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Theory article

Pricing and green innovation for national and private brands on the e-retail platform

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Appendix

A. Proof of **Corollary 4.1** and **Corollary 4.2:**

Scenario N Using the equation $\pi_m^N = q_m w_m^N$, through $p_m = w_m + n$, where $n > 0$, we have $w_m^N = d - p_m + \lambda p_r$; then using the equation $\pi_r^N = q_m(p_m^N - w_m^N) + q_r p_r^N + k(q_m + q_r)$, we have Hessian matrix $H_1 = \begin{bmatrix} \frac{\partial^2 \pi_r^N}{\partial p_m \partial p_m} & \frac{\partial^2 \pi_r^N}{\partial p_m \partial p_r} \\ \frac{\partial^2 \pi_r^N}{\partial p_r \partial p_m} & \frac{\partial^2 \pi_r^N}{\partial p_r \partial p_r} \end{bmatrix} = \begin{bmatrix} -4 & 4\lambda \\ 4\lambda & -2(1+\lambda^2) \end{bmatrix}$, from the $-4 < 0$ and $|H_1| > 0$ can guarantee that both p_m and p_r are concave, which ensures π_r^N existence of a maximum. Then from $\frac{\partial \pi_r^N}{\partial p_m} = 0$ and $\frac{\partial \pi_r^N}{\partial p_r} = 0$, we have $p_m^{N*} = \frac{k\lambda^2 - k + d(3-\lambda)}{4(1-\lambda)}$, $p_r^{N*} = \frac{d - k(1-\lambda)}{2 - 2\lambda}$. Substituting p_m^{N*} and p_r^{N*} into w_m^N, and we can get $w_m^{N*} = \frac{1}{4}(d + k - k\lambda)$. Finally substituting p_m^{N*}, p_r^{N*} and w_m^{N*} into π_m^N and π_r^N, we can get $\pi_r^{N*} = \frac{(3+\lambda)(d+k-k\lambda)^2}{8(1-\lambda)}$, $\pi_m^{N*} = \frac{1}{16}(d+k-k\lambda)^2$. By differentiating the equilibrium solution with respect to λ, we obtain: $\frac{\partial w_m^{N*}}{\partial \lambda} = -\frac{k}{4} < 0$, $\frac{\partial \pi_m^{N*}}{\partial \lambda} = -\frac{1}{8}k(d+k-k\lambda) < 0$ $\frac{\partial p_r^{N*}}{\partial \lambda} = \frac{d}{2(-1+\lambda)^2} < 0$, $\frac{\partial p_m^{N*}}{\partial \lambda} = -\frac{k}{4} + \frac{d}{2(-1+\lambda)^2} > 0$ for $d > \frac{k(1-\lambda)^2}{2}$,

$\frac{\partial \pi_r^{N^*}}{\partial \lambda} = \frac{(d+k-k\lambda)\left(2d+k\left(-1+\lambda^2\right)\right)}{4(-1+\lambda)^2} > 0$ for $d > \frac{k(1-\lambda)^2}{2}$, then we can get **Corollary 4.1.** By differentiating the equilibrium solution with respect to k, we obtain: $\frac{\partial w_m^{N^*}}{\partial k} = \frac{1-\lambda}{4} > 0$ $\frac{\partial p_m^{N^*}}{\partial k} = \frac{1}{4}(-1-\lambda) < 0$, $\frac{\partial p_r^{N^*}}{\partial k} = -\frac{1}{2} < 0$, $\frac{\partial \pi_m^{N^*}}{\partial k} = \frac{1}{8}(1-\lambda)(d+k-k\lambda) > 0$, $\frac{\partial \pi_r^{N^*}}{\partial k} = \frac{1}{4}(3+\lambda)(d+k-k\lambda) > 0$, then we can get **Corollary 4.2.**

B. Proof of **Corollary 4.3** to **Corollary 4.6:**

Scenario B using the equation $\pi_m^B = q_m w_m^B - \frac{1}{2}\eta\left(\alpha_m^B\right)^2$, through $p_m = w_m + n$, where $n > 0$, we have $w_m^B = \frac{\eta(d - p_m + \lambda p_r - \theta\lambda\alpha_r)}{\eta - \theta^2}$, $\alpha_m^B = \frac{\theta(d - p_m + \lambda p_r - \theta\lambda\alpha_r)}{\eta - \theta^2}$; substituting w_m^B and α_r^B into the equation $\pi_r^B = q_m(p_m^B - w_m^B) + q_r p_r^B - \frac{1}{2}\eta\left(\alpha_r^B\right)^2 + k(q_m + q_r)$ we have Hessian matrix

