



Research article

Dynamic control of a multi-product firm’s product and process innovation in a market with network externality

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Appendix 1.

Proof of Conclusion 3.2: Before proceeding, we aim to verify the second-order conditions, which ensures that the first-order conditions characterize a maximum. The current-value Hamiltonian is strictly concave with respect to the control variables (q_i, k, h) provided that $\alpha > 0, \beta > 0$, and $a_1 > a_2\phi m(v - \vartheta s)$. Specifically, we have $\frac{\partial^2 H}{\partial k^2} = -\alpha < 0, \frac{\partial^2 H}{\partial h^2} = -\beta < 0$, and $\frac{\partial^2 H}{\partial q_i^2} = -2a_1 + 2a_2\phi m(v - \vartheta s) < 0$ (This expression is for the special case where the network size is proportional to the output level. For the general case, we naturally have $\frac{\partial^2 H}{\partial q_i^2} = -2a_1 < 0$), and all cross-partial derivatives among the control variables are zero. Hence, the Hessian matrix is negative definite, and the first-order conditions are sufficient for an interior maximum. (i) from Condition 3.1, we deduce $a + a_2\varphi\nu Q(t) + \mu r(t) - c(t) > 0$. Differentiating output level (3.9) w.r.t. $s(t)$ gives $\frac{\partial q(t)}{\partial s(t)} = -\frac{a_2\varphi\vartheta Q(t)2a_1a_2\varphi\vartheta Q(t)+(n-1)[a+a_2\varphi\nu Q(t)+\mu r(t)-c(t)]+2a_1\mu}{[2a_1+(n-1)s(t)]^2} < 0$.

Besides, differentiating equation (2.1) w.r.t. $k(t)$ obtains $\frac{\partial s(t)}{\partial k(t)} = \frac{\partial \int_0^t \dot{s}(z)dz}{\partial k(t)} = -t < 0$. Since the reaction of $k(t)$ to a change in $q(t)$ is $\frac{\partial q(t)}{\partial k(t)} = \frac{\partial q(t)}{\partial s(t)} \frac{\partial s(t)}{\partial k(t)}$, we have $\frac{\partial q(t)}{\partial k(t)} > 0$.

(ii) Differentiating output level (3.9) w.r.t. $c(t)$ obtains $\frac{\partial q(t)}{\partial c(t)} = -\frac{1}{2a_1+(n-1)s(t)} < 0$. Besides, differentiating equation (2.2) w.r.t. effort $h(t)$ gives $\frac{\partial c(t)}{\partial h(t)} = \frac{\partial \int_0^t \dot{c}(z)dz}{\partial h(t)} = -t < 0$. Since $\frac{\partial q(t)}{\partial h(t)} = \frac{\partial q(t)}{\partial c(t)} \frac{\partial c(t)}{\partial h(t)}$, we have $\frac{\partial q(t)}{\partial h(t)} > 0$. □

Appendix 2.

Proof of Conclusion 3.3: (i) from Condition 3.1, we deduce $a + a_2\varphi vQ(t) + \mu r(t) - c(t) > 0$. Besides, in Conclusion 3.2, we have obtained $\frac{\partial s(t)}{\partial k(t)} < 0$ and $\frac{\partial c(t)}{\partial h(t)} < 0$. Differentiating output level (3.10) w.r.t. $s(t)$ gives $\frac{\partial p(t)}{\partial s(t)} = -\frac{2a_1^2[a_2\varphi\theta Q(t)+\mu]+a_1(n-1)[a+a_2\varphi vQ(t)+\mu r(t)-c(t)]}{[2a_1+(n-1)s(t)]^2} < 0$. Since $\frac{\partial p(t)}{\partial k(t)} = \frac{\partial p(t)}{\partial s(t)} \frac{\partial s(t)}{\partial k(t)}$, we have $\frac{\partial p(t)}{\partial k(t)} > 0$.

(ii) Differentiating equation (2.2) w.r.t. effort $h(t)$ gives $\frac{\partial c(t)}{\partial h(t)} = \frac{\partial \int_0^t \dot{c}(z)dz}{\partial h(t)} = -t < 0$, and differentiating the output level (3.10) w.r.t. $c(t)$ gives $\frac{\partial p(t)}{\partial c(t)} = \frac{a_1+(n-1)s(t)}{2a_1+(n-1)s(t)} > 0$. Since $\frac{\partial p(t)}{\partial h(t)} = \frac{\partial p(t)}{\partial c(t)} \frac{\partial c(t)}{\partial h(t)}$, we have $\frac{\partial p(t)}{\partial h(t)} < 0$. \square

Appendix 3.

