ -8\Phi\left(B_r^2-B_m^2\right)\left(3A^2+16B_m^2+8AB_r+5B_r^2-2B_m\,(7A+9B_r)\right) s \\ +16\Phi^2\,(B_r-B_m)(A-2B_m+B_r)^2\,(B_m+B_r) \end{pmatrix}}{8(1-\alpha)\alpha(B_r-B_m)(A-2B_m+B_r)^2(5B_m+3B_r)^2},$$ respectively.

Let $\Delta_r^{NMS} = \begin{pmatrix} 2\begin{pmatrix} -32B_m^4-B_m^3\,(A-65B_r)+B_mB_r\left(15A^2-33AB_r-46B_r^2\right) \\ +B_r^2\left(9A^2+15AB_r+8B_r^2\right)+B_m^2\left(8A^2-45AB_r+37B_r^2\right) \end{pmatrix} s^2 \\ -8\Phi\left(B_r^2-B_m^2\right)\left(3A^2+16B_m^2+8AB_r+5B_r^2-2B_m\,(7A+9B_r)\right) s \\ +16\Phi^2\left(B_r^2-B_m^2\right)(A-2B_m+B_r)^2 \end{pmatrix}$ and

$$\Delta_m^{NMS} = \begin{pmatrix} s^2\left(4B_m^3+2B_m^2\,(A+9B_r)+B_r\left(3A^2+12AB_r+13B_r^2\right)-B_m\left(A^2+18AB_r+33B_r^2\right)\right) \\ -2\Phi\,(3A-8B_m+5B_r)(B_r-B_m)(A-2B_m+B_r)\,s+4\Phi^2\,(B_r-B_m)(A-2B_m+B_r)^2 \end{pmatrix}.$$

A.4. Results of Model EMS

In the Model EMS, products are sold exclusively through the retail channel. Additionally, the manufacturer sells the EW directly to consumers along with the products, while the retailer also offers its own EW in the retail channel. In this model, the inverse demand function for the product is given by $p_p = v\,(1-\alpha) - p_e - \alpha\,(1-\alpha)\,B_m\left(q_{pr}+q_{pm}\right) - r\alpha\beta_m$, and the demand functions for the manufacturer's and retailer's EWs are given by $q_e = \frac{p_e - p_s - \alpha(1-\alpha)(B_r - B_m)\left(q_{pr}+q_{pm}\right) - r\alpha(\beta_r - \beta_m)}{(1-\alpha)\alpha(B_m - B_r)}$ and $q_s = \frac{r\alpha(B_r-B_m)-(A-B_m)(p_s+r\alpha\beta_r)+(A-B_r)(p_e+r\alpha\beta_m)}{(1-\alpha)\alpha(A-B_r)(B_r-B_m)}$, respectively.

In the third stage, after observing the retailer's product order quantity and the selling price of its EW, the manufacturer determines its direct sales quantity and the selling price of its EW. Differentiating $\pi_m\left(q_{pm}, p_e\right)$ with respect to q_{pm} and p_e and setting them equal to zero gives $q_{pm}\left(q_{pr}\right) = \frac{(v-c)(1-\alpha)-r\alpha-S-s-\alpha(1-\alpha)B_m q_{pr}}{2(1-\alpha)\alpha B_m}$ and $p_e\left(p_s, q_{pr}\right) = \frac{s+r\alpha(1-2\beta_m+\beta_r)+p_s+(1-\alpha)\alpha(B_r-B_m)q_{pr}}{2}$. Since the manufacturer maximizes profit over two decision variables in this stage, we verify the second-order condition through the Hessian matrix with respect to (q_{pm}, p_e):

$$H_m^{EMS} = \begin{pmatrix} -2(1-\alpha)\alpha B_m & 0 \\ 0 & -\dfrac{2}{(1-\alpha)\alpha(B_r - B_m)} \end{pmatrix}.$$

Given $0 < \alpha < 1$ and $0 < B_m < B_r$, the first leading principal minor is negative and $\det(H_m^{EMS}) = \frac{4B_m}{B_r - B_m} > 0$. Thus, H_m^{EMS} is negative definite, and the first-order conditions yield the unique joint maximizer for (q_{pm}, p_e).

In the second stage, anticipating the manufacturer's direct sales quantity and EW price, and having observed the manufacturer's wholesale price, the retailer determines its optimal product order quantity in the retail channel and its EW price to maximize its profit

$$\begin{aligned}\pi_r\left(q_{pr}, p_s\right) = &\left(\frac{S + (v+c)(1-\alpha) - p_s - (1-\alpha)\alpha B_r q_{pr} - r\alpha\beta_r}{2} - w_p\right) q_{pr} \\ &+ \left(p_s - \alpha(1-\beta_r) r\right) \frac{\begin{pmatrix}(A - B_r)\left(s + (1-\alpha)\alpha(B_r - B_m) q_{pr}\right) \\ + (A - 2B_m + B_r)(r\alpha - p_s - r\alpha\beta_r)\end{pmatrix}}{2(1-\alpha)\alpha(A - B_r)(B_r - B_m)}.\end{aligned}$$

Differentiating $\pi_r\left(q_{pr}, p_s\right)$ with respect to p_s and q_{pr} and setting them to zero gives $q_{pr}\left(w_p\right) = \frac{(v+c)(1-\alpha)-r\alpha+S-2w_p}{2(1-\alpha)\alpha B_r}$ and $p_s = \frac{2r\alpha(A-2B_m+B_r)(1-\beta_r)+s(A-B_r)}{2(A-2B_m+B_r)}$. For the retailer's joint choice of (q_{pr}, p_s), the Hessian matrix is

$$H_r^{EMS} = \begin{pmatrix} -(1-\alpha)\alpha B_r & 0 \\ 0 & -\dfrac{A - 2B_m + B_r}{(1-\alpha)\alpha(A - B_r)(B_r - B_m)} \end{pmatrix}.$$

Under $0 < B_m < B_r < A$ and $A - 2B_m + B_r > 0$, $H_{11} < 0$, and $\det(H_r^{EMS}) = \frac{B_r(A-2B_m+B_r)}{(A-B_r)(B_r-B_m)} > 0$. Therefore, H_r^{EMS} is negative definite, which confirms that the retailer's first-order conditions identify the unique joint maximum.