$$H_2 = \begin{bmatrix} \frac{\partial^2 \pi_r^B}{\partial p_m \partial p_m} & \frac{\partial^2 \pi_r^B}{\partial p_m \partial p_r} & \frac{\partial^2 \pi_r^B}{\partial p_m \partial \alpha_r} \\ \frac{\partial^2 \pi_r^B}{\partial p_r \partial p_m} & \frac{\partial^2 \pi_r^B}{\partial p_r \partial p_r} & \frac{\partial^2 \pi_r^B}{\partial p_r \partial \alpha_r} \\ \frac{\partial^2 \pi_r^B}{\partial \alpha_r \partial p_m} & \frac{\partial^2 \pi_r^B}{\partial \alpha_r \partial p_r} & \frac{\partial^2 \pi_r^B}{\partial \alpha_r \partial \alpha_r} \end{bmatrix} = \begin{bmatrix} \frac{2\eta(-2\eta+\theta^2)}{(\eta-\theta^2)^2} & \frac{2\eta(2\eta-\theta^2)\lambda}{(\eta-\theta^2)^2} & \frac{\eta\theta(-3\eta+\theta^2)\lambda}{(\eta-\theta^2)^2} \\ \frac{\eta\theta(-3\eta+\theta^2)\lambda}{(\eta-\theta^2)^2} & -2 - \frac{2(\eta^2+\eta\theta^2-\theta^4)\lambda^2}{(\eta-\theta^2)^2} & \theta - \frac{\theta(-2\eta+\theta^2)(\eta+\theta^2)\lambda^2}{(\eta-\theta^2)^2} \\ \frac{\eta\theta(-3\eta+\theta^2)\lambda}{(\eta-\theta^2)^2} & \theta - \frac{\theta(-2\eta+\theta^2)(\eta+\theta^2)\lambda^2}{(\eta-\theta^2)^2} & \eta\left(-1 - \frac{2\eta\theta^2\lambda^2}{(\eta-\theta^2)^2}\right) \end{bmatrix},$$

the $\frac{2\eta(-2\eta+\theta^2)}{(\eta-\theta^2)^2} < 0$, $\begin{vmatrix} \frac{2\eta(-2\eta+\theta^2)}{(\eta-\theta^2)^2} & \frac{2\eta(2\eta-\theta^2)\lambda}{(\eta-\theta^2)^2} \\ \frac{\eta\theta(-3\eta+\theta^2)\lambda}{(\eta-\theta^2)^2} & -2 - \frac{2(\eta^2+\eta\theta^2-\theta^4)\lambda^2}{(\eta-\theta^2)^2} \end{vmatrix} > 0$ and $|H_2| < 0$ for

$\left(4\eta > 3\theta^2 and 0 < \lambda \le \frac{1}{2}\right) \| \left(2\eta > \theta^2(1+\lambda) and \frac{1}{2} < \lambda < 1\right)$ can guarantee that both p_m, p_r and α_r are concave to ensures π_r^N existence of a maximum and all parameters maintain practical feasibility. Then from $\frac{\partial \pi_r^B}{\partial p_m} = 0$, $\frac{\partial \pi_r^B}{\partial p_r} = 0$ and $\frac{\partial \pi_r^B}{\partial \alpha_r} = 0$, we have

$$p_m^{B^*} = \frac{k(1-\lambda)\left(3\eta\theta^2 - 2\eta^2(1+\lambda) + \theta^4\left(-1+\lambda^2\right)\right) + d\left(2\eta^2(3-\lambda) - \eta\theta^2\left(5-2\lambda^2\right) + \theta^4\left(1-\lambda^2\right)\right)}{2(1-\lambda)\left(\left(2\eta-\theta^2\right)^2 + \theta^2(\eta-\theta^2)\lambda^2\right)},$$

$$p_r^{B^*} = \frac{d\eta\left(4\eta - \theta^2(2+\lambda)\right) - k(1-\lambda)\left(4\eta^2 - 2\theta^4\left(-1+\lambda^2\right) + \eta\theta^2\left(-6+\lambda+2\lambda^2\right)\right)}{2(1-\lambda)\left(\left(-2\eta+\theta^2\right)^2 + \theta^2(\eta-\theta^2)\lambda^2\right)},$$

