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Research article

A Stackelberg reinsurance- investment game on asset- liability management under CEV model

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Appendix A. Proof of Theorem 2.4

Proof. Under the conditions stated in Remark 2, all Itô integrals in the subsequent derivation are true martingales, and the supremum in the Hamilton–Jacobi–Bellman (HJB) equations are attainable. The verification argument is thus valid. The two parts of insurer and reinsurer in the theorem are similar, so we only prove the first part. Now, we assume that $D_1(t, \hat{x}_1, l_1, s)$ and $f_1(t, \hat{x}_1, l_1, s)$ satisfy the conditions in Theorem 2.4, and the optimal investment- reinsurance strategy in Eq (2.9) is obtained at u_1^*. We use two steps to finish the proof as follows:

(1) First, we prove that

$$D_1(t, \hat{x}_1, l_1, s) = V_1(t, \hat{x}_1, l_1, s), f_1(t, \hat{x}_1, l_1, s) = \mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^*,u_2}(T)].$$

According to Dynkin's formula and the conditions in (2.10), we have

$$\begin{aligned}
&\mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1(t, \hat{X}_1^{u_1^*,u_2}(T), l_1(T), s(T))] \\
&= f_1(t, \hat{x}_1, l_1, s) + \mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^T \mathcal{L}_1^{u_1^*,u_2} f_1(\nu, \hat{X}_1^{u_1^*}(\nu), l_1(\nu), s(\nu))d\nu] \\
&= f_1(t, \hat{x}_1, l_1, s).
\end{aligned} \tag{A1}$$

Moreover, considering $f_1(T, \hat{x}_1, l_1, s) = \hat{x}_1$ in the theorem, we obtain

$$\begin{aligned}
f_1(T, \hat{x}_1, l_1, s) &= \mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1(t, \hat{X}_1^{u_1^*,u_2}(T), l_1(T), s(T))] \\
&= \mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^*,u_2}(T)].
\end{aligned} \tag{A2}$$

Because the optimal strategy is the solution the optimal problem in the left-hand side of (2.9), combining with (2.10), we have

$$\mathcal{L}_1^{u_1^*,u_2} D_1(t,\hat{x}_1,l_1,s) - \mathcal{L}_1^{u_1^*,u_2}\frac{m_1}{2} f_1^2(t,\hat{x}_1,l_1,s) = 0. \tag{A3}$$

By the condition $D_1(T,\hat{x}_1,l_1,s) = \hat{x}_1$, and using Dynkin's formula again, we have

$$\begin{aligned}\mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^*,u_2}(T)] &= \mathbb{E}_{t,\hat{x}_1,l_1,s}[D_1(T,\hat{X}_1^{u_1^*,u_2}(T),l_1(T),s(T))] \\ &= D_1(t,\hat{x}_1,l_1,s) + \mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^T \mathcal{L}_1^{u_1^*,u_2} D_1(\nu,\hat{X}_1^{u_1^*}(\nu),l_1(\nu),s(\nu))d\nu].\end{aligned} \tag{A4}$$

Now, plugging (A3) into (A4), we can obtain

$$D_1(t,\hat{x}_1,l_1,s) = \mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^*,u_2}(T)] - \frac{m_1}{2}\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^T \mathcal{L}_1^{u_1^*,u_2} f_1^2(\nu,\hat{X}_1^{u_1^*,u_2}(\nu),l_1(\nu),s(\nu))d\nu]. \tag{A5}$$

Moreover, by the terminal condition, the Dynkin formula, and (A2), we can yield

$$\begin{aligned}\mathbb{E}_{t,\hat{x}_1,l_1,s}[(\hat{X}_1^{u_1^*,u_2}(T))^2] &= \mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1^2(T,\hat{X}_1^{u_1^*,u_2}(T),l_1(T),s(T))] \\ &= f_1^2(t,\hat{x}_1,l_1,s) + \mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^T \mathcal{L}_1^{u_1^*,u_2} f_1^2(\nu,\hat{X}_1^{u_1^*}(\nu),l_1(\nu),s(\nu))d\nu] \\ &= (\mathbb{E}_{t,\hat{x}_1,l_1,s}[(\hat{X}_1^{u_1^*,u_2}(T))])^2 + \mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^T \mathcal{L}_1^{u_1^*,u_2} f_1^2(\nu,\hat{X}_1^{u_1^*}(\nu),l_1(\nu),s(\nu))d\nu].\end{aligned} \tag{A6}$$

According to the equivalence, we know that

$$Var_{t,\hat{x}_1,l_1,s}[(\hat{X}_1^{u_1^*,u_2}(T)] = \mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^T \mathcal{L}_1^{u_1^*,u_2} f_1^2(\nu,\hat{X}_1^{u_1^*}(\nu),l_1(\nu),s(\nu))d\nu]. \tag{A7}$$

Finally, putting (A7) into (A5), we arrive at

$$\begin{aligned}D_1(t,\hat{x}_1,l_1,s) &= \mathbb{E}_{t,\hat{x}_1,l_1,s}[(\hat{X}_1^{u_1^*,u_2}(T)] - \frac{m_1}{2} Var_{t,\hat{x}_1,l_1,s}[(\hat{X}_1^{u_1^*,u_2}(T)] \\ &= \mathcal{J}_1^{u_1^*,u_2}(t,\hat{x}_1,l_1,s) = V_1(t,\hat{x}_1,l_1,s).\end{aligned} \tag{A8}$$

(2) Second, we prove that u_1^* is a reinsurance- investment equilibrium strategy, which is defined in Definition 2.2. However, $\mathcal{J}_1^{u_1,u_2}(t,\hat{x}_1,l_1,s) = \mathbb{E}_{t,\hat{x}_1,l_1,s}[(\hat{X}_1^{u_1,u_2}(T))] - \frac{m_1}{2} Var_{t,\hat{x}_1,l_1,s}[(\hat{X}_1^{u_1,u_2}(T))]$ and with the strategy u_1^ϵ in Definition 2.2. We can obtain