Proof of Conclusion 3.4: From Condition 3.1, we deduce $a + a_2\varphi vQ(t) + \mu r(t) - c(t) > 0$. Besides, in Conclusion 3.2, we have obtained $\frac{\partial s(t)}{\partial k(t)} < 0$ and $\frac{\partial c(t)}{\partial h(t)} < 0$.

(i) Differentiating equation (3.11) w.r.t. $s(t)$, and using Condition 3.1-(i), one can obtain $\frac{\partial \lambda_1(t)}{\partial s(t)} = \frac{n}{\tau-\sigma} \left\{ \frac{2a_1 a_2 \varphi \theta Q(t) + 2a_1 \mu + (n-1)[a + a_2 \varphi v Q(t) + \mu r(t) - c(t)]}{[2a_1 + (n-1)s(t)]^2} \right\} + \frac{2(n-1)[a + a_2 \varphi (v - \theta s(t))Q(t) - \mu(s(t) - r(t)) - c(t)]}{2a_1 + (n-1)s(t)} + \frac{a_2 \varphi \theta (\tau + \rho) Q(t) + \tau \mu}{\tau + \rho} > 0$. Since, $\frac{\partial \lambda_1(t)}{\partial k(t)} = \frac{\partial \lambda_1(t)}{\partial s(t)} \frac{\partial s(t)}{\partial k(t)}$, we have $\frac{\partial \lambda_1(t)}{\partial k(t)} < 0$.

Differentiating equations (3.12) and (3.13) w.r.t. $s(t)$, respectively, we have $\frac{\partial \lambda_2(t)}{\partial s(t)} = \frac{n[2a_1 a_2 \varphi \theta Q(t) + 2a_1 \mu + (n-1)[a + a_2 \varphi v Q(t) + \mu r(t) - c(t)]}{(\tau - \delta)[2a_1 + (n-1)s(t)]^2} > 0$ and $\frac{\partial \lambda_3(t)}{\partial s(t)} = -\frac{\mu n[2a_1 a_2 \varphi \theta Q(t) + 2a_1 \mu + (n-1)[a + a_2 \varphi v Q(t) + \mu r(t) - c(t)]}{(\tau + \rho)[2a_1 + (n-1)s(t)]^2} < 0$. Since $\frac{\partial \lambda_2(t)}{\partial k(t)} = \frac{\partial \lambda_2(t)}{\partial s(t)} \frac{\partial s(t)}{\partial k(t)}$ and $\frac{\partial \lambda_3(t)}{\partial k(t)} = \frac{\partial \lambda_3(t)}{\partial s(t)} \frac{\partial s(t)}{\partial k(t)}$, we have $\frac{\partial \lambda_2(t)}{\partial k(t)} < 0$ and $\frac{\partial \lambda_3(t)}{\partial k(t)} > 0$.

(ii) Differentiating equations (3.11)-(3.13) w.r.t. $c(t)$ gives $\frac{\partial \lambda_1(t)}{\partial c(t)} = \frac{n}{(\tau - \sigma)[2a_1 + (n-1)s(t)]} \left\{ \frac{n(n-1)[a + a_2 \varphi (v - \theta s(t))Q(t) - \mu(s(t) - r(t)) - c(t)]}{2a_1 + (n-1)s(t)} + \frac{a_2 \varphi \theta (\tau + \rho) Q(t) + \tau \mu}{\tau + \rho} \right\} > 0$, $\frac{\partial \lambda_2(t)}{\partial c(t)} = \frac{n}{(\tau - \delta)[2a_1 + (n-1)s(t)]} > 0$, and $\frac{\partial \lambda_3(t)}{\partial c(t)} = -\frac{n\mu}{(\tau + \rho)[2a_1 + (n-1)s(t)]} < 0$. Then, we have $\frac{\partial \lambda_1(t)}{\partial h(t)} < 0$, $\frac{\partial \lambda_2(t)}{\partial h(t)} < 0$, and $\frac{\partial \lambda_3(t)}{\partial h(t)} > 0$. \square

Appendix 4.

Proof of Conclusion 3.5: Differentiating equations (3.14) and (3.15) w.r.t. $s(t)$, $c(t)$, and $r(t)$, using Condition 3.1-(i), we have

$$\frac{\partial k(t)}{\partial s(t)} = \frac{(n-1)[a + a_2 \varphi (v - \theta s(t))Q(t) - \mu(s(t) - r(t)) - c(t)]}{\alpha[2a_1 + (n-1)s(t)]^3} \{4a_1 n[a_2 \varphi \theta Q(t) + \mu] + (2n - 1)[a + a_2 \varphi v Q(t) + \mu r(t) - c(t)]\} + \frac{[a_2 \varphi \theta (\tau + \rho) Q(t) + \mu \tau][2na_1[a_2 \varphi \theta Q(t) + \mu] + (n-1)[a + a_2 \varphi v Q(t) + \mu r(t) - c(t)]}{\alpha(\tau + \rho)[2a_1 + (n-1)s(t)]^2} > 0,$$