$a_r^{B^*} = \frac{\theta(d+k-k\lambda)\left(\eta(2+\lambda) - \theta^2(1+\lambda)\right)}{\left(2\eta-\theta^2\right)^2 + \theta^2(\eta-\theta^2)\lambda^2}$. Substituting $p_m^{B^*}$, $p_r^{B^*}$ and $\alpha_r^{B^*}$ into w_m^B and α_m^B, we can get $w_m^{B^*} = \frac{\eta(d+k-k\lambda)\left(2\eta-\theta^2(1+\lambda)\right)}{2\left(\left(2\eta-\theta^2\right)^2 + \theta^2(\eta-\theta^2)\lambda^2\right)}$, $\alpha_m^{B^*} = \frac{\theta(d+k-k\lambda)\left(2\eta-\theta^2(1+\lambda)\right)}{2\left(\left(2\eta-\theta^2\right)^2 + \theta^2(\eta-\theta^2)\lambda^2\right)}$. Finally substituting $p_m^{B^*}$, $p_r^{B^*}$, $\alpha_r^{B^*}$, $w_m^{B^*}$ and $\alpha_m^{B^*}$ into π_m^B and π_r^B, we can get

$\pi_m^{B*}=\frac{\eta\left(2\eta-\theta^2\right)\left(d+k-k\lambda\right)^2\left(2\eta-\theta^2\left(1+\lambda\right)\right)^2}{8\left(\left(2\eta-\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}$, $\pi_r^{B*}=\frac{\eta\left(d+k-k\lambda\right)^2\left(2\eta\left(3+\lambda\right)-3\theta^2\left(1+\lambda\right)\right)}{4\left(1-\lambda\right)\left(\left(2\eta-\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)}$.

By differentiating the equilibrium solution with respect to λ, we obtain **Corollary 4.3**: $\frac{\partial w_m^{B*}}{\partial\lambda}=\frac{\eta\left(-2k\eta\left(4\eta^2+\theta^4\left(1+\lambda+\lambda^2\right)-\eta\theta^2\left(4+\lambda\left(2+\lambda\right)\right)\right)-d\theta^2\left(4\eta^2\left(1+\lambda\right)+\theta^4\left(1+\lambda\right)^2-\eta\theta^2\left(4+\lambda\left(6+\lambda\right)\right)\right)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}<0$,

$\frac{\partial\alpha_m^{B*}}{\partial\lambda}=-\frac{d\left(1+k\right)\theta^3\left(4\eta^2\left(1+\lambda\right)+\theta^4\left(1+\lambda\right)^2-\eta\theta^2\left(4+\lambda\left(6+\lambda\right)\right)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}<0$,

$\frac{\partial p_r^{B*}}{\partial\lambda}=\frac{\eta(-k\theta^2\left(-1+\lambda\right)^2\left(\eta^2\left(4+8\lambda\right)+\theta^4\left(1+\lambda\left(4+\lambda\right)\right)-\eta\theta^2\left(4+\lambda\left(12+\lambda\right)\right)\right)+d\left(16\eta^3+\theta^6\left(-1+\lambda\right)\left(3+\lambda\right)\left(1+2\lambda\right)+4\eta^2\theta^2\left(-7+\lambda\left(-2+3\lambda\right)\right)+\eta\theta^4\left(16+\lambda\left(12-\lambda\left(17+2\lambda\right)\right)\right)\right))}{2\left(-1+\lambda\right)^2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$,

$\frac{\partial\pi_m^{B*}}{\partial\lambda}=\frac{\eta\left(d+k-k\lambda\right)(k\eta\left(-1+\lambda\right)\left(4\eta-\theta^2\left(2+\lambda\right)\right)\left(2\eta\left(1+\lambda\right)-\theta^2\left(1+2\lambda\right)\right)+d\left(16\eta^3+3\theta^6\left(-1+\lambda\right)\left(1+\lambda\right)^2+2\eta^2\theta^2\left(2+\lambda\right)\left(-7+\lambda\left(2+\lambda\right)\right)+\eta\theta^4\left(16+\lambda\left(9-\lambda\left(11+5\lambda\right)\right)\right)\right))}{2\left(-1+\lambda\right)^2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}<0$;