In the first stage, by anticipating the retailer's optimal product order quantity and the selling price of its EW, the manufacturer sets the wholesale price to maximize its profit

$$\pi_m\left(w_p\right) = \left(w_p - (1-\alpha)c\right) q_{pr} + \left(p_p - (1-\alpha)c - S\right) q_{pm} + \left(p_e - \alpha(1-\beta_m) r - s\right) q_{em},$$

with the constraint $q_{pr} \geqslant 0$ and $q_{pm} \geqslant 0$, i.e., $w_p \leqslant \frac{(v+c)(1-\alpha)-r\alpha+S}{2}$ and $\frac{\begin{pmatrix}(1-\alpha)\left((c+v)B_m - 2(v-c)B_r\right) + 2sB_r \\ + r\alpha(2B_r - B_m) + S(B_m + 2B_r)\end{pmatrix}}{2B_m} \leqslant w_p$. These two constraints are applied to ensure non-negative sales in both the retail and direct channels. The endpoints of the inequality must satisfy $\frac{\left(\begin{array}{l}(1-\alpha)((c+v)B_m-2(v-c)B_r)+2sB_r\\ +r\alpha(2B_r-B_m)+S(B_m+2B_r)\end{array}\right)}{2B_m}<\frac{(v+c)(1-\alpha)-r\alpha+S}{2}$, which simplifies to $\Phi-s-S>0$. After substituting the retailer's optimal response functions into the manufacturer's profit function, the manufacturer's objective is a quadratic function of w_p. The second-order condition is satisfied, since $\frac{\partial^2\pi_m}{\partial w_p^2}<0$. Therefore, the first-order condition, together with the constraints above, yields the constrained optimal wholesale price. Solving this constrained optimization problem gives $w_p^{EMS}=\frac{(3((v+c)(1-\alpha)-r\alpha)-S)(A-2B_m+B_r)-s(A-B_r)}{6(A-2B_m+B_r)}$. Substituting this into the constraints shows that the first inequality always holds, while the second inequality simplifies to the prerequisite $s<\frac{6\Phi B_r(A-2B_m+B_r)}{B_m(5A-21B_r)+12B_r(A+B_r)-8B_m^2}$. The equilibrium outcomes and optimal profits for the manufacturer and retailer are as follows.

(1) When the constraint $S\le\frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)}$ is satisfied, we have $p_{s,1}^{EMS*}=\frac{s(A-B_r)+2r\alpha(A-2B_m+B_r)(1-\beta_r)}{2(A-2B_m+B_r)}$, $w_{p,1}^{EMS*}=\frac{(3((v+c)(1-\alpha)-r\alpha)-S)(A-2B_m+B_r)-s(A-B_r)}{6(A-2B_m+B_r)}$, $q_{pm,1}^{EMS*}=\frac{2(A-2B_m+B_r)(3(\Phi-S)B_r-2SB_m)-s(B_m(A-13B_r)+6B_r(A+B_r))}{12(1-\alpha)\alpha B_mB_r(A-2B_m+B_r)}$, $q_{pr,1}^{EMS*}=\frac{4S(A-2B_m+B_r)+s(A-B_r)}{6(1-\alpha)\alpha B_r(A-2B_m+B_r)}$, $p_{e,1}^{EMS*}=\frac{s(2B_r(5A+B_r)-B_m(A+11B_r))+4(B_r-B_m)(A-2B_m+B_r)S+12r\alpha B_r(A-2B_m+B_r)(1-\beta_m)}{12B_r(A-2B_m+B_r)}$,

$$\pi_{r,1}^{EMS*}=\frac{\left(\begin{array}{l}(A-B_r)(2B_r(5A+4B_r)-B_m(A+17B_r))s^2\\ +8S(B_r-B_m)(A-2B_m+B_r)((A-B_r)s+2(A-2B_m+B_r)S)\end{array}\right)}{72(1-\alpha)\alpha B_r(A-2B_m+B_r)^2(B_r-B_m)},$$

and

$$\pi_{m,1}^{EMS*}=\frac{\left(\begin{array}{l}s^2\left(12B_r^2(A+B_r)^2-8B_mB_r\left(A^2+7AB_r+4B_r^2\right)+B_m^2\left(-A^2+26AB_r+23B_r^2\right)\right)\\ +4(B_r-B_m)(A-2B_m+B_r)\left(\begin{array}{l}2s((B_m(A-7B_r)+3B_r(A+B_r))S-3\Phi B_r(A-2B_m+B_r))\\ +(A-2B_m+B_r)\left((4B_m+3B_r)S^2-6\Phi B_rS+3\Phi^2B_r\right)\end{array}\right)\end{array}\right)}{48(1-\alpha)\alpha B_mB_r(A-2B_m+B_r)^2(B_r-B_m)}.$$

(2) When the constraint $\frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)}<S<\Phi-s$ is satisfied, we have $q_{pr,2}^{EMS*}=\frac{\Phi-S-s}{(1-\alpha)\alpha B_m}$, $w_{p,2}^{EMS*}=\frac{(1-\alpha)((c+v)B_m-2(v-c)B_r)+2sB_r+r\alpha(2B_r-B_m)+S(B_m+2B_r)}{2B_m}$, $q_{pm,2}^{EMS*}=0$, $p_{s,2}^{EMS*}=\frac{2r\alpha(A-2B_m+B_r)(1-\beta_r)+s(A-B_r)}{2(A-2B_m+B_r)}$,

$$p_{e,2}^{EMS*}=\frac{\left(\begin{array}{l}s\left(B_m(5A+7B_r)-8B_m^2-2B_r(A+B_r)\right)\\ 2(A-2B_m+B_r)((B_m-B_r)S+(\Phi B_r-B_m(\Phi-2r\alpha(1-\beta_m))))\end{array}\right)}{4B_m(A-2B_m+B_r)},$$