$$\begin{aligned}&\mathcal{J}_1^{u_1^\epsilon,u_2}(t,\hat{x}_1,l_1,s) \\ &= \mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^\epsilon,u_2}(T)] - \frac{m_1}{2} Var_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^\epsilon,u_2}(T)] \\ &= \mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^\epsilon,u_2}(T) - \frac{m_1}{2}(X_1^{u_1^\epsilon,u_2}(T))^2] + \frac{m_1}{2}(\mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^\epsilon,u_2}(T)])^2 \\ &= \mathbb{E}_{t,\hat{x}_1,l_1,s}\Big[\mathbb{E}_{t+\epsilon,\hat{x}_1^{\hat{u}_1,u_2}(t+\epsilon),l_1(t+\epsilon),s(t+\epsilon)}[\hat{X}_1^{u_1^*,u_2}(T) - \frac{m_1}{2}(X_1^{u_1^*,u_2}(T))^2]\Big] \\ &+ \frac{m_1}{2}(\mathbb{E}_{t,\hat{x}_1,l_1,s}\Big[\mathbb{E}_{t+\epsilon,\hat{x}_1^{\hat{u}_1,u_2}(t+\epsilon),l_1(t+\epsilon),s(t+\epsilon)}[\hat{X}_1^{u_1^*,u_2}(T)]\Big])^2\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{t,\hat{x}_1,l_1,s}[\mathcal{J}_1^{u_1^*,u_2}(t+\epsilon, \hat{x}_1^{\hat{u}_1,u_2^*}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))] \\
&- \frac{m_1}{2}\mathbb{E}_{t,\hat{x}_1,l_1,s}\left[(\mathbb{E}_{t+\epsilon,\hat{x}_1^{\hat{u}_1,u_2^*}(t+\epsilon),l_1(t+\epsilon),s(t+\epsilon)}[\hat{X}_1^{u_1^*,u_2}(T)])^2\right] \\
&+ \frac{m_1}{2}\left(\mathbb{E}_{t,\hat{x}_1,l_1,s}\left[\mathbb{E}_{t+\epsilon,\hat{x}_1^{\hat{u}_1,u_2^*}(t+\epsilon),l_1(t+\epsilon),s(t+\epsilon)}[\hat{X}_1^{u_1^*,u_2}(T)]\right]\right)^2 \\
&= \mathbb{E}_{t,\hat{x}_1,l_1,s}[\mathcal{J}_1^{u_1^*,u_2}(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))] \\
&- \frac{m_1}{2}\mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1^2(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))] \\
&+ \frac{m_1}{2}(\mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))])^2. \qquad \text{(A9)}
\end{aligned}
$$

Then, we substitute extra terms $f^2(t, \hat{x}_1, l_1, s)$ and $D(t, \hat{x}_1, l_1, s)$ in Eq (A9) and apply the result in (1), to get

$$
\begin{aligned}
&\mathcal{J}_1^{u_1^\epsilon,u_2}(t, \hat{x}_1, l_1, s) \\
&= \mathbb{E}_{t,\hat{x}_1,l_1,s}[D_1^{u_1^*,u_2}(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))] \\
&- \frac{m_1}{2}[\mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1^2(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))] - f^2(t, \hat{x}_1, l_1, s)] \\
&+ \frac{m_1}{2}[E_{t,\hat{x}_1,l_1,s}[f_1(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))])^2 - f^2(t, \hat{x}_1, l_1, s)]. \qquad \text{(A10)}
\end{aligned}
$$

For any $u_1 \in \mathcal{U}$, enough small $\epsilon > 0$, and $D_1(t, \hat{x}_1, l_1, s) \in C^{1,2,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$, we can define an operator as follows:

$$
\mathcal{L}_{1\epsilon}^{u_1,u_2} D_1(t, \hat{x}_1, l_1, s) := \mathbb{E}_{t,\hat{x}_1,l_1,s}[D_1(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))] - D_1(t, \hat{x}_1, l_1, s). \qquad \text{(A11)}
$$

Then, $\forall \epsilon > 0$, so we obtain

$$
\mathcal{L}_1^{\hat{u}_1,u_2} D_1(t, \hat{x}_1, l_1, s) = \lim_{\varepsilon \to 0} \frac{\mathcal{L}_1^{\hat{u}_1,u_2} D_1(t, \hat{x}_1, l_1, s)}{\varepsilon}. \qquad \text{(A12)}
$$

According to (A11), (A10) becomes

$$
\begin{aligned}
&\mathcal{J}_1^{u_1^\epsilon,u_2}(t, \hat{x}_1, l_1, s) = D_1(t, \hat{x}_1, l_1, s) + \mathcal{L}_{1\epsilon}^{\hat{u}_1,u_2} D_1(t, \hat{x}_1, l_1, s) - \frac{m_1}{2}\mathcal{L}_{1\epsilon}^{\hat{u}_1,u_2} f_1^2(t, \hat{x}_1, l_1, s) \\
&+ \frac{m_1}{2}[(\mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1^2(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))])^2 - f^2(t, \hat{x}_1, l_1, s)]. \qquad \text{(A13)}
\end{aligned}
$$

By the same Dynkin formula, we have

$$
\begin{aligned}
&\mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))] \\
&= f(t, \hat{x}_1, l_1, s) + \mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon} \mathcal{L}_1^{\hat{u}_1,u_2} f_1(\nu, \hat{X}_1^{\hat{u}_1,u_2}(\nu), l_1(\nu), s(\nu))d\nu]. \qquad \text{(A14)}
\end{aligned}
$$

and this indicates that

$$
[\mathbb{E}_{t,\hat{x}_1,l_1,s}[f_1(t+\epsilon, \hat{X}_1^{\hat{u}_1,u_2}(t+\epsilon), l_1(t+\epsilon), s(t+\epsilon))]]^2 - f_1^2(t, \hat{x}_1, l_1, s)
$$

$$= 2f(t,\hat{x}_1,l_1,s)\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon}\mathcal{L}_1^{\hat{u}_1,u_2}f_1(\nu,\hat{X}_1^{\hat{u}_1,u_2}(\nu),l_1(\nu),s(\nu))d\nu]$$
$$+ [\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon}\mathcal{L}_1^{\hat{u}_1,u_2}f_1(\nu,\hat{X}_1^{\hat{u}_1,u_2}(\nu),l_1(\nu),s(\nu))d\nu]^2. \tag{A15}$$

Plugging (A15) into (A13), we obtain

$$\begin{aligned}
&\mathcal{J}_1^{u_1^\epsilon,u_2}(t,\hat{x}_1,l_1,s)\\
&= D_1(t,\hat{x}_1,l_1,s) + \mathcal{L}_{1\epsilon}^{\hat{u}_1,u_2}D_1(t,\hat{x}_1,l_1,s) - \frac{m_1}{2}\mathcal{L}_{1\epsilon}^{\hat{u}_1,u_2}f_1^2(t,\hat{x}_1,l_1,s)\\
&+ m_1 f(t,\hat{x}_1,l_1,s)\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon}\mathcal{L}_1^{\hat{u}_1,u_2}f_1(\nu,\hat{X}_1^{\hat{u}_1,u_2}(\nu),l_1(\nu),s(\nu))d\nu]\\
&+ \frac{m_1}{2}[\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon}\mathcal{L}_1^{\hat{u}_1,u_2}f_1(\nu,\hat{X}_1^{\hat{u}_1,u_2}(\nu),l_1(\nu),s(\nu))d\nu]]^2.
\end{aligned} \tag{A16}$$

From the extend HJB equation in (2.9), we have

$$\mathcal{L}_1^{\hat{u}_1,u_2}D_1(t,\hat{x}_1,l_1,s) - \mathcal{L}_1^{\hat{u}_1,u_2}\frac{m_1}{2}f_1^2(t,\hat{x}_1,l_1,s) + m_1 f_1(t,\hat{x}_1,l_1,s)\mathcal{L}_1^{\hat{u}_1,u_2}f_1(t,\hat{x}_1,l_1,s) \le 0. \tag{A17}$$