$\frac{\partial p_m^{B*}}{\partial\lambda}=\frac{-2k\eta^2\left(-1+\lambda\right)^2\left(4\eta^2+\theta^4\left(1+\lambda+\lambda^2\right)-\eta\theta^2\left(4+\lambda\left(2+\lambda\right)\right)\right)++d\left(16\eta^4+\theta^8\left(-1+\lambda^2\right)^2+4\eta^3\theta^2\left(-9+\lambda+3\lambda^2-\lambda^3\right)-3\eta\theta^6\left(3-4\lambda^2+\lambda^4\right)+\eta^2\theta^4\left(28+\lambda\left(-2+\lambda\left(-23+2\lambda\left(2+\lambda\right)\right)\right)\right)\right)}{2\left(-1+\lambda\right)^2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$ f

or $\eta_0<\eta<\theta^2$ and $\lambda_1<\lambda<1$ or $\theta^2\le\eta$ when $d_1<d$;

$\frac{\partial\alpha_r^{B*}}{\partial\lambda}=\frac{d\left(1+k\right)\left(\eta-\theta^2\right)\left(4\eta^2\theta+\theta^5\left(1+\lambda\right)^2-\eta\theta^3\left(2+\lambda\right)^2\right)}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}<0$ for $\eta_0<\eta<\theta^2$ and $0<\lambda\le2-\sqrt{3}$,

or $\eta_1<\eta<\theta^2$ and $2-\sqrt{3}<\lambda\le\frac{1}{2}$, or $\theta^2<\eta<\eta_1$ and $\frac{1}{2}<\lambda<1$;

$\frac{\partial\pi_r^{B*}}{\partial\lambda}=\frac{\eta\left(d+k-k\lambda\right)(k\eta\left(-1+\lambda\right)\left(4\eta-\theta^2\left(2+\lambda\right)\right)\left(2\eta\left(1+\lambda\right)-\theta^2\left(1+2\lambda\right)\right)+d\left(16\eta^3+3\theta^6\left(-1+\lambda\right)\left(1+\lambda\right)^2+2\eta^2\theta^2\left(2+\lambda\right)\left(-7+\lambda\left(2+\lambda\right)\right)+\eta\theta^4\left(16+\lambda\left(9-\lambda\left(11+5\lambda\right)\right)\right)\right))}{2\left(-1+\lambda\right)^2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$ for

$d_2<d$.

By differentiating the equilibrium solution with respect to θ, we obtain **Corollary 4.4**: $\frac{\partial p_r^{B*}}{\partial\theta}=\frac{\eta\theta\left(d+k-k\lambda\right)\left(-8\eta\theta^2\left(1+\lambda\right)+4\eta^2\left(2+\lambda\right)+\theta^4\left(1+\lambda\right)\left(2+\lambda\right)\right)}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$;

$\frac{\partial\alpha_r^{B*}}{\partial\theta}=\frac{d\left(1+k\right)\left(-\theta^6\left(-1+\lambda\right)\left(1+\lambda\right)^2+4\eta^3\left(2+\lambda\right)+\eta\theta^4\left(1+\lambda\right)\left(2+\lambda\right)\left(-1+2\lambda\right)-\eta^2\theta^2\left(4+\lambda\left(8+\lambda\left(2+\lambda\right)\right)\right)\right)}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$,

$\frac{\partial\pi_r^{B*}}{\partial\theta}=\frac{\eta\theta\left(d+k-k\lambda\right)^2\left(3\theta^4\left(1+\lambda\right)^2-4\eta\theta^2\left(1+\lambda\right)\left(3+\lambda\right)+2\eta^2\left(6+\lambda\left(4+\lambda\right)\right)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$;

$\frac{\partial w_m^{B*}}{\partial\theta}=\frac{\partial p_m^{B*}}{\partial\theta}=-\frac{\eta\theta\left(d+k-k\lambda\right)\left(\theta^4\left(-1+\lambda\right)\left(1+\lambda\right)^2-4\eta\theta^2\left(-1+\lambda^2\right)+2\eta^2\left(-2+\lambda\left(2+\lambda\right)\right)\right)}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$ for $0<\lambda<0.227$,

or

$0.227<\lambda<\sqrt{3}-1$ and $\eta_2<\eta$;

$\frac{\partial\alpha_m^{B*}}{\partial\theta}=\frac{d\left(1+k\right)\left(8\eta^3-\theta^6\left(-1+\lambda\right)\left(1+\lambda\right)^2-\eta\theta^4\left(1+\lambda\right)\left(2+\left(-6+\lambda\right)\lambda\right)-2\eta^2\theta^2\left(2+\lambda\left(6+\lambda\right)\right)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}>0$ for

$0<\lambda\le0.255$, or $0.255<\lambda<1$ and $\eta_3<\eta$; then

$\frac{\partial \pi_m^{B^*}}{\partial \theta} = -\frac{\eta(d+k-k\lambda)^2\left(2\eta-\theta^2(1+\lambda)\right)\left(-\theta^7(-1+\lambda)(1+\lambda)^2+8\eta^3\theta\left(-1+\lambda(2+\lambda)\right)+3\eta\theta^5\left(-2+\lambda^2(3+\lambda)\right)-6\eta^2\theta^3\left(-2+\lambda(2+3\lambda)\right)\right)}{4\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^3} > 0$ for

$0<\lambda\le 0.141$, or $0.141<\lambda<\sqrt{2}-1$ and $\eta_4<\eta$.