$$\pi_{r,2}^{EMS*}=\frac{\left(\begin{array}{l}s^2\left(4B_r^2(A+B_r)-4B_mB_r(A+3B_r)+B_m^2(A+7B_r)\right)+\\ 4(B_r-B_m)B_r(A-2B_m+B_r)\left(S^2-2\Phi S+\Phi^2-2s(\Phi-S)\right)\end{array}\right)}{8(1-\alpha)\alpha B_m^2(B_r-B_m)(A-2B_m+B_r)},$$

and

$$\pi_{m,2}^{EMS*}=\frac{\left(\begin{array}{l}s^2\left(64B_m^3B_r-12B_r^2(A+B_r)^2+B_m^2\left(A^2-58AB_r-119B_r^2\right)+4B_mB_r\left(3A^2+20AB_r+17B_r^2\right)\right)\\ +4(B_r-B_m)(A-2B_m+B_r)s\left(\begin{array}{l}\Phi\left(4B_m^2+6B_r(A+B_r)-B_m(A+15B_r)\right)\\ +\left(4B_m^2-B_m(3A-11B_r)-6B_r(A+B_r)\right)S\end{array}\right)\end{array}\right)}{16(1-\alpha)\alpha B_m^2(A-2B_m+B_r)^2(B_r-B_m)}.$$

Let $\Delta_{r,1}^{EMS}=\left(\begin{array}{l}(A-B_r)(2B_r(5A+4B_r)-B_m(A+17B_r))s^2\\ +8(A-B_r)(B_r-B_m)(A-2B_m+B_r)sS+16(B_r-B_m)(A-2B_m+B_r)^2S^2\end{array}\right)$,

$$\Delta_{r,2}^{EMS} = \begin{pmatrix} s^2\left(4B_r^2(A+B_r) - 4B_mB_r(A+3B_r) + B_m^2(A+7B_r)\right) \\ +8B_r(B_m-B_r)\,s\,(\Phi(A-2B_m+B_r) - (A-2B_m+B_r)S) \\ +4(B_r-B_m)\begin{pmatrix} B_r(A-2B_m+B_r)S^2 - 2\Phi B_r(A-2B_m+B_r)S \\ +\Phi^2 B_r(A-2B_m+B_r) \end{pmatrix} \end{pmatrix},$$

$$\Delta_{m,1}^{EMS} = \begin{pmatrix} s^2\cdot\left(12B_r^2(A+B_r)^2 - 8B_mB_r(A^2+7AB_r+4B_r^2) - B_m^2(A^2-26AB_r-23B_r^2)\right)+ \\ 4(B_r-B_m)(A-2B_m+B_r)\times\Big(2s\,(B_m(A-7B_r)+3B_r(A+B_r))\,S+ \\ (A-2B_m+B_r)\cdot\left((4B_m+3B_r)S^2 - 6\Phi B_rS + 3\Phi^2B_r - 6\Phi B_rs\right)\Big) \end{pmatrix} \text{ and}$$

$$\Delta_{m,2}^{EMS} = \begin{pmatrix} s^2\cdot\Big(64B_m^3B_r - 12B_r^2(A+B_r)^2 + B_m^2(A^2-58AB_r-119B_r^2) \\ +4B_mB_r(3A^2+20AB_r+17B_r^2)\Big)+ \\ 4(B_r-B_m)(A-2B_m+B_r)\times\Big(s\cdot\Big(\Phi(4B_m^2+6B_r(A+B_r)-B_m(A+15B_r)) \\ +(4B_m^2 - B_m(3A-11B_r) - 6B_r(A+B_r))S\Big)+ \\ (A-2B_m+B_r)\cdot\left(2\Phi(B_m+3B_r)S - 3(B_m+B_r)S^2 - \Phi^2(3B_r-B_m)\right)\Big) \end{pmatrix}.$$

B. Proofs

Proof of Proposition 1. The difference between π_m^{NMS*} and π_m^{NNS*} can be expressed as

$$\pi_m^{NMS*} - \pi_m^{NNS*} = \frac{\begin{pmatrix} 2B_r\begin{pmatrix} 4B_m^3 + 2B_m^2(A+9B_r) + B_r\left(3A^2+12AB_r+13B_r^2\right) \\ -B_m\left(A^2+18AB_r+33B_r^2\right) \end{pmatrix}s^2 \\ +4\Phi(B_m-B_r)B_r\left(3A^2+16B_m^2+8AB_r+5B_r^2-2B_m(7A+9B_r)\right)s \\ +5\Phi^2(B_m-B_r)^2(A-2B_m+B_r)^2 \end{pmatrix}}{8(1-\alpha)\alpha B_r(A-2B_m+B_r)^2(B_r-B_m)(5B_m+3B_r)}.$$

When the conditions $\frac{\sqrt{5}A}{5} < B_r < A$ and $B_m \leqslant \frac{(5B_r-A)-\sqrt{5}(A-B_r)}{4}$ are satisfied, $\pi_m^{NMS} \geqslant \pi_m^{NNS}$ holds. However, under the conditions $\frac{\sqrt{5}A}{5} < B_r < A$ and $\frac{(5B_r-A)-\sqrt{5}(A-B_r)}{4} < B_m < B_r$, $\pi_m^{NMS} \geqslant \pi_m^{NNS}$ holds only if $s \leqslant s_1$ or $s_2 \leqslant s < \Phi$; in contrast, $\pi_m^{NMS} < \pi_m^{NNS}$ holds when $s_1 < s < s_2$. Here, s_1 and s_2 are defined as

$$s_1 = \frac{\Phi(B_r-B_m)\begin{pmatrix} 4B_r\left(3A^2+16B_m^2+8AB_r+5B_r^2-2B_m(7A+9B_r)\right) \\ -2(A-2B_m+B_r)\sqrt{2B_r(5B_m+3B_r)(A^2-4B_m^2-2B_m(A-5B_r)-5B_r^2)} \end{pmatrix}}{2\left(2B_r\left(4B_m^3+2B_m^2(A+9B_r)+B_r\left(3A^2+12AB_r+13B_r^2\right)-B_m\left(A^2+18AB_r+33B_r^2\right)\right)\right)} \text{ and}$$

$$s_2 = \frac{\Phi(B_r-B_m)\begin{pmatrix} 4B_r\left(3A^2+16B_m^2+8AB_r+5B_r^2-2B_m(7A+9B_r)\right) \\ +2(A-2B_m+B_r)\sqrt{2B_r(5B_m+3B_r)(A^2-4B_m^2-2B_m(A-5B_r)-5B_r^2)} \end{pmatrix}}{2\left(2B_r\left(4B_m^3+2B_m^2(A+9B_r)+B_r\left(3A^2+12AB_r+13B_r^2\right)-B_m\left(A^2+18AB_r+33B_r^2\right)\right)\right)}.$$

Proof of Proposition 2. Under manufacturer encroachment, the comparison between π_m^{EMS} and π_m^{ENS}, is divided into four cases, contingent on the values of S and s.