Together with (A12), we have

$$\begin{aligned}
&\mathcal{L}_{1\epsilon}^{\hat{u}_1,u_2}D_1(t,\hat{x}_1,l_1,s) - \mathcal{L}_{1\epsilon}^{\hat{u}_1,u_2}\frac{m_1}{2}f_1^2(t,\hat{x}_1,l_1,s)\\
&+ m_1 f_1(t,\hat{x}_1,l_1,s)\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon}\mathcal{L}_1^{\hat{u}_1,u_2}f_1(\nu,\hat{X}_1^{\hat{u}_1,u_2}(\nu),l_1(\nu),s(\nu))d\nu)] < 0.
\end{aligned} \tag{A18}$$

Plugging the above equation into (A16), we get

$$\begin{aligned}
&\mathcal{J}_1^{u_1^\epsilon,u_2}(t,\hat{x}_1,l_1,s)\\
&= D_1(t,\hat{x}_1,l_1,s) + \frac{m_1}{2}[\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon}\mathcal{L}_1^{\hat{u}_1,u_2}f_1(\nu,\hat{X}_1^{\hat{u}_1,u_2}(\nu),l_1(\nu),s(\nu))d\nu]]^2 + o(\epsilon)\\
&= \mathcal{J}_1^{u_1^*,u_2}(t,\hat{x}_1,l_1,s) + \frac{m_1}{2}[\mathbb{E}_{t,\hat{x}_1,l_1,s}[\int_t^{t+\epsilon}\mathcal{L}_1^{\hat{u}_1,u_2}f_1(\nu,\hat{X}_1^{\hat{u}_1,u_2}(\nu),l_1(\nu),s(\nu))d\nu]]^2 + o(\epsilon).
\end{aligned} \tag{A19}$$

Now, it is easy to get the result

$$\liminf_{\varepsilon\to 0}\frac{\mathcal{J}_1^{u_1^*,u_2}(t,\hat{x}_1,l_1,s) - \mathcal{J}_1^{u_1^\epsilon,u_2}(t,\hat{x}_1,l_1,s)}{\varepsilon} \ge 0. \tag{A20}$$

Therefore, u_1^* is a reinsurance- investment equilibrium strategy. The proof of the reinsuer's strategy is the same as that of the insurer, so we omit it here.

Appendix B. Proof of Theorem 3.1

Proof. According to the solution steps of the Stackelberg game, the proof of Theorem 3.1 is carried out in three steps as follows:

Step 1: As the leader in the Stackelberg game, the reinsurer first announces its admissible strategy $(q_2(.), \pi_2(.)) \in U_2$.

Step 2: After observing the leader's strategy, the insurer (as the follower) solves its own optimization problem under the mean-variance framework. Using the variational operator (2.7), Eq (2.9) is simplified as follows:

$$\begin{aligned}
&\sup_{u_1 \in U_1} \Bigg\{ \frac{\partial D_1}{\partial t} + \Big[r(\hat{x}_1(t) + l_1(t) - k_1 l_2(t)) + (\eta q_1(t) + \theta - \eta)(a + \lambda' h) + \lambda' h q_1(t) \\
&\quad - k_1(1 - q_2(t))((1+\eta)\lambda' h + a\eta) - \alpha_1 + k_1\alpha_2 + (\alpha - r)(\pi_1(t) - k_1\pi_2(t)) \Big] \frac{\partial D_1}{\partial \hat{x}_1} \\
&\quad + \alpha s \frac{\partial D_1}{\partial s} + \alpha_1 \frac{\partial D_1}{\partial l_1} + \frac{1}{2}(\sigma_{11}^2 s^{2\beta} + \sigma_{12}^2) \left(\frac{\partial^2 D_1}{\partial l_1^2} - m_1 \left(\frac{\partial f_1}{\partial l_1} \right)^2 \right) + \frac{1}{2} \big[(\sigma\pi_1(t) \\
&\quad - \sigma_{11} - k_1\pi_2(t)\sigma + k_1\sigma_{21})^2 s^{2\beta} + b^2(q_1(t) - k_1(1 - q_2(t)))^2 + (\sigma_{12} - k_1\sigma_{22})^2 \big] \\
&\quad \times \left[\frac{\partial^2 D_1}{\partial \hat{x}_1^2} - m_1 \left(\frac{\partial f_1}{\partial \hat{x}_1} \right)^2 \right] + \frac{1}{2}\sigma^2 s^{2\beta+2} \left(\frac{\partial^2 D_1}{\partial s^2} - m_1 \left(\frac{\partial f_1}{\partial s} \right)^2 \right) + \big[\sigma_{11}(\sigma\pi_1(t) \\
&\quad - \sigma_{11} - k_1\pi_2(t)\sigma + k_1\sigma_{21}) s^{2\beta} - \sigma_{12}(\sigma_{12} - k_1\sigma_{22}) \big] \left[\frac{\partial^2 D_1}{\partial \hat{x}_1 \partial l_1} - m_1 \frac{\partial f_1}{\partial \hat{x}_1} \frac{\partial f_1}{\partial l_1} \right] \\
&\quad + \big[\sigma(\sigma\pi_1(t) - \sigma_{11} - k_1\pi_2(t)\sigma + k_1\sigma_{21}) s^{2\beta+1} \big] \times \left[\frac{\partial^2 D_1}{\partial \hat{x}_1 \partial s} - m_1 \frac{\partial f_1}{\partial \hat{x}_1} \frac{\partial f_1}{\partial s} \right] \\
&\quad + \sigma\sigma_{11} s^{2\beta+1} \left(\frac{\partial^2 D_1}{\partial l_1 \partial s} - m_1 \frac{\partial f_1}{\partial l_1} \frac{\partial f_1}{\partial s} \right) \Bigg\} = 0. \qquad \text{(B1)}
\end{aligned}$$

To solve Eq (2.9) and Eq (B1), we conjecture that the solution for the insurer takes the following form:

$$\begin{aligned}
D_1(t, \hat{x_1}, l_1, s) &= A(t)\hat{x}_1 + \frac{B(t)}{m_1} l_1 + \frac{C(t)}{m_1} s^{-2\beta}, A(T) = 1, B(T) = C(T) = 0; \\
f_1(t, \hat{x_1}, l_1, s) &= \tilde{A}(t)\hat{x}_1 + \frac{\tilde{B}(t)}{m_1} l_1 + \frac{\tilde{C}(t)}{m_1} s^{-2\beta}, \tilde{A}(T) = 1, \tilde{B}(T) = \tilde{C}(T) = 0.
\end{aligned}$$

The partial derivatives of the above two functions can be obtained through calculation as follows:

$$\begin{aligned}
&\frac{\partial D_1}{\partial t} = A'(t)\hat{x}_1 + \frac{l_1}{m_1} B'(t) + C'(t)\frac{s^{-2\beta}}{m_1}, \quad \frac{\partial D_1}{\partial \hat{x}_1} = A(t), \frac{\partial D_1}{\partial l_1} = \frac{B(t)}{m_1}, \quad \frac{\partial D_1}{\partial s} = -2\beta \frac{C(t)}{m_1} s^{-2\beta-1}, \\
&\frac{\partial f_1}{\partial t} = \tilde{A}'(t)\hat{x}_1 + \tilde{B}'(t)\frac{l_1}{m_1} + \tilde{C}'(t)\frac{s^{-2\beta}}{m_1}, \quad \frac{\partial f_1}{\partial \hat{x}_1} = \tilde{A}(t), \frac{\partial f_1}{\partial l_1} = \frac{\tilde{B}(t)}{m_1}, \quad \frac{\partial f_1}{\partial s} = -2\beta \frac{\tilde{C}(t)}{m_1} s^{-2\beta-1}.
\end{aligned}$$