$$\frac{\partial k(t)}{\partial c(t)} = \frac{2n(n-1)[a + a_2 \varphi (v - \theta s(t))Q(t) - \mu(s(t) - r(t)) - c(t)]}{\alpha[2a_1 + (n-1)s(t)]^2} + \frac{n[a_2 \varphi \theta (\tau + \rho) Q(t) + \mu \tau]}{\alpha(\tau + \rho)[2a_1 + (n-1)s(t)]} > 0,$$

$$\frac{\partial k(t)}{\partial r(t)} = -\frac{2n\mu(n-1)[a + a_2 \varphi (v - \theta s(t))Q(t) - \mu(s(t) - r(t)) - c(t)]}{\alpha[2a_1 + (n-1)s(t)]^2} - \frac{n\mu[a_2 \varphi \theta (\tau + \rho) Q(t) + \mu \tau]}{\alpha(\tau + \rho)[2a_1 + (n-1)s(t)]} < 0,$$

$$\frac{\partial h(t)}{\partial s(t)} = \frac{n[2a_1 a_2 \varphi \theta Q(t) + 2a_1 \mu + (n-1)[a + a_2 \varphi v Q(t) + \mu r(t) - c(t)]}{\beta[2a_1 + (n-1)s(t)]^2} > 0,$$

$$\frac{\partial h(t)}{\partial c(t)} = \frac{n}{\beta[2a_1 + (n-1)s(t)]} > 0, \text{ and}$$

$$\frac{\partial h(t)}{\partial r(t)} = -\frac{n\mu}{\beta[2a_1 + (n-1)s(t)]} < 0. \quad \square$$

Appendix 5.

Proof of Conclusion 3.6: Differentiating equations (3.14) and (3.15) w.r.t. n , we have

$$\frac{\partial k(t)}{\partial n} = -\frac{[2a_1 - s(t)][a + a_2 \varphi (v - \theta s(t))Q(t) - \mu(s(t) - r(t)) - c(t)]}{\alpha[2a_1 + (n-1)s(t)]^2} \left\{ \frac{[n(2a_1 + 1) - 1][a + a_2 \varphi (v - \theta s(t))Q(t) - \mu(s(t) - r(t)) - c(t)]}{2a_1 + (n-1)s(t)} + \frac{a_2 \varphi \theta (\tau + \rho) Q(t) + \mu \tau}{\tau + \rho} \right\}, \text{ and}$$

$$\frac{\partial \dot{h}(t)}{\partial n} = - \frac{[2a_1 - s(t)]\{a + a_2\varphi[v - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t)\}}{\beta[2a_1 + (n-1)s(t)]^2}.$$

Further, using Condition 3.1, we can obtain $\frac{\partial \dot{k}(t)}{\partial n} \geq (<)0$ and $\frac{\partial \dot{h}(t)}{\partial n} \geq (<)0$ if and only if $s(t) \geq (<)2a_1$. □

Appendix 6.

Proof of Conclusion 3.7: Differentiating equations (3.16) and (3.17) w.r.t. s and c , respectively, and using Condition 3.1, we have

$$\begin{aligned} \frac{\partial k(s,c)}{\partial s} &= - \frac{n[2a_1 a_2 \varphi \vartheta Q(t) + (n-1)[a_2 \varphi v Q(t) + a - c(t)]}{\alpha(\tau - \sigma)[2a_1 + (n-1)s(t)]} \left\{ \frac{(n-1)[a_2 \varphi(v - \vartheta s(t))Q(t) - c(t) + a]}{2a_1 + (n-1)s(t)} + \frac{a_2 \varphi \vartheta(\tau + \rho)Q(t) + \mu\tau}{\tau + \rho} \right\} - \\ &\frac{n(n-1)\{a_2 \varphi[v - \vartheta s(t)]Q(t) - c(t) + a\}\{2a_1 a_2 \varphi \vartheta Q(t) + (n-1)[a_2 \varphi v Q(t) + a - c(t)]\}}{\alpha(\tau - \sigma)[2a_1 + (n-1)s(t)]^2} < 0, \\ \frac{\partial k(s,c)}{\partial c} &= \frac{-n}{\alpha(\tau - \sigma)[2a_1 + (n-1)s(t)]} \left\{ \frac{(n-1)[a_2 \varphi(v - \vartheta s(t))Q(t) - c(t) + a]}{2a_1 + (n-1)s(t)} + \frac{a_2 \varphi \vartheta(\tau + \rho)Q(t) + \mu\tau}{\tau + \rho} \right\} - \frac{n(n-1)\{a_2 \varphi[v - \vartheta s(t)]Q(t) - c(t) + a\}}{\alpha(\tau - \sigma)[2a_1 + (n-1)s(t)]^2} < 0, \\ \frac{\partial h(s,c)}{\partial s} &= - \frac{n[2a_1 a_2 \varphi \vartheta Q(t) + (n-1)[a_2 \varphi v Q(t) + a - c(t)]}{\beta(\tau - \delta)[2a_1 + (n-1)s(t)]^2} < 0, \text{ and} \\ \frac{\partial h(s,c)}{\partial c} &= \frac{n\{a_2 \varphi[v - \vartheta s(t)]Q(t) - c(t) + a\}}{\beta(\tau - \delta)[2a_1 + (n-1)s(t)]} = \frac{-n}{\beta(\tau - \delta)[2a_1 + (n-1)s(t)]} < 0 \end{aligned}$$