- $\frac{\Phi}{2} \leqslant S \leqslant \frac{\Phi(A+5B_r)}{3A+7B_r}$.
(i) $s \leqslant \frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)}$. We have $\pi_{m,1}^{EMS} > \pi_{m,1}^{ENS}$.
(ii) $\frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)} < s < \Phi - S$.
$\pi_{m,2}^{EMS} \geqslant \pi_{m,1}^{ENS}$ holds for $\frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)} < s \leqslant s_3$, and $\pi_{m,2}^{EMS} < \pi_{m,1}^{ENS}$ holds for $s_3 < s < \Phi - S$. Here, where s_3 is defined as

$$s_3 = \frac{2(A-2B_m+B_r)\begin{pmatrix} 3\left(B_r-B_m\right)B_r\left(A+B_r\right)\begin{pmatrix}\Phi\begin{pmatrix}-4B_m^2-6B_r\left(A+B_r\right)\\ +B_m\left(A+15B_r\right)\end{pmatrix}\\ +S\begin{pmatrix}-4B_m^2+B_m\left(3A-11B_r\right)\\ +6B_r\left(A+B_r\right)\end{pmatrix}\end{pmatrix}\\ -B_m\sqrt{3B_m\left(B_r-B_m\right)B_r\Delta_1}\end{pmatrix}}{3B_r(A+B_r)\begin{pmatrix}64B_m^3B_r-12B_r^2(A+B_r)^2+B_m^2\left(A^2-58AB_r-119B_r^2\right)\\ +4B_mB_r\left(3A^2+20AB_r+17B_r^2\right)\end{pmatrix}} \text{ and}$$

$$\Delta_1 = S^2\Big[16B_m^2B_r(3A+7B_r)^2+B_m(3A^4-90A^3B_r-536A^2B_r^2-1390AB_r^3-1187B_r^4)+2B_r(9A^4+48A^3B_r+174A^2B_r^2+352AB_r^3+217B_r^4)\Big]-8\Phi B_rS\Big[3A^4+12A^3B_r+38A^2B_r^2+108AB_r^3+79B_r^4+4B_m^2(3A^2+22AB_r+35B_r^2)-B_m(11A^3+56A^2B_r+199AB_r^2+214B_r^3)\Big]+\Phi^2\Big[16B_m^2B_r(A+5B_r)^2+B_m(A^4-30A^3B_r-80A^2B_r^2-426AB_r^3-617B_r^4)+2B_r(3A^4+16A^3B_r+26A^2B_r^2+128AB_r^3+115B_r^4)\Big].$$

- $\frac{\Phi(A+5B_r)}{3A+7B_r} < S < \frac{\Phi\left((A+3B_r)^2+(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)}$.
(i) $s \leqslant \frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)}$. $\pi_{m,1}^{EMS*} > \pi_{m,2}^{ENS*}$ always holds.
(ii) $\frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)} < s < \Phi - S$.
$\pi_{m,2}^{EMS*} \geqslant \pi_{m,2}^{ENS*}$ holds for $\frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)} < s \leqslant s_4$, and $\pi_{m,2}^{EMS*} < \pi_{m,2}^{ENS*}$ holds for $s_4 < s < \Phi - S$. Here, s_4 is defined as

$$s_4 = \frac{\begin{pmatrix}4\left(A+B_r\right)\left(A-2B_m+B_r\right)\left(B_r-B_m\right)B_r\left(A+B_r\right)\\ \times\begin{pmatrix}\Phi\begin{pmatrix}4B_m^2-B_m\left(A+15B_r\right)\\ +6B_r\left(A+B_r\right)\end{pmatrix}\\ +S\begin{pmatrix}4B_m^2-B_m\left(3A-11B_r\right)\\ -6B_r\left(A+B_r\right)\end{pmatrix}\end{pmatrix}+B_m\sqrt{2\left(B_m-B_r\right)B_r\Delta_2}\end{pmatrix}}{2B_r(A+B_r)^2\begin{pmatrix}12B_r^2(A+B_r)^2-64B_m^3B_r-B_m^2\left(A^2-58AB_r-119B_r^2\right)\\ -4B_mB_r\left(3A^2+20AB_r+17B_r^2\right)\end{pmatrix}} \text{ and}$$

$$\Delta_2 = S^2\begin{pmatrix} 8B_m^3B_r(3A+7B_r)^2 - 2B_r^2\left(3A^2+10AB_r+7B_r^2\right)^2 \\ +B_m^2\left(A^4-65A^3B_r-487A^2B_r^2-1075AB_r^3-774B_r^4\right) \\ +B_mB_r\left(15A^4+188A^3B_r+702A^2B_r^2+1012AB_r^3+483B_r^4\right)\end{pmatrix}$$
$$-\Phi\begin{pmatrix} 16B_m^3B_r\left(3A^2+22AB_r+35B_r^2\right) \\ -4B_r^2(A+B_r)^2\left(3A^2+22AB_r+35B_r^2\right) \\ +B_m^2\left(A^4-36A^3B_r-458A^2B_r^2-1284AB_r^3-1103B_r^4\right) \\ +8B_mB_r\left(A^4+19A^3B_r+93A^2B_r^2+161AB_r^3+86B_r^4\right)\end{pmatrix}S$$
$$+\Phi^2B_r\begin{pmatrix} 8B_m^3(A+5B_r)^2-2B_r\left(A^2+6AB_r+5B_r^2\right)^2 \\ -3B_m^2\left(A^3+33A^2B_r+123AB_r^2+131B_r^3\right) \\ +B_m\left(A^4+28A^3B_r+186A^2B_r^2+404AB_r^3+245B_r^4\right)\end{pmatrix}.$$

Proof of Proposition 3. The difference between π_m^{ENS*} and π_m^{NNS*} is analyzed in many cases according to the values of S and B_r.