Substituting the above-derived expressions into Eq (B1), we have:

$$\begin{aligned}
&\sup_{u_1 \in U_1} \Bigg\{ A'(t)\hat{x}_1 + B'(t)\frac{l_1}{m_1} + C'(t)\frac{s^{-2\beta}}{m_1} + \big[r(\hat{x}_1 + l_1 - k_1 l_2) + (\eta q_1 + \theta \\
&\quad - \eta)(a + \lambda' h) + \lambda' h q_1 - k_1(1 - q_2)((1+\eta)\lambda' h + a\eta) - \alpha_1 + k_1\alpha_2 + (\alpha
\end{aligned}$$

$$
\begin{aligned}
&- r)(\pi_1 - k_1\pi_2)]A(t) + \alpha_1\frac{B(t)}{m_1} - 2\beta\alpha s\frac{C(t)}{m_1}s^{-2\beta-1} + \frac{1}{2}(\sigma_{11}^2 s^{2\beta} + \sigma_{12}^2)\\
&\times\left(-\frac{\tilde{B}(t)^2}{m_1}\right) + \frac{1}{2}[(\sigma\pi_1 - \sigma_{11} - k_1\pi_2\sigma + k_1\sigma_{21})^2 s^{2\beta} + b^2(q_1 - k_1(1-q_2))^2\\
&+ (\sigma_{12} - k_1\sigma_{22})^2](-m_1\tilde{A}(t)^2) + \frac{1}{2}\sigma^2 s^{2\beta+2}(2\beta(2\beta+1)\frac{C(t)}{m_1}s^{-2\beta-2}\\
&- 4\beta^2\frac{\tilde{C}(t)^2}{m_1}s^{-4\beta-2}) + [\sigma_{11}(\sigma\pi_1 - \sigma_{11} - k_1\sigma\pi_2 + k_1\sigma_{21})s^{2\beta} - \sigma_{12}(\sigma_{12}\\
&- k_1\sigma_{22})](-\tilde{A}(t)\tilde{B}(t)) + [\sigma(\sigma\pi_1 - \sigma_{11} - k_1\sigma\pi_2 + k_1\sigma_{21})s^{2\beta+1}]\\
&\times(2\beta\tilde{A}(t)\tilde{C}(t)s^{-2\beta-1}) + \sigma\sigma_{11}s^{2\beta+1}(2\beta\tilde{B}(t)\frac{\tilde{C}(t)}{m_1}s^{-2\beta-1})\Bigg\} = 0. \qquad \text{(B2)}
\end{aligned}
$$

By the first-order condition for maximizing the value in Eq(B2), we derive:

$$
\begin{aligned}
&(a-r)A(t) - \sigma m_1\tilde{A}(t)^2 s^{2\beta}(\sigma\pi_1 - \sigma_{11} - k_1\pi_2\sigma + k_1\sigma_{21})\\
&- \tilde{A}(t)\tilde{B}(t)\sigma\sigma_{11}s^{2\beta} + 2\beta\tilde{A}(t)\tilde{C}(t)\sigma^2 = 0;
\end{aligned}
$$

$$
[\eta(a+\lambda' h) + \lambda' h]A(t) - b^2(q_1 - k_1(1-q_2))m_1\tilde{A}(t)^2 = 0.
$$

By the above two equations, we know that investment strategy and reinsurance strategy are independent. We can have that

$$
\pi_1^* = \frac{\sigma_{11} - k_1\sigma_{21}}{\sigma} - \frac{\tilde{A}(t)\tilde{B}(t)\sigma\sigma_{11}s^{2\beta} - (\alpha - r)A(t) - 2\beta\sigma^2\tilde{A}(t)\tilde{C}(t)}{m_1\sigma^2\tilde{A}(t)^2 s^{2\beta}} + k_1\pi_2.
$$

$$
q_1^* = \frac{[(1+\eta)\lambda' h + a\eta]A(t)}{m_1 b^2\tilde{A}(t)^2} + k_1(1-q_2).
$$

Substitute

$$
\pi_1 - k_1\pi_2 = \frac{\sigma_{11} - k_1\sigma_{21}}{\sigma} - \frac{\tilde{A}(t)\tilde{B}(t)\sigma\sigma_{11}s^{2\beta} - (\alpha - r)A(t) - 2\beta\sigma^2\tilde{A}(t)\tilde{C}(t)}{m_1\sigma^2\tilde{A}(t)^2 s^{2\beta}},
$$

$$
q_1 - k_1(1-q_2) = \frac{[(1+\eta)\lambda' h + a\eta]A(t)}{m_1 b^2\tilde{A}(t)^2},
$$

into Eq (B2) to obtain

$$
\begin{aligned}
&A'(t)\hat{x}_1 + \frac{B'(t)}{m_1}l_1 + \frac{C'(t)}{m_1}s^{-2\beta} + \Bigg[r(\hat{x}_1 + l_1 - k_1 l_2) + \frac{((1+\eta)\lambda' h + a\eta)^2 A(t)}{m_1 b^2\tilde{A}^2(t)}\\
&+ (\theta - \eta)(a + \lambda' h) - \alpha_1 + k_1\alpha_2 + (\alpha - r)\Bigg(\frac{\sigma_{11} - k_1\sigma_{21}}{\sigma} - \frac{\tilde{B}(t)\sigma_{11}}{m_1\sigma\tilde{A}(t)} + \frac{(\alpha - r)A(t)}{m_1\sigma^2\tilde{A}^2(t)}s^{-2\beta}\\
&+ \frac{2\beta\tilde{C}(t)}{m_1\tilde{A}(t)}s^{-2\beta}\Bigg)\Bigg]A(t) - 2\alpha\beta\frac{C(t)}{m_1}s^{-2\beta} + \frac{\alpha_1}{m_1}B(t) - \frac{1}{2}(\sigma_{11}^2 s^{2\beta} + \sigma_{12}^2)\frac{\tilde{B}^2(t)}{m_1}\\
&+ \frac{1}{2}\Bigg[\frac{\tilde{B}^2(t)\sigma_{11}^2}{m_1^2\tilde{A}^2(t)}s^{2\beta} - 2\frac{\tilde{B}(t)\sigma_{11}}{m_1\tilde{A}(t)}\Bigg(\frac{(\alpha - r)A(t)}{m_1\sigma\tilde{A}^2(t)} + \frac{2\beta\sigma\tilde{C}(t)}{m_1\tilde{A}(t)}\Bigg) + \Bigg(\frac{(\alpha - r)A(t)}{m_1\sigma\tilde{A}^2(t)}
\end{aligned}
$$