Appendix 7.

Proof of Proposition 3.8: Solving system (3.20) under the steady state conditions $\dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{c}(t) = \dot{r}(t) = 0$, and denoting steady-state equilibrium by the superscript “ f ”, we have $k^f = \sigma s^f$,

$$h^f = \delta c^f, c^f = \frac{n[a + a_2 \varphi(v - \vartheta s^f)Q]}{\beta(\tau - \delta)\delta[2a_1 + (n-1)s^f] + n}, s^f = r^f = -\frac{K_2}{3K_1} + \sqrt[3]{-\frac{Y}{2} + \sqrt{(\frac{Y}{2})^2 + (\frac{Z}{3})^3}} + \sqrt[3]{-\frac{Y}{2} - \sqrt{(\frac{Y}{2})^2 + (\frac{Z}{3})^3}},$$

where $Z = \frac{3K_1 K_3 - K_2^2}{3K_1^2}$, $Y = \frac{2K_2^3 - 9K_1 K_2 K_3 + 27K_1^2 K_4}{27K_1^3}$, in which $K_1 = n\alpha\sigma(\tau - \sigma)(\tau + \rho)[\beta(\tau - \delta)\delta(n - 1)]^2$, $K_2 = 2n\alpha\sigma\beta\delta(n - 1)(\tau - \delta)(\tau - \sigma)(\tau + \rho)[2a_1\beta(\tau - \delta)\delta + n] + n(n - 1)a_2\varphi\vartheta Q\delta^2\beta^2(\tau - \delta)^2[(n - 1)a_2\varphi\vartheta(\tau + \rho)Q + n\mu\tau]$, $K_3 = n\alpha\sigma(\tau - \sigma)(\tau + \rho)[2a_1\beta(\tau - \delta)\delta + n]^2 + a_2\varphi\vartheta Q\delta\beta(\tau - \delta)n^2[a_2\varphi\vartheta(\tau + \rho)Q + \mu\tau][2a_1\beta(\tau - \delta)\delta + n] + n(n - 1)\delta^2\beta^2(\tau - \delta)^2[(2 - n)a_2\varphi\vartheta(\tau + \rho)Q - n\mu\tau][a + a_2\varphi v Q]$, $K_4 = -n(n - 1)(\tau + \rho)[\beta(\tau - \delta)\delta]^2[a + a_2\varphi v Q]^2 - \delta\beta(\tau - \delta)[a_2\varphi\vartheta(\tau + \rho)Q + \mu\tau]n^2[2a_1\beta(\tau - \delta)\delta + n][a + a_2\varphi v Q]$.

Now, we analyze the stability properties of $\{k^f, h^f, s^f, c^f, r^f\}$. Let J^f denote the Jacobian matrix associated with the canonical system (3.20), that is,

$$J^f = \frac{\partial(\dot{k}, \dot{h}, \dot{s}, \dot{c}, \dot{r})}{\partial(k, h, s, c, r)} = \begin{bmatrix} \tau - \sigma & 0 & X_1 & X_2 & -\mu X_2 \\ 0 & \tau - \delta & X_3 & X_4 & -\mu X_3 \\ -1 & 0 & \sigma & 0 & 0 \\ 0 & -1 & 0 & \delta & 0 \\ 0 & 0 & \rho & 0 & -\rho \end{bmatrix}$$