- $S \leqslant \frac{\Phi(A+5B_r)}{3A+7B_r}$.
  (i) $B_r < \frac{(4\sqrt{2}-3)A}{23}$. Thus, we have $\pi_{m,1}^{ENS*} < \pi_m^{NNS*}$.
  (ii) $\frac{(2\sqrt{3}-1)A}{11} \leqslant B_r < \frac{\sqrt{5}A}{5}$.
  $\pi_{m,1}^{ENS*} \geqslant \pi_m^{NNS*}$ holds for $S \leqslant \frac{\Phi\left(8B_r(A+2B_r)-(A+B_r)\sqrt{6\left(A^2-5B_r^2\right)}\right)}{2\left(3A^2+12AB_r+13B_r^2\right)}$ or $\frac{\Phi\left(8B_r(A+2B_r)+(A+B_r)\sqrt{6\left(A^2-5B_r^2\right)}\right)}{2\left(3A^2+12AB_r+13B_r^2\right)} \leqslant S \leqslant \frac{\Phi(A+5B_r)}{3A+7B_r}$. Moreover, $\pi_{m,1}^{ENS*} < \pi_m^{NNS*}$ holds for $\frac{\Phi\left(8B_r(A+2B_r)-(A+B_r)\sqrt{6\left(A^2-5B_r^2\right)}\right)}{2\left(3A^2+12AB_r+13B_r^2\right)} < S < \frac{\Phi\left(8B_r(A+2B_r)+(A+B_r)\sqrt{6\left(A^2-5B_r^2\right)}\right)}{2\left(3A^2+12AB_r+13B_r^2\right)}$.
  (iii) $\frac{(4\sqrt{2}-3)A}{23} \leqslant B_r < \frac{(2\sqrt{3}-1)A}{11}$. $\pi_{m,1}^{ENS*} \geqslant \pi_m^{NNS}$ holds for $\frac{\Phi\left(8B_r(A+2B_r)+(A+B_r)\sqrt{6\left(A^2-5B_r^2\right)}\right)}{2\left(3A^2+12AB_r+13B_r^2\right)} \leq S \leq \frac{\Phi\left((A+3B_r)^2+(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)}$. Moreover, $\pi_{m,1}^{ENS*} < \pi_m^{NNS}$ holds for $S < \frac{\Phi\left(8B_r(A+2B_r)+(A+B_r)\sqrt{6\left(A^2-5B_r^2\right)}\right)}{2\left(3A^2+12AB_r+13B_r^2\right)}$
  (iv) $\frac{\sqrt{5}A}{5} \leqslant B_r < A$. $\pi_{m,1}^{ENS*} \geqslant \pi_m^{NNS*}$ always holds.

- $\frac{\Phi(A+5B_r)}{3A+7B_r} < S < \Phi$.
  (i) $B_r < \frac{(4\sqrt{2}-3)A}{23}$. $\pi_{m,2}^{ENS*} < \pi_m^{NNS*}$ holds for $\frac{\Phi(A+5B_r)}{3A+7B_r} < S < \frac{\Phi\left((A+3B_r)^2-(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)}$ or $\frac{\Phi\left((A+3B_r)^2+(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)} < S < \Phi$. Moreover, $\pi_{m,2}^{ENS*} \geqslant \pi_m^{NNS*}$ holds for $\frac{\Phi\left((A+3B_r)^2-(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)} \leqslant S \leqslant \frac{\Phi\left((A+3B_r)^2+(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)}$.
  (ii) $\frac{(4\sqrt{2}-3)A}{23} \leqslant B_r < A$. $\pi_{m,2}^{ENS*} \geqslant \pi_m^{NNS*}$ holds for $\frac{\Phi(A+5B_r)}{3A+7B_r} < S \leqslant \frac{\Phi\left((A+3B_r)^2+(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)}$ and $\pi_{m,2}^{ENS*} < \pi_m^{NNS*}$ holds for $\frac{\Phi\left((A+3B_r)^2+(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)} < S < \Phi$.

Proof of Proposition 4. The difference between π_m^{NMS*} and π_m^{EMS*} is analyzed in the following cases. To keep the comparison analytically tractable, we impose $\frac{\sqrt{5}A}{5} < B_r < A$, $B_m \leqslant \frac{(5B_r-A)-\sqrt{5}(A-B_r)}{4}$ and $s < \frac{6\Phi B_r(A-2B_m+B_r)}{B_m(5A-21B_r)+12B_r(A+B_r)-8B_m^2}$.

- $S \leqslant \frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)}$. $\pi_{m,1}^{EMS*} - \pi_m^{NMS*} > 0$ always holds.
- $\frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)} < S < \Phi - s$. $\pi_{m,2}^{EMS*} \geqslant \pi_m^{NMS*}$ holds for $\frac{\left(6\Phi B_r(A-2B_m+B_r) - s\left(B_m(A-13B_r)+6B_r(A+B_r)\right)\right)}{2(A-2B_m+B_r)(2B_m+3B_r)} < S$ and

$$S \leqslant \frac{(A-2B_m+B_r)\begin{pmatrix}(5B_m+3B_r)\begin{pmatrix}2\Phi(A-2B_m+B_r)(B_m+3B_r)\\ +s\left(4B_m^2-6B_r(A+B_r)+B_m(-3A+11B_r)\right)\end{pmatrix}\\ +B_m\begin{pmatrix}4\Phi(A-2B_m+B_r)\\ -s(3A-8B_m+5B_r)\end{pmatrix}\sqrt{2B_m(5B_m+3B_r)}\end{pmatrix}}{6(A-2B_m+B_r)^2(B_m+B_r)(5B_m+3B_r)},$$ and

$$\pi_{m,2}^{EMS*} < \pi_m^{NMS*} \text{ holds for } \frac{(A-2B_m+B_r)\begin{pmatrix}(5B_m+3B_r)\begin{pmatrix}2\Phi(A-2B_m+B_r)(B_m+3B_r)\\ +s\begin{pmatrix}4B_m^2-6B_r(A+B_r)\\ +B_m(-3A+11B_r)\end{pmatrix}\end{pmatrix}\\ +B_m\begin{pmatrix}4\Phi(A-2B_m+B_r)\\ -s(3A-8B_m+5B_r)\end{pmatrix}\sqrt{2B_m(5B_m+3B_r)}\end{pmatrix}}{6(A-2B_m+B_r)^2(B_m+B_r)(5B_m+3B_r)} < S < \Phi - s.$$