$$+\frac{2\beta\sigma\tilde{C}(t)}{m_1\tilde{A}(t)}\Big)^2 s^{-2\beta}+\frac{[a\eta+(1+\eta)\lambda' h]^2A^2(t)}{m_1^2b^2\tilde{A}^4(t)}+(\sigma_{12}-k_1\sigma_{22})^2\Big](-m_1\tilde{A}^2(t))$$
$$+\sigma^2\beta(2\beta+1)\frac{C(t)}{m_1}-2\sigma^2\beta^2\frac{\tilde{C}^2(t)}{m_1}s^{-2\beta}+\Big[(\sigma_{11}^2-k_1\sigma_{11}\sigma_{21}-\frac{\tilde{B}(t)\sigma_{11}^2}{\tilde{A}(t)m_1}$$
$$+\frac{(\alpha-r)\sigma_{11}A(t)}{m_1\sigma\tilde{A}^2(t)}s^{-2\beta}+\frac{2\beta\sigma\sigma_{11}\tilde{C}(t)}{\tilde{A}(t)m_1}s^{-2\beta}-\sigma_{11}+k_1\sigma_{21})s^{2\beta}+\sigma_{12}(\sigma_{12}$$
$$-k_1\sigma_{22})\Big](-\tilde{A}(t)\tilde{B}(t))+\Big(-\frac{\tilde{B}(t)\sigma\sigma_{11}}{m_1\tilde{A}(t)}+\frac{(\alpha-r)A(t)}{m_1\tilde{A}^2(t)}s^{-2\beta}+\frac{2\beta\sigma^2\tilde{C}(t)}{m_1\tilde{A}(t)}s^{-2\beta}\Big)$$
$$\times(2\beta\tilde{A}(t)\tilde{C}(t))+2\sigma\sigma_{11}\beta\tilde{B}(t)\frac{\tilde{C}(t)}{m_1}=0. \tag{B3}$$

Eq (2.10) can be simplified to

$$\tilde{A}'(t)\hat{x}_1+\frac{\tilde{B}'(t)}{m_1}l_1+\frac{\tilde{C}'(t)}{m_1}s^{-2\beta}+\Big[r(\hat{x}_1+l_1-k_1l_2)+(\eta q_1+\theta-\eta)(a+\lambda' h)$$
$$+\lambda' hq_1-k_1(1-q_2)\Big]\Big((1+\eta)\lambda' h+a\eta\Big)-\alpha_1+k_1\alpha_2+(\alpha-r)(\pi_1-k_1\pi_2) \tag{B4}$$
$$\times\tilde{A}(t)+\alpha_1\frac{\tilde{B}(t)}{m_1}-2\alpha\beta\frac{\tilde{C}(t)}{m_1}s^{-2\beta}+\beta(2\beta+1)\sigma^2\frac{\tilde{C}(t)}{m_1}=0.$$

Separate the variables for Eq (B3) and Eq (B4):

$$\begin{cases}A'(t)+rA(t)=0,\\ \dfrac{B'(t)}{m_1}+rA(t)=0,\\ \dfrac{C'(t)}{m_1}+\dfrac{(\alpha-r)^2A(t)^2}{m_1\sigma^2\tilde{A}(t)^2}+\dfrac{2\beta A(t)\tilde{C}(t)(\alpha-r)}{m_1\tilde{A}(t)}-2\alpha\beta\dfrac{C(t)}{m_1}-\dfrac{1}{2}m_1\tilde{A}(t)^2(\dfrac{\alpha-r}{m_1\sigma^2\tilde{A}(t)^2}\\ \quad+\dfrac{2\beta\sigma\tilde{C}(t)}{m_1\tilde{A}(t)^2})^2-2\sigma^2\beta^2\dfrac{\tilde{C}(t)^2}{m_1}+\dfrac{2\beta(\alpha-r)\tilde{C}(t)A(t)}{m_1\tilde{A}(t)}+\dfrac{4\beta^2\sigma^2\tilde{C}(t)^2}{m_1}=0,\\ \tilde{A}'(t)+r\tilde{A}(t)=0,\\ \dfrac{\tilde{B}'(t)}{m_1}+r\tilde{A}(t)=0,\\ \dfrac{\tilde{C}'(t)}{m_1}-2\alpha\beta\dfrac{\tilde{C}(t)}{m_1}=0.\end{cases}$$

Combining with the boundary conditions $A(T)=\tilde{A}(T)=1$, and $B(T)=\tilde{B}(T)=0$, $C(T)=\tilde{C}(T)=0$, we have

$$\begin{cases}A(t)=e^{r(T-t)},\\ B(t)=m_1(e^{r(T-t)}-1),\\ C(t)=-\dfrac{2(\alpha-r)(2\alpha-1)\beta\sigma^3t}{\sigma^4}e^{2\alpha\beta t}\Big(1-\dfrac{T}{t}e^{2\alpha\beta(T-t)}\Big),\\ \tilde{A}(t)=e^{r(T-t)},\\ \tilde{B}(t)=m_1(e^{r(T-t)}-1),\\ \tilde{C}(t)=e^{2\alpha\beta t}\Big(1-e^{2\alpha\beta(T-t)}\Big).\end{cases}$$

Step 3: The reinsurer knows the insurer's strategy that determines its optimal strategy $(q_2^*(.), \pi_2^*(.)) \in U_2$. Using the variational operator from Eq (2.8), the reinsurer simplifies Eq (2.11):

$$\begin{aligned}
&\sup_{u_2 \in U_2} \Big\{ \frac{\partial D_2}{\partial t} + [r(\hat{y}_1(t) + l_2(t) - k_2 l_1(t)) - \alpha_2 + k_2\alpha_1 + (1 - q_2(t))((1 \\
&+ \eta)\lambda' h + a\eta) - k_2(\eta q_1(t) + \theta - \eta)(a + \lambda' h) + (\alpha - r)(\pi_2(t) \\
&- k_2\pi_1(t)) - k_2\lambda' h q_1(t)]\frac{\partial D_2}{\partial \hat{y}_1} + \alpha_2 \frac{\partial D_2}{\partial l_2} + \alpha s \frac{\partial D_2}{\partial s} + \frac{1}{2}(\sigma_{21}^2 s^{2\beta} \\
&+ \sigma_{22}^2)(\frac{\partial^2 D_2}{\partial l_2^2} - m_2\left(\frac{\partial f_2}{\partial l_2}\right)^2) + \frac{1}{2}[(\sigma\pi_2(t) - \sigma_{21} - k_2\pi_1(t)\sigma \\
&+ k_2\sigma_{11})^2 s^{2\beta} + b^2(-k_2 q_1(t) + 1 - q_2(t))^2 + (\sigma_{22} - k_2\sigma_{12})^2] \\
&\times \left[\frac{\partial^2 D_2}{\partial \hat{y}_1^2} - m_2\left(\frac{\partial f_2}{\partial \hat{y}_1}\right)^2\right] + \frac{1}{2}\sigma^2 s^{2\beta+2}(\frac{\partial^2 D_2}{\partial s^2} - m_2(\frac{\partial f_2}{\partial s})^2) \\
&+ [\sigma(\sigma\pi_2(t) - \sigma_{21} - k_2\pi_1(t)\sigma + k_2\sigma_{11})s^{2\beta+1}](\frac{\partial^2 D_2}{\partial \hat{y}_1 \partial s} - m_2\frac{\partial f_2}{\partial \hat{y}_1}\frac{\partial f_2}{\partial s}) \\
&- [\sigma_{21}(\sigma\pi_2(t) - \sigma_{21} - k_2\pi_1(t)\sigma + k_2\sigma_{11})s^{2\beta} + \sigma_{22}(\sigma_{22} - k_2\sigma_{12})] \\
&\times (\frac{\partial^2 D_2}{\partial \hat{y}_1 \partial l_2} - m_2\frac{\partial f_2}{\partial \hat{y}_1}\frac{\partial f_2}{\partial l_2}) + \sigma\sigma_{21}s^{2\beta+1}(\frac{\partial^2 D_2}{\partial l_2 \partial s} - m_2\frac{\partial f_2}{\partial l_2}\frac{\partial f_2}{\partial s})\Big\} = 0.
\end{aligned} \tag{B5}$$