By differentiating the equilibrium solution with respect to η, we obtain **Corollary 4.5**: $\frac{\partial p_r^{B^*}}{\partial \eta} = \frac{\partial \alpha_r^{B^*}}{\partial \eta} = -\frac{\theta^2(d+k-k\lambda)\left(-8\eta\theta^2(1+\lambda)+4\eta^2(2+\lambda)+\theta^4(1+\lambda)(2+\lambda)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2} < 0$,

$\frac{\partial \pi_r^{B^*}}{\partial \eta} = -\frac{\theta^2(d+k-k\lambda)^2\left(3\theta^4(1+\lambda)^2-4\eta\theta^2(1+\lambda)(3+\lambda)+2\eta^2\left(6+\lambda(4+\lambda)\right)\right)}{4\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2} < 0$;

$\frac{\partial w_m^{B^*}}{\partial \eta} = \frac{\partial p_m^{B^*}}{\partial \eta} = \frac{\theta^2(d+k-k\lambda)\left(\theta^4(-1+\lambda)(1+\lambda)^2-4\eta\theta^2\left(-1+\lambda^2\right)+2\eta^2\left(-2+\lambda(2+\lambda)\right)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2} < 0$ for

$0<\lambda\le 0.227$, or $0.227<\lambda<\sqrt{3}-1$ and $\eta_1<\eta$;

$\frac{\partial \alpha_m^{B^*}}{\partial \eta} = \frac{d(1+k)\theta\left(-8\eta^2+8\eta\theta^2(1+\lambda)+\theta^4(1+\lambda)\left(-2+(-2+\lambda)\lambda\right)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2} < 0$ for $\eta_0<\eta\le\frac{3\theta^2}{2}$ and

$0<\lambda<\lambda_2$, or $\frac{3\theta^2}{2}<\eta$, where λ_2 is the smallest real root of the cubic equation in x given by $-8\eta^2+8\eta\theta^2-2\theta^4+\left(8\eta\theta^2-4\theta^4\right)x-\theta^4x^2+\theta^4x^3=0$;

$\frac{\partial \pi_m^{B^*}}{\partial \eta} = -\frac{(d+k-k\lambda)^2\left(-2\eta+\theta^2(1+\lambda)\right)\left(-\theta^8(-1+\lambda)(1+\lambda)^2+8\eta^3\theta^2\left(-1+\lambda(2+\lambda)\right)+3\eta\theta^6\left(-2+\lambda^2(3+\lambda)\right)-6\eta^2\theta^4\left(-2+\lambda(2+3\lambda)\right)\right)}{8\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^3} < 0$ for

$0<\lambda\le 0.141$, or $0.141<\lambda<\sqrt{2}-1$ and $\eta_2<\eta$.

By differentiating the equilibrium solution with respect to k, we obtain **Corollary 4.6**:

$\frac{\partial w_m^{B^*}}{\partial k} = \frac{\eta(1-\lambda)\left(2\eta-\theta^2(1+\lambda)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)} > 0$, $\frac{\partial \pi_r^{B^*}}{\partial k} = \frac{\eta(d+k-k\lambda)\left(-3\theta^2(1+\lambda)+2\eta(3+\lambda)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)} > 0$,

$\frac{\partial \pi_m^{B^*}}{\partial k} = \frac{\eta\left(2\eta-\theta^2\right)(1-\lambda)(d+k-k\lambda)\left(-2\eta+\theta^2(1+\lambda)\right)^2}{4\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2} > 0$,

$\frac{\partial \alpha_m^{B^*}}{\partial k} = \frac{d\theta\left(2\eta-\theta^2(1+\lambda)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)} > 0$, $\frac{\partial \alpha_r^{B^*}}{\partial k} = \frac{\theta(1-\lambda)\left(-\theta^2(1+\lambda)+\eta(2+\lambda)\right)}{\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2} > 0$;