Note that, in above Jacobian matrix,

$$\begin{aligned} X_1, X_2, X_3, X_4 \text{ are given by, respectively, } X_1 &= \frac{(n-1)[a + a_2 \varphi(v - \vartheta s^f)Q - c^f][4a_1 n(a_2 \varphi \vartheta Q + \mu) + (2n-1)(a + a_2 \varphi v Q + \mu r^f - c^f)]}{\alpha[2a_1 + (n-1)s^f]^3} + \\ &\frac{[a_2 \varphi \vartheta(\tau + \rho) + \mu\tau][2na_1(a_2 \varphi \vartheta Q + \mu) + (n-1)(a + a_2 \varphi v Q + \mu r^f - c^f)]}{\alpha(\tau + \rho)[2a_1 + (n-1)s^f]^2} > 0, X_2 = \frac{2n(n-1)[a + a_2 \varphi(v - \vartheta s^f)Q - c^f]}{\alpha[2a_1 + (n-1)s^f]^2} + \frac{n[a_2 \varphi \vartheta(\tau + \rho)Q + \mu\tau]}{\alpha(\tau + \rho)[2a_1 + (n-1)s^f]} > 0, \\ X_3 &= \frac{n[2a_1 a_2 \varphi \vartheta Q + 2a_1 \mu + (n-1)(a + a_2 \varphi v Q + \mu r^f - c^f)]}{\beta[2a_1 + (n-1)s^f]^2} > 0, \text{ and } X_4 = \frac{n}{\beta[2a_1 + (n-1)s^f]} > 0. \end{aligned}$$

To establish the saddle-point property, we need to demonstrate that the Jacobian matrix J^f of the dynamic control system (3.20) has exactly one negative real eigenvalue. This requires satisfying two conditions: (i) the determinant of Jacobian matrix J^f must be less than zero, i.e., $Det J^f < 0$; (ii) the trace of the Jacobian matrix J^f must be greater than zero, i.e., $T(J^f) > 0$. From the Jacobian matrix J^f ,

after careful calculation, we obtain $T(J^f) = 2\tau - \rho$ and $Det(J^f) = \rho\{X_2X_3(1 - \mu) - [X_4 + (\tau - \delta)\delta][X_1 - \mu X_2 + (\tau - \sigma)\sigma]\} < 0$. Consequently, the steady-state equilibrium $\{k^f, h^f, s^f, c^f, r^f\}$ is stable (unstable) in the saddle-point sense iff $2\tau - \rho > (<)0$. \square

Appendix 8.

Proof of Conclusion 3.11: Differentiating expressions (3.35) and (3.36) w.r.t. m , and using Condition 3.9, we have

$$\frac{\partial k(s,c)}{\partial m} = \left\{ \frac{a_2\varphi n[a-c(t)]}{\alpha(\tau-\sigma)[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s(t)]^2} \right\} \left\{ \frac{2\mu\tau[v-\vartheta s(t)]}{\tau+\rho} + \frac{[a-c(t)]\{4(a_2\varphi\vartheta m+n-1)(v-\vartheta s(t))+\vartheta(2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s(t))\}}{2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s(t)} \right\} > 0;$$

$$\frac{\partial h(s,c)}{\partial m} = \frac{2a_2\varphi n[a-c(t)][v-\vartheta s(t)]}{\beta(\tau-\delta)[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s(t)]^2} > 0. \quad \square$$

Appendix 9.

Proof of Proposition 3.12: Solving the system (3.37) under the steady state conditions $\dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{c}(t) = \dot{r}(t) = 0$, and denoting the steady-state equilibrium by the superscript "f", we have

$$k^{f'} = \sigma s^{f'}, h^{f'} = \delta c^{f'}, s^{f'} = r^{f'} = \frac{an-[2\beta\delta(\tau-\delta)(a_1-a_2\varphi m\nu)+n]c^{f'}}{\beta\delta(\tau-\delta)(2a_2\varphi m\vartheta+n-1)c^{f'}}, \text{ and } c^{f'} = -\frac{W_2}{3W_1} + \sqrt[3]{-\frac{V}{2} + \sqrt{\left(\frac{V}{2}\right)^2 + \left(\frac{U}{3}\right)^3}} + \sqrt[3]{-\frac{V}{2} - \sqrt{\left(\frac{V}{2}\right)^2 + \left(\frac{U}{3}\right)^3}}, \text{ where } U = \frac{3W_1W_3-W_2^2}{3W_1^2}, V = \frac{2W_2^3-9W_1W_2W_3+27W_1^2W_4}{27W_1^3}, \text{ in which } W_1 = \beta(\tau+\rho)(a_2\varphi\vartheta m+n-1)(2a_2\varphi m\vartheta+n-1)^2\delta^3(\tau-\delta)^3, W_3 = n\sigma\alpha(\tau-\sigma)(\tau+\rho)(2a_2\varphi m\vartheta+n-1)[n+2\beta\delta(\tau-\delta)(a_1-a_2\varphi m\nu)], \text{ and } W_4 = -a\sigma\alpha(\tau-\sigma)(\tau+\rho)(2a_2\varphi m\vartheta+n-1)n^2.$$