To solve Eq (2.11) and Eq (B5), similarly, we conjecture that the reinsurer's value function takes the following form:

$$\begin{aligned}
D_2(t, \hat{y}_1, l_2, s) &= D(t)\hat{y}_1 + \frac{E(t)}{m_2}l_2 + \frac{F(t)}{m_2}s^{-2\beta}, D(T) = 1, E(T) = F(T) = 0; \\
f_2(t, \hat{y}_1, l_2, s) &= \tilde{D}(t)\hat{y}_1 + \frac{\tilde{E}(t)}{m_2}l_2 + \frac{\tilde{F}(t)}{m_2}s^{-2\beta}, \tilde{D}(T) = 1, \tilde{E}(T) = \tilde{F}(T) = 0.
\end{aligned}$$

The partial derivatives of the above function are

$$\begin{aligned}
\frac{\partial D_2}{\partial t} &= D'(t)\hat{y}_1 + \frac{E'(t)}{m_2}l_2 + \frac{F'(t)}{m_2}s^{-2\beta}, & \frac{\partial D_2}{\partial l_2} &= \frac{E(t)}{m_2}, \\
\frac{\partial D_2}{\partial \hat{y}_1} &= D(t), & \frac{\partial D_2}{\partial s} &= -2\beta\frac{F(t)}{m_2}s^{-2\beta-1}, \\
\frac{\partial f_2}{\partial t} &= \tilde{D}'(t)\hat{y}_1 + \frac{\tilde{E}'(t)}{m_2}l_2 + \frac{\tilde{F}'(t)}{m_2}s^{-2\beta}, & \frac{\partial f_2}{\partial l_2} &= \frac{\tilde{E}(t)}{m_2}, \\
\frac{\partial f_2}{\partial \hat{y}_1} &= \tilde{D}(t), & \frac{\partial f_2}{\partial s} &= -2\beta\frac{\tilde{F}(t)}{m_2}s^{-2\beta-1}.
\end{aligned}$$

By substituting the above partial derivatives into Eq (B5), we obtain

$$\begin{aligned}\sup_{u_2\in U_2}\Big\{&D'(t)\hat{y}_1+\frac{E'(t)}{m_2}l_2+\frac{F'(t)}{m_2}s^{-2\beta}+[r(\hat{y}_1+l_2-k_2l_1)-\alpha_2+k_2\alpha_1\\&+(1-q_2)((1+\eta)\lambda'h+a\eta)-k_2(\eta q_1+\theta-\eta)(a+\lambda'h)+(\alpha\\&-r)(\pi_2(t)-k_2\pi_1(t))-k_2\lambda'hq_1(t)]D(t)+\alpha_2\frac{E(t)}{m_2}-2\alpha\beta\frac{F(t)}{m_2}s^{-2\beta}\\&-\frac{1}{2}(\sigma_{12}^2s^{2\beta}+\sigma_{22}^2)\frac{\widetilde{E}(t)^2}{m_2}+\frac{1}{2}[(\sigma\pi_2(t)-\sigma_{21}-k_2\pi_1(t)\sigma+k_2\sigma_{11})^2s^{2\beta}\\&+b^2(-k_2q_1(t)+1-q_2(t))^2+(\sigma_{22}-k_2\sigma_{12})^2](-m_2\tilde{D}(t)^2)+\beta(2\beta\\&+1)\sigma^2\frac{F(t)}{m_2}-2m_2\beta^2\sigma^2\frac{\tilde{F}(t)^2}{m_2^2}s^{-2\beta}+[\sigma_{12}(\sigma\pi_2(t)-\sigma_{21}-k_2\pi_1(t)\sigma\\&+k_2\sigma_{11})s^{2\beta}+\sigma_{22}(\sigma_{22}-k_2\sigma_{12})](\tilde{D}(t)\tilde{E}(t))+[\sigma(\pi_2(t)\sigma-\sigma_{21}\\&-k_2\pi_1(t)\sigma+k_2\sigma_{11})s^{2\beta+1}](2\beta\tilde{D}(t)\tilde{F}(t)s^{-2\beta-1})+2\sigma\sigma_{21}\beta\tilde{E}(t)\frac{\tilde{F}(t)}{m_2}=0\Big\}.\end{aligned}\tag{B6}$$

Applying the first-order condition,

$$\begin{aligned}&(\alpha-r)D(t)-m_2\sigma\tilde{D}(t)^2(\sigma\pi_2(t)-\sigma_{21}-k_2\pi_1(t)\sigma\\&+k_2\sigma_{11})s^{2\beta}+\tilde{D}(t)\tilde{E}(t)\sigma_{12}\sigma s^{2\beta}+2\beta\sigma^2\tilde{D}(t)\tilde{F}(t)=0.\end{aligned}$$

$$-[(1+\eta)\lambda'h+a\eta]D(t)+b^2(-k_2q_1+1-q_2)(m_2\tilde{D}(t)^2)=0.$$

We get the reinsurer's optimal strategy as follows:

$$\pi_2^*=k_2\pi_1+\frac{(\alpha-r)D(t)+\tilde{D}(t)\tilde{E}(t)\sigma_{12}\sigma s^{2\beta}+2\beta\sigma^2\tilde{D}(t)\tilde{F}(t)}{m_2\sigma^2\tilde{D}(t)^2s^{2\beta}}+\frac{\sigma_{21}-k_2\sigma_{11}}{\sigma};$$

$$q_2^*=1-k_2q_1-\frac{[(1+\eta)\lambda'h+a\eta]D(t)}{m_2b^2\tilde{D}(t)^2}.$$

By substituting

$$\pi_2-k_2\pi_1=\frac{(\alpha-r)D(t)+\tilde{D}(t)\tilde{E}(t)\sigma_{12}\sigma s^{2\beta}+2\beta\sigma^2\tilde{D}(t)\tilde{F}(t)}{m_2\sigma^2\tilde{D}(t)^2s^{2\beta}}+\frac{\sigma_{21}-k_2\sigma_{11}}{\sigma},$$