$\frac{\partial p_m^{B^*}}{\partial k} = \frac{3\eta\theta^2-2\eta^2(1+\lambda)+\theta^4\left(-1+\lambda^2\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)} > 0$ for $0<\lambda<\frac{1}{8}$ and

$\eta_0<\eta<\frac{1}{4}\left(\frac{3\theta^2}{1+\lambda}+\sqrt{\frac{\theta^4\left(1+8\lambda\left(-1+\lambda+\lambda^2\right)\right)}{(1+\lambda)^2}}\right)$; $\frac{\partial p_r^{B^*}}{\partial k} = \frac{\eta\left(4\eta-\theta^2(2+\lambda)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)} - 1 > 0$ for $0<\lambda<\frac{1}{2}$ and

$\eta_0<\eta<\frac{1}{8}\left(-\theta^2\left(-6+\lambda+2\lambda^2\right)+\theta^2\sqrt{4+\lambda\left(-12+\lambda\left(9+4\lambda(1+\lambda)\right)\right)}\right)$.

Follow a similar logic in proving **Corollary 4.3** to **Corollary 4.6**, we can prove **Corollary 4.7** to **Corollary 4.14**. Hence, we omit the tedious details of the corresponding Corollarys' proves.

C. **Proof** of **Proposition 5.1:**

By comparison, we have

$$w_m^{M^*}-w_m^{B^*}=\frac{1}{2}\eta\left(d+k-k\lambda\right)\left(\frac{1}{2\eta-\theta^2}+\frac{-2\eta+\theta^2\left(1+\lambda\right)}{\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2}\right)>0,$$

$$w_m^{M^*}-w_m^{N^*}=\frac{\theta^2\left(d+k-k\lambda\right)}{8\eta-4\theta^2}>0,\ w_m^{N^*}-w_m^{E^*}=\frac{\theta^2\lambda\left(2+\lambda\right)\left(d+k-k\lambda\right)}{4\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0,\text{ and}$$

$$w_m^{B^*}-w_m^{N^*}=\frac{\theta^2\left(d+k-k\lambda\right)\left(\theta^2\left(-1+\lambda^2\right)+\eta\left(2-\lambda\left(2+\lambda\right)\right)\right)}{4\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)}>0\text{ if }0<\lambda\le 3-\sqrt{7}\text{ or}$$

$3-\sqrt{7}<\lambda\le\sqrt{3}-1$ and $\frac{\theta^2\left(1-\lambda^2\right)}{2-\lambda\left(2+\lambda\right)}<\eta$; by comparison, we have

$$p_m^{M^*}-p_m^{B^*}=\frac{\eta\theta^2\lambda\left(d+k-k\lambda\right)\left(-\theta^2\left(1+\lambda\right)+\eta\left(2+\lambda\right)\right)}{2\left(\left(2\eta-\theta^2\right)^3+\theta^2\left(2\eta^2-3\eta\theta^2+\theta^4\right)\lambda^2\right)}>0,\ p_m^{M^*}-p_m^{N^*}=\frac{\theta^2\left(d+k-k\lambda\right)}{8\eta-4\theta^2}>0,$$

$$p_m^{N^*}-p_m^{E^*}=\frac{\theta^2\lambda\left(2+\lambda\right)\left(d+k-k\lambda\right)}{4\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0,$$

$$p_m^{B^*}-p_m^{N^*}=\frac{\theta^2\left(d+k-k\lambda\right)\left(\theta^2\left(-1+\lambda^2\right)+\eta\left(2-\lambda\left(2+\lambda\right)\right)\right)}{4\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)}>0\text{ for }0<\lambda\le 3-\sqrt{7}\text{ or}$$

$3-\sqrt{7}<\lambda\le\sqrt{3}-1$ and $\frac{\theta^2\left(1-\lambda^2\right)}{2-\lambda\left(2+\lambda\right)}<\eta$;

$$d_m^{M^*}-d_m^{B^*}=\frac{1}{2}\eta\left(d+k-k\lambda\right)\left(\frac{1}{2\eta-\theta^2}+\frac{-2\eta+\theta^2\left(1+\lambda\right)}{\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2}\right)>0,$$

$$d_m^{M^*}-d_m^{N^*}=\frac{\theta^2\left(d+k-k\lambda\right)}{8\eta-4\theta^2}>0,\ d_m^{N^*}-d_m^{E^*}=\frac{\theta^2\lambda\left(2+\lambda\right)\left(d+k-k\lambda\right)}{4\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0,$$

$$d_m^{B^*}-d_m^{N^*}=\frac{\theta^2\left(d+k-k\lambda\right)\left(\theta^2\left(-1+\lambda^2\right)+\eta\left(2-\lambda\left(2+\lambda\right)\right)\right)}{4\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)}>0\text{ for }0<\lambda\le 3-\sqrt{7}\text{ or}$$

$3-\sqrt{7}<\lambda\le\sqrt{3}-1$ and $\frac{\theta^2\left(1-\lambda^2\right)}{2-\lambda\left(2+\lambda\right)}<\eta$.