Let $J^{f'}$ denote the Jacobian matrix associated with the canonical system (3.37), namely,

$$J^{f'} = \frac{\partial(k,h,s,c,r)}{\partial(k,h,s,c,r)} = \begin{bmatrix} \tau - \sigma & 0 & Y_1 & Y_2 & -\mu Y_2 \\ 0 & \tau - \delta & Y_3 & Y_4 & -\mu Y_3 \\ & 0 & \sigma & 0 & 0 \\ 0 & -1 & 0 & \delta & 0 \\ 0 & 0 & \rho & 0 & -\rho \end{bmatrix} \text{ where } Y_1, Y_2, Y_3, \text{ and } Y_4 \text{ are given by, respectively,}$$

$$Y_1 = \frac{n}{\alpha} \left\{ \frac{\mu[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}] + (a-c^{f'})[(2a_2\varphi m\vartheta+n-1)]}{[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}]^2} \right\} \left\{ \frac{2(a_2\varphi\vartheta m+n-1)(a-c^{f'})}{2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}} + \frac{\mu\tau}{\tau+\rho} \right\} > 0,$$

$$Y_2 = \left\{ \frac{n}{\alpha[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}]} \right\} \left\{ \frac{2(a_2\varphi\vartheta m+n-1)(a-c^{f'})}{2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}} + \frac{\mu\tau}{\tau+\rho} \right\} > 0,$$

$$Y_3 = \frac{n\mu[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}] + (2a_2\varphi m\vartheta+n-1)(a-c^{f'})}{\beta[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}]^2} > 0, \text{ and}$$

$$Y_4 = \frac{n}{\beta[2(a_1-a_2\varphi m\nu)+(2a_2\varphi m\vartheta+n-1)s^{f'}]} > 0.$$

From the above Jacobian matrix $J^{f'}$, one can obtain $T(J^{f'}) = 2\tau - \rho$, $Det(J^{f'}) = \rho\{Y_2Y_3(1 - \mu) - [Y_4 + (\tau - \delta)\delta][Y_1 - \mu Y_2 + (\tau - \sigma)\sigma]\} < 0$. Consequently, it appears that the steady-state equilibrium $\{k^{f'}, h^{f'}, s^{f'}, c^{f'}, r^{f'}\}$ is stable (unstable) in the saddle-point sense iff $2\tau - \rho > (<)0$. \square

Appendix 10.

Proof of Proposition 4.1: The government's Hamiltonian function of optimization problem (4.4) is

$$\begin{aligned}
 H = & (n+1)[a + a_2\varphi(v - \vartheta s(t))Q(t) - \mu(s(t) - r(t))]q(t) - nc(t)q(t) \\
 & - \frac{(2n+1)}{2}[a_1 + (n-1)s(t)]q^2(t) - \frac{\alpha}{2}k^2(t) - \frac{\beta}{2}h^2(t) + \omega_1(t)[-k(t) + \sigma s(t)] \\
 & + \omega_2(t)[-h(t) + \delta c(t)] + \omega_3(t)[\rho(s(t) - r(t))]
 \end{aligned} \tag{A.1}$$

From Hamiltonian function (A.1), first-order conditions and co-state equations are

$$\frac{\partial H}{\partial k(t)} = -\alpha k(t) - \omega_1(t) = 0 \tag{A.2}$$

$$\frac{\partial H}{\partial h(t)} = -\beta h(t) - \omega_2(t) = 0 \tag{A.3}$$

$$\dot{\omega}_1(t) = \tau\omega_1(t) - \frac{\partial H}{\partial s(t)} \tag{A.4}$$

$$= (\tau - \sigma)\omega_1(t) + \frac{(2n+1)(n-1)}{2}q^2(t) + (n+1)[\mu + a_2\varphi\vartheta Q(t)]q(t) - \rho\omega_3(t)$$

$$\dot{\omega}_2(t) = \tau\omega_2(t) - \frac{\partial H}{\partial c(t)} = (\tau - \delta)\omega_2(t) + nq(t) \tag{A.5}$$

$$\dot{\omega}_3(t) = \tau\omega_3(t) - \frac{\partial H}{\partial r(t)} = (\tau + \rho)\omega_3(t) - (n+1)\mu q(t) \tag{A.6}$$

where the transversally conditions are $\lim_{t \rightarrow \infty} \omega_1(t)s(t)e^{-\tau t} = 0$, $\lim_{t \rightarrow \infty} \omega_2(t)c(t)e^{-\tau t} = 0$, and $\lim_{t \rightarrow \infty} \omega_3(t)r(t)e^{-\tau t} = 0$. Now, solving differential equation (A.6) w.r.t. $\omega_3(t)$, and using transversally condition $\lim_{t \rightarrow \infty} \omega_3(t)r(t)e^{-\tau t} = 0$ gives $\omega_3(t) = \frac{(n+1)\mu q(t)}{\tau + \rho}$. From differential equations (A.2)-(A.6), and using $\omega_3(t) = \frac{(n+1)\mu q(t)}{\tau + \rho}$ and output level (3.30), we have