Proof of Remark 2. As shown in Propositions 3 and 4, we compare the encroachment thresholds with and without manufacturer-provided EWs. The comparison is conducted under $\frac{\sqrt{5}A}{5} < B_r < A$, $B_m \leqslant \frac{(5B_r-A)-\sqrt{5}(A-B_r)}{4}$, and $s < \frac{6\Phi B_r(A-2B_m+B_r)}{B_m(5A-21B_r)+12B_r(A+B_r)-8B_m^2}$. Below the threshold Δ_3, the manufacturer will choose to encroach on the market even if the manufacturer does not offer an EW. Below Δ_4, the manufacturer will choose to encroach on the market when the manufacturer offers its own EW. Here, Δ_3 and Δ_4 are defined as $\Delta_3 = \frac{\Phi\left((A+3B_r)^2+(A+B_r)\sqrt{B_r(A+3B_r)}\right)}{2(A+2B_r)(A+3B_r)}$ and

$$\Delta_4 = \frac{(A-2B_m+B_r)\begin{pmatrix}(5B_m+3B_r)\left(2\Phi(A-2B_m+B_r)(B_m+3B_r)+s\begin{pmatrix}4B_m^2-6B_r(A+B_r)\\ +B_m(-3A+11B_r)\end{pmatrix}\right)\\ +B_m\left(4\Phi(A-2B_m+B_r)-s(3A-8B_m+5B_r)\right)\sqrt{2B_m(5B_m+3B_r)}\end{pmatrix}}{6(A-2B_m+B_r)^2(B_m+B_r)(5B_m+3B_r)}.$$

$\Delta_4 - \Delta_3 = \frac{f(s)}{6(A-2B_m+B_r)(B_m+B_r)(A+2B_r)(A+3B_r)(5B_m+3B_r)}$, in which

$$f(s) = s\begin{pmatrix}(5B_m+3B_r)\left(A^2+5AB_r+6B_r^2\right)\left(4B_m^2-B_m(3A-11B_r)-6B_r(A+B_r)\right)\\ -B_m(3A-8B_m+5B_r)\sqrt{2B_m(5B_m+3B_r)}\left(A^2+5AB_r+6B_r^2\right)\end{pmatrix}$$
$$+\Phi\begin{pmatrix}(A+3B_r)(5B_m+3B_r)\left(3B_r(A+B_r)^2+2B_m^2(A+5B_r)-B_m\left(A^2+12AB_r+11B_r^2\right)\right)\\ -3(A+B_r)(A-2B_m+B_r)\sqrt{B_r(A+3B_r)}\left(5B_m^2+8B_mB_r+3B_r^2\right)\\ +4\sqrt{2}B_m(A-2B_m+B_r)\sqrt{B_m(5B_m+3B_r)}\left(A^2+5AB_r+6B_r^2\right)\end{pmatrix}$$

Because $\begin{pmatrix}(5B_m+3B_r)\left(A^2+5AB_r+6B_r^2\right)\left(4B_m^2-B_m(3A-11B_r)-6B_r(A+B_r)\right)\\ -B_m(3A-8B_m+5B_r)\sqrt{2B_m(5B_m+3B_r)}\left(A^2+5AB_r+6B_r^2\right)\end{pmatrix} < 0$, $f(s)$ is monotonically decreasing. Furthermore, $f(s=0) > 0$ and $f\left(s = \frac{6\Phi B_r(A-2B_m+B_r)}{B_m(5A-21B_r)+12B_r(A+B_r)-8B_m^2}\right) < 0$. We have

$$s^{*}=\frac{\Phi\begin{pmatrix}(A-2B_m+B_r)\begin{pmatrix}4B_m\sqrt{2B_m(5B_m+3B_r)}\left(A^2+5AB_r+6B_r^2\right)\\-3(A+B_r)\sqrt{B_r(A+3B_r)}\left(5B_m^2+8B_mB_r+3B_r^2\right)\end{pmatrix}\\+(A+3B_r)(5B_m+3B_r)\left(3B_r(A+B_r)^2+2B_m^2(A+5B_r)-B_m\left(A^2+12AB_r+11B_r^2\right)\right)\end{pmatrix}}{\begin{pmatrix}B_m\sqrt{2B_m(5B_m+3B_r)}(3A-8B_m+5B_r)\left(A^2+5AB_r+6B_r^2\right)\\-(5B_m+3B_r)\left(A^2+5AB_r+6B_r^2\right)\left(4B_m^2-B_m(3A-11B_r)-6B_r(A+B_r)\right)\end{pmatrix}},$$ such that

when $s \leqslant s^*$, $f(s) \geqslant 0$, i.e., $\Delta_4 \geqslant \Delta_3$. When $s^* < s < \frac{6\Phi B_r(A-2B_m+B_r)}{B_m(5A-21B_r)+12B_r(A+B_r)-8B_m^2}$, $f(s) < 0$, i.e., $\Delta_4 < \Delta_3$.

Proof of Lemma 5. $p_s^{NMS*} - p_s^{NNS*} = \frac{s(A-B_r)}{2(A-2B_m+B_r)} > 0$ always holds.

Proof of Lemma 6. The following price comparison is made under $\frac{A}{3} \leqslant B_r < A$, $B_m \leqslant \frac{3B_r(A+B_r)\left((A+5B_r)-2\sqrt{B_r(A+3B_r)}\right)}{2\left(A^2+6AB_r+13B_r^2\right)}$, and $\frac{\Phi}{2} \leq S \leq \frac{\Phi(A+5B_r)}{3A+7B_r}$.