D. **Proof** of **Proposition 5.2:**

By comparison, we have

$$p_r^{N^*}=p_r^{M^*},\ p_r^{E^*}-p_r^{B^*}=\frac{\theta^4\lambda\left(d+k-k\lambda\right)\left(2\eta-\theta^2\left(1+\lambda\right)\right)}{2\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0,$$

$$p_r^{B^*}-p_r^{N^*}=\frac{\theta^2\left(d+k-k\lambda\right)\left(-\theta^2\left(1+\lambda\right)+\eta\left(2+\lambda\right)\right)}{8\eta^2+2\eta\theta^2\left(-4+\lambda^2\right)-2\theta^4\left(-1+\lambda^2\right)}>0,\text{ so }p_r^{E^*}>p_r^{B^*}>p_r^{N^*}=p_r^{M^*}.$$

E. **Proof** of **Proposition 5.3:**

By comparison, we have

$$d_r^{E*}-d_r^{B*}=\frac{\eta\theta^2\lambda(d+k-k\lambda)\left(2\eta-\theta^2(1+\lambda)\right)}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0,$$

$$d_r^{B*}-d_r^{N*}=\frac{1}{4}(d+k-k\lambda)\left(\frac{4\eta\left(-\theta^2(1+\lambda)+\eta(2+\lambda)\right)}{\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2}-2-\lambda\right)>0,$$

$$d_r^{N*}-d_r^{M*}=\frac{\theta^2\lambda(d+k-k\lambda)}{8\eta-4\theta^2}>0, \text{ so } d_r^{E*}>d_r^{B*}>d_r^{N*}>d_r^{M*}.$$

F. **Proof** of **Proposition 5.4:**

By comparison, we have

$$\alpha_r^{E*}-\alpha_r^{B*}=\frac{\theta^3\lambda(d+k-k\lambda)\left(2\eta-\theta^2(1+\lambda)\right)}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0,$$

$$\alpha_m^{M*}-\alpha_m^{B*}=\frac{\theta^3\lambda(d+k-k\lambda)\left(-\theta^2(1+\lambda)+\eta(2+\lambda)\right)}{2\left(\left(2\eta-\theta^2\right)^3+\theta^2\left(2\eta^2-3\eta\theta^2+\theta^4\right)\lambda^2\right)}>0,$$

$$\alpha_r^{B*}-\alpha_m^{M*}=\theta(d+k-k\lambda)\left(\frac{1}{2\left(-2\eta+\theta^2\right)}+\frac{-\theta^2(1+\lambda)+\eta(2+\lambda)}{\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2}\right)>0, \text{ so}$$

$$\alpha_r^{E*}>\alpha_r^{B*}>\alpha_m^{M*}>\alpha_m^{B*}.$$

G. **Proof** of **Proposition 5.5:**

By comparison, we

have $\pi_m^{M*}-\pi_m^{N*}=\frac{\theta^2(d+k-k\lambda)^2}{32\eta-16\theta^2}>0$;

$$\pi_m^{M*}-\pi_m^{B*}=\frac{1}{8}\eta(d+k-k\lambda)^2\left(\frac{1}{2\eta-\theta^2}-\frac{\left(2\eta-\theta^2\right)\left(-2\eta+\theta^2(1+\lambda)\right)^2}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}\right)>0;$$

$$\pi_m^{N*}-\pi_m^{E*}=\frac{1}{16}(d+k-k\lambda)^2\left(1-\frac{4\left(-2\eta+\theta^2(1+\lambda)\right)^2}{\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)^2}\right)>0;$$

$$\pi_m^{B*}-\pi_m^{E*}=\frac{1}{8}(d+k-k\lambda)^2\left(-2\eta+\theta^2(1+\lambda)\right)^2\left(\frac{\eta\left(2\eta-\theta^2\right)}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}-\frac{2}{\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)^2}\right)>0;$$

$$\pi_m^{N*}-\pi_m^{B*}=\frac{1}{16}(d+k-k\lambda)^2\left(1-\frac{2\eta\left(2\eta-\theta^2\right)\left(-2\eta+\theta^2(1+\lambda)\right)^2}{\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}\right)>0 \text{ for } \sqrt{3}-\sqrt{4-\sqrt{3}}<\lambda\le\sqrt{2}-1 \text{ and}$$