$$-k_2q_1+1-q_2=\frac{[(1+\eta)\lambda'h+a\eta]D(t)}{m_2b^2\tilde{D}(t)^2},$$

into Eq (B6), we have:

$$\begin{aligned}&D'(t)\hat{y_1}+\frac{E'(t)}{m_2}l_2+\frac{F'(t)}{m_2}s^{-2\beta}+\alpha_2\frac{E(t)}{m_2}-2\alpha\beta\frac{F(t)}{m_2}s^{-2\beta}-\frac{1}{2}(\sigma_{21}^2s^{2\beta}\\&+\sigma_{22}^2)\frac{\widetilde{E}(t)^2}{m_2}+\beta(2\beta+1)\sigma^2\frac{F(t)}{m_2}-2\beta^2\sigma^2\frac{\tilde{F}(t)^2}{m_2}s^{-2\beta}+[r(\hat{y_1}+l_2\end{aligned}$$

$$
\begin{aligned}
&- k_2 l_1) - \alpha_2 + k_2\alpha_1 - k_2(\theta-\eta)(a+\lambda' h) + \frac{[(1+\eta)\lambda' h + a\eta]^2 D(t)}{m_2 b^2 \tilde{D}(t)^2} \\
&+ \frac{(\alpha - r)(\sigma_{21} - k_2\sigma_{11})}{\sigma} + \frac{(\alpha-r)^2 D(t)}{m_2\sigma^2\tilde{D}(t)^2 s^{2\beta}} + \frac{(\alpha-r)\tilde{D}(t)\tilde{E}(t)\sigma_{12}\sigma s^{2\beta}}{m_2\sigma^2\tilde{D}(t)^2 s^{2\beta}} \\
&+ \frac{2\beta\sigma^2(\alpha-r)\tilde{D}(t)\tilde{F}(t)}{m_2\sigma^2\tilde{D}(t)^2 s^{2\beta}}]D(t) + \frac{1}{2}[(\frac{(\alpha-r)D(t)}{m_2\sigma\tilde{D}(t)^2 s^{2\beta}} + \frac{\tilde{D}(t)\tilde{E}(t)\sigma_{12}\sigma s^{2\beta}}{m_2\sigma\tilde{D}(t)^2 s^{2\beta}} \\
&+ \frac{2\beta\sigma^2\tilde{D}(t)\tilde{F}(t)}{m_2\sigma\tilde{D}(t)^2 s^{2\beta}})^2 s^{2\beta} + \frac{((1+\eta)\lambda' h + a\eta)^2 D(t)^2}{m_2^2 b^2 \tilde{D}(t)^4} + (\sigma_{22} - k_2\sigma_{12})^2] \\
&\times(-m_2\tilde{D}(t)^2) + [\sigma_{12}(\frac{(\alpha-r)D(t) + \tilde{D}(t)\tilde{E}(t)\sigma_{12}\sigma s^{2\beta} + 2\beta\sigma^2\tilde{D}(t)\tilde{F}(t)}{m_2\sigma\tilde{D}(t)^2}) \\
&+ \sigma_{22}(\sigma_{22} - k_2\sigma_{12})](\tilde{D}(t)\tilde{E}(t)) + 2\beta(\frac{(\alpha-r)D(t) + \tilde{D}(t)\tilde{E}(t)\sigma_{12}\sigma s^{2\beta}}{m_2\tilde{D}(t)^2 s^{2\beta}} \\
&\frac{2\beta\sigma^2\tilde{D}(t)\tilde{F}(t)}{m_2\tilde{D}(t)^2 s^{2\beta}})\tilde{D}(t)\tilde{F}(t) + 2\sigma\sigma_{21}\beta\frac{\tilde{E}(t)\tilde{F}(t)}{m_2} = 0. \qquad \text{(B7)}
\end{aligned}
$$

Eq (2.12) can be simplified to

$$
\begin{aligned}
&\tilde{D}'(t)\hat{y}_1 + \frac{\tilde{E}'(t)}{m_2}l_2 + \frac{\tilde{F}'(t)}{m_2}s^{-2\beta} + [r(\hat{y}_1 + l_2 - k_2 l_1) + k_2(\theta-\eta)(a+\lambda' h) \\
&\quad - \alpha_2 + k_2\alpha_1 + (1 - q_2(t) - k_2 q_1(t))((1+\eta)\lambda' h + a\eta) + (\alpha - r)(\pi_2(t) \qquad \text{(B8)} \\
&\quad - k_2\pi_1(t))]\tilde{D}(t) + \alpha_2\frac{\tilde{E}(t)}{m_2} - 2\alpha\beta\frac{\tilde{F}(t)}{m_2}s^{-2\beta} + \beta(2\beta+1)\sigma^2\frac{\tilde{F}(t)}{m_2} = 0.
\end{aligned}
$$

Separating the variables yields

$$
\begin{aligned}
D'(t) + rD(t) &= 0, \\
\frac{E'(t)}{m_2} + rD(t) &= 0, \\
F'(t) - 2\alpha\beta F(t) + \frac{(\alpha-r)^2 D^2(t)}{\sigma^2\tilde{D}(t)^2} + \frac{2(\alpha-r)\beta D(t)\tilde{F}(t)}{\tilde{D}(t)} - \frac{(\alpha-r)^2 D(t)^2}{2\sigma^2} & \\
-2\beta^2\sigma^2\tilde{F}(t)^2\tilde{D}(t)^2 - 2\beta(\alpha-r)D(t)\tilde{F}(t)\tilde{D}(t) - 2\beta^2\sigma^2\tilde{F}(t)^2 & \\
+\frac{2\beta(\alpha-r)D(t)\tilde{F}(t)}{\tilde{D}(t)} + 4\beta^2\tilde{F}(t) &= 0, \\
\tilde{D}'(t) + r\tilde{D}(t) &= 0 \\
\frac{\tilde{E}'(t)}{m_2} + r\tilde{D}(t) &= 0 \\
\frac{(\alpha-r)^2 D(t)}{m_2\sigma^2\tilde{D}(t)} + \frac{2\beta(\alpha-r)\tilde{F}(t)}{m_2} - 2\alpha\beta\frac{\tilde{F}(t)}{m_2} &= 0.
\end{aligned}
$$