- $s \leq \frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)}$.
  $p_{s,1}^{EMS*} - p_{s,1}^{ENS*} = \frac{(A-B_r)(s(A+B_r)-(\Phi-S)(A-2B_m+B_r))}{2(A+B_r)(A-2B_m+B_r)}$
  (i) $\frac{A}{3} < B_r \leq \frac{5A}{9}$. $p_{s,1}^{EMS*} \leq p_{s,1}^{ENS*}$ always holds.
  (ii) $\frac{5A}{9} < B_r < A$. Under the condition that $\frac{\Phi}{2} \leq S < \frac{\Phi(13B_r-A)}{3A+17B_r}$, $p_{s,1}^{EMS*} \leq p_{s,1}^{ENS*}$ holds for $s \leq \frac{(\Phi-S)(A-2B_m+B_r)}{A+B_r}$, and $p_{s,1}^{EMS*} > p_{s,1}^{ENS*}$ holds for $\frac{(\Phi-S)(A-2B_m+B_r)}{A+B_r} < s \leq \frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)}$.
  However, under the condition that $\frac{\Phi(13B_r-A)}{3A+17B_r} \leq S \leq \frac{\Phi(A+5B_r)}{3A+7B_r}$, $p_{s,1}^{EMS*} \leq p_{s,1}^{ENS*}$ always holds.

- $\frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)} < s < \Phi - S$.
  $p_{s,2}^{EMS*} - p_{s,1}^{ENS*} = \frac{(A-B_r)(s(A+B_r)-(\Phi-S)(A-2B_m+B_r))}{2(A+B_r)(A-2B_m+B_r)}$
  (a) $\frac{A}{3} < B_r < \frac{5A}{9}$. We have $p_{s,2}^{EMS*} < p_{s,1}^{ENS*}$ for $\frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)} < s < \frac{(\Phi-S)(A-2B_m+B_r)}{A+B_r}$, and $p_{s,2}^{EMS*} \geq p_{s,1}^{ENS*}$ for $\frac{(\Phi-S)(A-2B_m+B_r)}{A+B_r} \leq s < \Phi - S$. (b) $\frac{5A}{9} \leq B_r < A$. If $\frac{\Phi}{2} \leq S \leq \frac{\Phi(13B_r-A)}{3A+17B_r}$, $p_{s,2}^{EMS*} \geq p_{s,1}^{ENS*}$ always holds. However, if $\frac{\Phi(13B_r-A)}{3A+17B_r} < S \leq \frac{\Phi(A+5B_r)}{3A+7B_r}$, $p_{s,2}^{EMS} < p_{s,1}^{ENS}$ for $\frac{(A-2B_m+B_r)(6B_r\Phi-2(2B_m+3B_r)S)}{B_m(A-13B_r)+6B_r(A+B_r)} < s < \frac{(\Phi-S)(A-2B_m+B_r)}{A+B_r}$, and $p_{s,2}^{EMS*} \geq p_{s,1}^{ENS*}$ for $\frac{(\Phi-S)(A-2B_m+B_r)}{A+B_r} \leq s < \Phi - S$.

Proof of Proposition 7. The difference between p_s^{ENS*} and p_s^{NNS*} can be expressed as two cases.

- $S \leq \frac{\Phi(A+5B_r)}{3A+7B_r}$. $p_{s,1}^{ENS*} - p_s^{NNS*} = \frac{(\Phi-S)(A-B_r)}{2(A+B_r)} > 0$ always holds.

- $\frac{\Phi(A+5B_r)}{3A+7B_r} < S < \Phi$. $p_{s,2}^{ENS*} - p_s^{NNS*} = \frac{(\Phi-S)(A-B_r)}{2(A+B_r)} > 0$ always holds.

Proof of Proposition 8. The following comparison is restricted to $\frac{\sqrt{5}A}{5} < B_r < A$, $B_m \leq \frac{(5B_r-A)-\sqrt{5}(A-B_r)}{4}$, and $s < \frac{6\Phi B_r(A-2B_m+B_r)}{B_m(5A-21B_r)+12B_r(A+B_r)-8B_m^2}$.
We have $p_{s,1}^{EMS*} = p_s^{NMS*}$ for $s < S \leq \frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)}$, and $p_{s,2}^{EMS*} = p_s^{NMS*}$ for $\frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)} < S < \Phi - s$.

Proof of Proposition 9. The difference between p_e^{EMS*} and p_e^{NMS*} is analyzed in many cases.

- $s < S \leq \frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)}$.

$p_{e,1}^{EMS*} - p_e^{NMS*} = \frac{(B_r - B_m)\begin{pmatrix} 4(A - 2B_m + B_r)(5B_m + 3B_r)S \\ +s(B_m(5A - 29B_r) + 12B_r(A + B_r)) - 12\Phi B_r(A - 2B_m + B_r) \end{pmatrix}}{12B_r(A - 2B_m + B_r)(5B_m + 3B_r)}$. (i) $s \le \frac{12\Phi B_r(A-2B_m+B_r)}{B_m(25A-33B_r)+24B_r(A+B_r)-40B_m^2}$.
$p_{e,1}^{EMS*} \le p_e^{NMS*}$ holds for $s < S \le \frac{12\Phi B_r(A-2B_m+B_r)-s(B_m(5A-29B_r)+12B_r(A+B_r))}{4(A-2B_m+B_r)(5B_m+3B_r)}$, and
$p_{e,1}^{EMS} > p_e^{NMS}$ holds for $\frac{\left(12\Phi B_r(A - 2B_m + B_r) - s(B_m(5A - 29B_r) + 12B_r(A + B_r))\right)}{4(A-2B_m+B_r)(5B_m+3B_r)} < S \le \frac{\left(6\Phi B_r(A - 2B_m + B_r) - s(B_m(A - 13B_r) + 6B_r(A + B_r))\right)}{2(A-2B_m+B_r)(2B_m+3B_r)}$.

(ii) $\frac{12\Phi B_r(A-2B_m+B_r)}{B_m(25A-33B_r)+24B_r(A+B_r)-40B_m^2} < s < \frac{6\Phi B_r(A-2B_m+B_r)}{B_m(5A-21B_r)+12B_r(A+B_r)-8B_m^2}$.
There is $p_{e,1}^{EMS*} > p_e^{NMS*}$ for $s < S \le \frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)}$, and $p_{s,2}^{EMS*} = p_s^{NMS*}$ for $\frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)} < S < \Phi - s$.