$\eta_0<\eta<\eta_7$, or $\sqrt{2}-1<\lambda<1$ where η_7 is the third smallest real root of the cubic equation in x given by $\theta^6-2\theta^6\lambda^2+\theta^6\lambda^4+\left(-6\theta^4+4\theta^4\lambda+12\theta^4\lambda^2-2\theta^4\lambda^4\right)x$
$+\left(12\theta^2-16\theta^2\lambda-20\theta^2\lambda^2+\theta^2\lambda^4\right)x^2+\left(-8+16\lambda+8\lambda^2\right)x^3=0$.

H. **Proof** of **Proposition 5.6:**

By comparison, we

have $\pi_r^{B*}-\pi_r^{E*}=\dfrac{(d+k-k\lambda)^2\left(-2\eta\theta+\theta^3(1+\lambda)\right)^2}{4\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0$,

$\pi_r^{E*}-\pi_r^{M*}=\dfrac{(d+k-k\lambda)^2\left(\eta\theta^2\left(2+\lambda(4+\lambda)\right)-\theta^4(1+\lambda)^2\right)}{4\left(2\eta-\theta^2\right)\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)}>0$, $\pi_r^{M*}-\pi_r^{N*}=\dfrac{\theta^2(d+k-k\lambda)^2}{16\eta-8\theta^2}>0$,

so $\pi_r^{B*}>\pi_r^{E*}>\pi_r^{M*}>\pi_r^{N*}$.

I. **Proof** of **Proposition 5.7:**

By comparison, we

have $\pi_s^{N*}=\pi_m^{N*}+\pi_r^{N*}=\dfrac{(7+\lambda)(d+k-k\lambda)^2}{16(1-\lambda)}$,

$\pi_s^{B*}=\pi_m^{B*}+\pi_r^{B*}=\dfrac{\eta(d+k-k\lambda)^2\left(7\theta^6(-1+\lambda)(1+\lambda)^2+8\eta^3(7+\lambda)-6\eta\theta^4(1+\lambda)(-7+2\lambda(1+\lambda))+4\eta^2\theta^2\left(-21+\lambda(-9+\lambda(5+\lambda))\right)\right)}{8(1-\lambda)\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2}$,

$\pi_s^{M*}=\pi_m^{M*}+\pi_r^{M*}=\dfrac{(d+k-k\lambda)^2\left(-2\theta^2(1+\lambda)+\eta(7+\lambda)\right)}{8\left(2\eta-\theta^2\right)(1-\lambda)}$,

$\pi_s^{E*}=\pi_m^{E*}+\pi_r^{E*}=\dfrac{(d+k-k\lambda)^2\left(4\eta^2(7+\lambda)-\theta^4(1+\lambda)\left(-3+2\lambda^2\right)+2\eta\theta^2\left(-10+\lambda\left(-4+\lambda(5+\lambda)\right)\right)\right)}{4(1-\lambda)\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)^2}$,

$\pi_s^{B*}-\pi_s^{E*}=\dfrac{(d+k-k\lambda)^2\left(6\left(2\eta-\theta^2\right)^3+2\theta^2\left(-2\eta+\theta^2\right)\left(-8\eta+5\theta^2\right)\lambda^2+\theta^4\left(5\eta-4\theta^2\right)\lambda^4\right)\left(-2\eta\theta+\theta^3(1+\lambda)\right)^2}{8\left(\left(-2\eta+\theta^2\right)^2+\theta^2\left(\eta-\theta^2\right)\lambda^2\right)^2\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)^2}>0$,

$\pi_s^{E*}-\pi_s^{M*}=\dfrac{(d+k-k\lambda)^2\left(-2\theta^6(-1+\lambda)(1+\lambda)^3+8\eta^2\theta^2(1+2\lambda)+\eta\theta^4\left(-8-16\lambda+8\lambda^3+\lambda^4\right)\right)}{8\left(2\eta-\theta^2\right)\left(4\eta+\theta^2\left(-2+\lambda^2\right)\right)^2}>0$,

$\pi_s^{M*}-\pi_s^{N*}=\dfrac{3\theta^2(d+k-k\lambda)^2}{32\eta-16\theta^2}>0$, so $\pi_s^{B*}>\pi_s^{E*}>\pi_s^{M*}>\pi_s^{N*}$.

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