Combining with the boundary conditions $D(T) = \tilde{D}(T) = 1$, and $E(T) = \tilde{E}(T) = 0$, $F(T) = \tilde{F}(T) = 0$, we have

$$
D(t) = \tilde{D}(T) = e^{r(T-t)},
$$

$$E(t) = \tilde{E}(t) = m_2(e^{r(T-t)} - 1),$$

$$\tilde{F}(t) = \frac{1}{2\beta r}\frac{(\alpha - r)^2}{\sigma^2}e^{2\beta r(t-T)}(1 - e^{2\beta r(T-t)}),$$

$$F(t) = e^{2\alpha\beta t}(\int_a^b -H(t)e^{-2\alpha\beta t}dt + C)$$

where

$$\begin{aligned} H(t) = {} & \frac{(\alpha - r)^2}{\sigma^2} + \frac{3(\alpha - r)}{r}\frac{(\alpha - r)^2}{\sigma^2}e^{2\beta r(t-T)} - \frac{3(\alpha - r)}{r}\frac{(\alpha - r)^2}{\sigma^2} \\ & +(\frac{\alpha - r}{r} - \frac{1}{2})\frac{(\alpha - r)^2}{\sigma^2}e^{2r(T-t)} + (\frac{2 - \sigma^2}{2r^2}\frac{(\alpha - r)^4}{\sigma^4}e^{4\beta r(t-T)} \\ & -\frac{\sigma^2}{2r^2}\frac{(\alpha - r)^4}{\sigma^4}e^{2r(2\beta-1)(t-T)})(1 - 2e^{2\beta r(T-t)} + e^{4\beta r(T-t)}) \\ & -\frac{\alpha - r}{r}\frac{(\alpha - r)^2}{\sigma^2}e^{2r(1-\beta)(T-t)} \end{aligned}$$

In summary, we have

$$\begin{cases} \pi_1 - k_1\pi_2 = \frac{\sigma_{11} - k_1\sigma_{21}}{\sigma} - \frac{m_1(e^{r(T-t)}-1)\sigma\sigma_{11}s^{2\beta} - (\alpha - r) - 2\beta\sigma^2 e^{2\alpha\beta t}(1 - e^{2\alpha\beta(T-t)})}{m_1\sigma^2 s^{2\beta}e^{r(T-t)}}, \\ \pi_2 - k_2\pi_1 = \frac{\sigma_{21} - k_2\sigma_{11}}{\sigma} + \frac{m_2(e^{r(T-t)}-1)\sigma\sigma_{12}s^{2\beta} + (\alpha - r) + \frac{(\alpha - r)^2}{r}e^{2\beta r(t-T)}(1 - e^{2r\beta(T-t)})}{m_2\sigma^2 s^{2\beta}e^{r(T-t)}}. \end{cases}$$

Thus,

$$\begin{cases} \pi_1^* = \frac{\sigma_{11}}{\sigma} + \frac{(\alpha - r)(m_2 + m_1k_1) + m_1m_2\sigma s^{2\beta}(k_1\sigma_{12} - \sigma_{11})(e^{r(T-t)} - 1) + m_1k_1\frac{(\alpha - r)^2}{r}(e^{2\beta r(t-T)} - 1)}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}} \\ -\frac{2m_2\beta\sigma^2 e^{2\alpha\beta t}(1 - e^{2\alpha\beta(T-t)})}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}, \\ \pi_2^* = \frac{\sigma_{21}}{\sigma} + \frac{(\alpha - r)(m_1 + m_2k_2) + m_1m_2\sigma s^{2\beta}(\sigma_{12} - k_2\sigma_{11})(e^{r(T-t)} - 1) + m_1\frac{(\alpha - r)^2}{r}(e^{2\beta r(t-T)} - 1)}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}} \\ +\frac{2m_2k_2\beta\sigma^2 e^{2\alpha\beta t}(1 - e^{2\alpha\beta(T-t)})}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}. \end{cases}$$

We have

$$\begin{cases} q_1 - k_1(1 - q_2) = \frac{(1+\eta)\lambda' h + a\eta}{m_1b^2e^{r(T-t)}}, \\ -k_2q_1 + 1 - q_2 = \frac{(1+\eta)\lambda' h + a\eta}{m_2b^2e^{r(T-t)}}. \end{cases}$$

Thus,

$$\begin{cases} q_1^* = \frac{(k_1 + m_2 - k_1k_2m_2)[(1+\eta)\lambda' h + a\eta]}{(1 - k_1k_2)m_1m_2b^2e^{r(T-t)}}, \\ q_2^* = 1 - \frac{[(1+\eta)\lambda' h + a\eta](m_1 + m_2k_2)}{(1 - k_1k_2)m_1m_2b^2e^{r(T-t)}}. \end{cases}$$

The optimal investment strategies for the insurer and the reinsurer are

$$\begin{cases} \pi_1^* = \frac{\sigma_{11}}{\sigma} + \frac{(\alpha - r)(m_2 + m_1k_1) + m_1m_2\sigma s^{2\beta}(k_1\sigma_{12} - \sigma_{11})(e^{r(T-t)} - 1) + m_1k_1\frac{(\alpha - r)^2}{r}(e^{2\beta r(t-T)} - 1)}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}} \\ -\frac{2m_2\beta\sigma^2 e^{2\alpha\beta t}(1 - e^{2\alpha\beta(T-t)})}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}, \\ \pi_2^* = \frac{\sigma_{21}}{\sigma} + \frac{(\alpha - r)(m_1 + m_2k_2) + m_1m_2\sigma s^{2\beta}(\sigma_{12} - k_2\sigma_{11})(e^{r(T-t)} - 1) + m_1\frac{(\alpha - r)^2}{r}(e^{2\beta r(t-T)} - 1)}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}} \\ +\frac{2m_2k_2\beta\sigma^2 e^{2\alpha\beta t}(1 - e^{2\alpha\beta(T-t)})}{(1 - k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}. \end{cases}$$

The optimal reinsurance strategies for the insurer and the reinsurer are

$$\begin{cases} q_1^* = \frac{(k_1+m_2-k_1k_2m_2)[(1+\eta)\lambda' h+a\eta]}{(1-k_1k_2)m_1m_2b^2e^{r(T-t)}}, \\ q_2^* = 1-\frac{[(1+\eta)\lambda' h+a\eta](m_1+m_2k_2)}{(1-k_1k_2)m_1m_2b^2e^{r(T-t)}}. \end{cases}$$

If we assume no cooperation or competition, which corresponds to $k_1 = k_2 = 0$, then

$$\begin{cases} \pi_1^* = \frac{\sigma_{11}}{\sigma} + \frac{(\alpha-r)m_2-m_1m_2\sigma s^{2\beta}\sigma_{11}(e^{r(T-t)}-1)-2m_2\beta\sigma^2e^{2\alpha\beta t}(1-e^{2\alpha\beta(T-t)})}{m_1m_2\sigma^2s^{2\beta}e^{r(T-t)}}, \\ \pi_2^* = \frac{\sigma_{21}}{\sigma} + \frac{(\alpha-r)m_1+m_1m_2\sigma s^{2\beta}\sigma_{12}(e^{r(T-t)}-1)+m_1\frac{(\alpha-r)^2}{r}(e^{2\beta r(t-T)}-1)}{m_1m_2\sigma^2s^{2\beta}e^{r(T-t)}}. \end{cases}$$

$$\begin{cases} q_1^* = \frac{(1+\eta)\lambda' h+a\eta}{m_1b^2e^{r(T-t)}}, \\ q_2^* = 1-\frac{(1+\eta)\lambda' h+a\eta}{m_2b^2e^{r(T-t)}}. \end{cases}$$

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