- $\frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)} < S < \Phi - s$.
$p_{e,2}^{EMS*} - p_e^{NMS*} = \frac{(B_r - B_m)\begin{pmatrix} s\left(12B_m^2 - 7B_m(A - B_r) - 6B_r(A + B_r)\right) \\ -2(A - 2B_m + B_r)(5B_m + 3B_r)S + 6\Phi(A - 2B_m + B_r)(B_m + B_r) \end{pmatrix}}{4B_m(A-2B_m+B_r)(5B_m+3B_r)}$.
We have $p_{e,2}^{EMS*} \ge p_e^{NMS*}$ for
$\frac{6\Phi B_r(A-2B_m+B_r)-s(B_m(A-13B_r)+6B_r(A+B_r))}{2(A-2B_m+B_r)(2B_m+3B_r)} < S \le \frac{\begin{pmatrix} 6\Phi(A - 2B_m + B_r)(B_m + B_r) \\ +s\left(12B_m^2 - 7B_m(A - B_r) - 6B_r(A + B_r)\right) \end{pmatrix}}{2(A-2B_m+B_r)(5B_m+3B_r)}$, and
$p_{e,2}^{EMS*} < p_e^{NMS*}$ for $\frac{\begin{pmatrix} 6\Phi(A - 2B_m + B_r)(B_m + B_r) \\ +s\left(12B_m^2 - 7B_m(A - B_r) - 6B_r(A + B_r)\right) \end{pmatrix}}{2(A-2B_m+B_r)(5B_m+3B_r)} < S < \Phi - s$.

Sensitivity analysis of the key parameters. To transparently report the robustness checks, Table 1 summarizes the sensitivity analysis conducted for Propositions 1–4 and Remarks 1–2. The analysis varies the direct selling cost S, the EW selling cost s, and the consumer co-payment rates β_r and β_m. The table reports the parameter values tested, the corresponding theoretical intervals, the resulting numerical equilibrium outcomes, and the associated robustness implications. All tests are conducted within the feasible regions specified in the corresponding propositions and remarks.

Table 1. Sensitivity analysis of key parameters.

Result	Parameter values tested	Covered theoretical interval	Numerical equilibrium outcome	Robustness implication
Proposition 1	$v = 3$, $r = 2$, $c = 0.5$, $\alpha = 0.4$, $\beta_r = 0.1$; $\beta_m = 0.04, 0.09$; $s \in [0, 0.12]$	Low-B_r case, where the comparison between Models NMS and NNS is not globally monotone in s.	For $\beta_m = 0.04$, manufacturer providing EW is profitable for $s < 0.0260$ and $0.1038 < s < 0.1200$; for $\beta_m = 0.09$, the corresponding intervals are $s < 0.0044$ and $0.0209 < s < 0.1200$.	The qualitative conclusion of Proposition 1 remains robust.
Proposition 2	$v = 2.05$, $r = 2$, $c = 0.2$, $\alpha = 0.1$, $\beta_r = 0.75$, $\beta_m = 0.06$; $S = 0.8, 1.05$	The two values of S are selected from the first and second direct selling cost intervals stated in Proposition 2.	In both intervals, Model EMS is preferred only when s is below the relevant threshold; the admissible EW selling cost range is smaller when $S = 1.05$ than when $S = 0.8$.	The qualitative conclusion of Proposition 2 remains robust.
Proposition 3	$v = 3$, $r = 2$, $c = 0.5$, $\alpha = 0.2$; $\beta_r = 0.80, 0.15$; $S \in [0, \Phi]$	$\beta_r = 0.80$ belongs to the high-B_r interval; $\beta_r = 0.15$ belongs to the low-B_r interval with a nonmonotone ENS—NNS comparison.	For $\beta_r = 0.80$, encroachment is profitable for $S \leq 1.3197$. For $\beta_r = 0.15$, the Models ENS and NNS comparison generates two profitable intervals separated by a no encroachment interval.	The qualitative conclusion that retailer EW reshapes the encroachment region remains robust.
Proposition 4	$v = 3$, $r = 2$, $c = 0.5$, $\alpha = 0.2$, $\beta_r = 0.80$, $\beta_m = 0.20$; $s = 0.20, 0.60$	Both values of s satisfy the feasibility condition and cover the intervals stated in Proposition 4.	When $s = 0.20$, Model EMS is preferred for $0.20 < S \leq 1.2131$; when $s = 0.60$, Model EMS is preferred for $0.60 < S \leq 0.8640$.	The qualitative conclusion of Proposition 4 remains robust.
Remark 1	$v = 2.05$, $r = 2$, $c = 0.2$, $\alpha = 0.1$, $\beta_m = 0.01$; $\beta_r = 0.80, 0.95$	Both values satisfy the common comparison region $2A/3 \leq B_r < A$ and the corresponding feasibility condition for B_m.	In both cases, encroachment shrinks the manufacturer's EW provision region in the s-S plane, as shown in Figure **??**.	The qualitative conclusion of Remark 1 remains robust.
Remark 2	$v = 3$, $r = 2$, $c = 0.5$, $\alpha = 0.2$, $\beta_r = 0.80$; $\beta_m = 0.20, 0.60$	Both values satisfy $\sqrt{5}A/5 < B_r < A$ and the upper bound on B_m, and they represent different degrees of EW differentiation.	The area ratio between the encroachment region with the manufacturer-provided EW and that without the manufacturer-provided EW is about 0.574 for $\beta_m = 0.20$ and 0.590 for $\beta_m = 0.60$.	The qualitative conclusion of Remark 2 remains robust.

Overall, the sensitivity analysis confirms that the qualitative conclusions of Propositions 1–4 and Remarks 1–2 remain unchanged over the tested parameter ranges. Variations in the direct selling cost S, the EW selling cost s, and the consumer co-payment rates β_r and β_m shift the equilibrium thresholds and the sizes of the corresponding strategy regions, but do not alter the underlying equilibrium structure or the economic insights.