$$\begin{aligned}
 \dot{k}(t) = & (\tau - \sigma)k(t) - \frac{(2n+1)(n-1)}{2\alpha} \left\{ \frac{a+a_2\varphi[v-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)}{2a_1+(n-1)s(t)} \right\}^2 \\
 & - \frac{(n+1)}{\alpha} [\mu + a_2\varphi\vartheta Q(t) - \frac{\rho\mu}{\tau+\rho}] \left\{ \frac{a+a_2\varphi[v-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)}{2a_1+(n-1)s(t)} \right\}
 \end{aligned} \tag{A.7}$$

$$\dot{h}(t) = (\tau - \delta)h(t) - \frac{n\{a + a_2\varphi[v - \vartheta s(t)]Q(t) - \mu[s(t) - r(t)] - c(t)\}}{\beta[2a_1 + (n-1)s(t)]} \tag{A.8}$$

Combining differential equations (2.1)-(2.3) and (A.7)-(A.8) gives

$$\left\{ \begin{aligned}
 \dot{k}(t) &= (\tau - \sigma)k(t) - \frac{(2n+1)(n-1)}{2\alpha} \left\{ \frac{a+a_2\varphi[v-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)}{2a_1+(n-1)s(t)} \right\}^2 \\
 &\quad - \frac{(n+1)}{\alpha} [\mu + a_2\varphi\vartheta Q(t) - \frac{\rho\mu}{\tau+\rho}] \left\{ \frac{a+a_2\varphi[v-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)}{2a_1+(n-1)s(t)} \right\} \\
 \dot{h}(t) &= (\tau - \delta)h(t) - \frac{n\{a+a_2\varphi[v-\vartheta s(t)]Q(t)-\mu[s(t)-r(t)]-c(t)\}}{\beta[2a_1+(n-1)s(t)]} \\
 \dot{s}(t) &= -k(t) + \sigma s(t) \\
 \dot{c}(t) &= -h(t) + \delta c(t) \\
 \dot{r}(t) &= \rho[s(t) - r(t)]
 \end{aligned} \right. \tag{A.9}$$

Solving system (A.9) under steady state conditions $\dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{c}(t) = \dot{r}(t) = 0$, and denoting the steady-state equilibrium by superscript "g", we have $k^g = \sigma s^g$, $h^g = \delta c^g$, $r^g = s^g =$

$-\frac{F_2}{3F_1} + \sqrt[3]{-\frac{M}{2} + \sqrt{(\frac{M}{2})^2 + (\frac{N}{3})^3}} + \sqrt[3]{-\frac{M}{2} - \sqrt{(\frac{M}{2})^2 + (\frac{N}{3})^3}}$, $c^g = \frac{n[a+a_2\varphi(v-\vartheta s^g)Q]}{\beta(\tau-\delta)\delta[2a_1+(n-1)s^g]+n}$, where $M = \frac{3F_1F_3-F_2^2}{3F_1^2}$, $N = \frac{2F_2^3-9F_1F_2F_3+27F_1^2F_4}{27F_1^3}$, in which $F_1 = (\tau - \sigma)\sigma\beta^2\delta^2(n-1)^2(\tau - \delta)^2$, $F_2 = (n-1)\beta\delta(\tau - \delta)\{2(\tau - \sigma)\sigma[2a_1\beta\delta(\tau - \delta) + n] - \frac{\beta\delta(\tau-\delta)(2n+1)a_2^2\vartheta^2\varphi^2Q^2}{2\alpha} - \frac{(n+1)}{\alpha}\beta\delta(\tau - \delta)(\mu + a_2\varphi\vartheta Q - \frac{\rho\mu}{\tau+\rho})a_2\vartheta\varphi Q\}$, $F_3 = \frac{1}{\alpha}(n-1)(a + a_2\varphi Qv)\beta^2\delta^2(\tau - \delta)^2[(2n+1) + (n+1)(\mu + a_2\varphi\vartheta Q - \frac{\rho\mu}{\tau+\rho})] - [2a_1\beta\delta(\tau - \delta) + n]\beta\delta(\tau - \delta)a_2\vartheta\varphi Q[\frac{(n+1)}{\alpha}(\mu + a_2\varphi\vartheta Q - \frac{\rho\mu}{\tau+\rho})] + (\tau - \sigma)\sigma[2a_1\beta\delta(\tau - \delta) + n]^2$, $F_4 = \beta\delta(\tau - \delta)(a + a_2\varphi Qv)\{[\frac{(n+1)}{\alpha}(\mu + a_2\varphi\vartheta Q - \frac{\rho\mu}{\tau+\rho})][2a_1\beta\delta(\tau - \delta) + n] - \beta\delta(\tau - \delta)(a + a_2\varphi Qv)\frac{(2n+1)(n-1)}{2\alpha}\}$.

Let J^g denote the Jacobian matrix associated with the canonical system (A.9), that is,

$$J^g = \frac{\partial(\dot{k}, \dot{h}, \dot{s}, \dot{c}, \dot{r})}{\partial(k, h, s, c, r)} = \begin{bmatrix} \tau - \sigma & 0 & Z_1 & Z_2 & -\mu Z_2 \\ 0 & \tau - \delta & Z_3 & Z_4 & -\mu Z_3 \\ -1 & 0 & \sigma & 0 & 0 \\ 0 & -1 & 0 & \delta & 0 \\ 0 & 0 & \rho & 0 & -\rho \end{bmatrix}$$

where $Z_1 = \{-\frac{(2n+1)(n-1)[a+a_2\varphi(v-\vartheta s^g)Q-c^g]}{[2a_1+(n-1)s^g]} - \frac{(n+1)}{\alpha}[\mu + a_2\varphi\vartheta Q - \frac{\rho\mu}{\tau+\rho}]\}\{\frac{[2a_1a_2\varphi\vartheta Q+2a_1\mu+(n-1)[a+a_2\varphi Qv+\mu r^g-c^g]]}{[2a_1+(n-1)s^g]^2}\}$, $Z_2 = \frac{(2n+1)(n-1)}{\alpha}\{\frac{a+a_2\varphi(v-\vartheta s^g)Q-c^g}{2a_1+(n-1)s^g}\} + \frac{(n+1)}{\alpha}[\mu + a_2\varphi\vartheta Q - \frac{\rho\mu}{\tau+\rho}]\{\frac{a+a_2\varphi(v-\vartheta s^g)Q}{2a_1+(n-1)s^g}\}$, $Z_3 = \frac{n[2a_1a_2\varphi\vartheta Q+2a_1\mu+(n-1)(a+a_2\varphi Qv+\mu r^g-c^g)]}{\beta[2a_1+(n-1)s^g]^2}$, $Z_4 = \frac{n}{\beta[2a_1+(n-1)s^g]}$.

From the Jacobian matrix J^g , we have $T(J^g) = 2\tau - \rho$, $Det(J^g) = \rho\{Z_2Z_3(1 - \mu) - [Z_4 + (\tau - \delta)\delta][Z_1 - \mu Z_2 + (\tau - \sigma)\sigma]\} < 0$. It appears that the steady-state equilibrium $\{k^g, h^g, s^g, c^g, r^g\}$ is stable (unstable) in the saddle-point sense iff $2\tau - \rho > (<)0$. \square

Appendix 11.

Proof of Proposition 4.2: The government's Hamiltonian function of optimization problem (4.8) is

$$\begin{aligned}
 H = & (n+1)[a - \mu(s(t) - r(t))]q(t) - \frac{(2n+1)}{2}[a_1 - a_2\varphi m(v - \vartheta s(t)) \\
 & + (n-1)s(t)]q^2(t) - nc(t)q(t) - \frac{\alpha}{2}k^2(t) - \frac{\beta}{2}h^2(t) + \omega'_1(t)[-k(t) + \sigma s(t)] \\
 & + \omega'_2(t)[-h(t) + \delta c(t)] + \omega'_3(t)[\rho(s(t) - r(t))]
 \end{aligned} \tag{A.10}$$

From Hamiltonian function (A.10), we can get following first-order conditions and co-state equations

$$\frac{\partial H}{\partial k(t)} = -\alpha k(t) - \omega'_1(t) = 0 \tag{A.11}$$

$$\frac{\partial H}{\partial h(t)} = -\beta h(t) - \omega'_2(t) = 0 \tag{A.12}$$

$$\begin{aligned}
 \dot{\omega}'_1(t) &= \tau\omega'_1(t) - \frac{\partial H}{\partial s(t)} \\
 &= (\tau - \sigma)\omega'_1(t) + \frac{(2n+1)}{2}[a_2\varphi m\vartheta + (n-1)]q^2(t) + (n+1)\mu q(t) - \rho\omega'_3(t)
 \end{aligned} \tag{A.13}$$

$$\dot{\omega}'_2(t) = \tau\omega'_2(t) - \frac{\partial H}{\partial c(t)} = (\tau - \delta)\omega'_2(t) + nq(t) \tag{A.14}$$

$$\dot{\omega}'_3(t) = \tau\omega'_3(t) - \frac{\partial H}{\partial r(t)} = (\tau + \rho)\omega'_3(t) - (n+1)\mu q(t) \tag{A.15}$$

where the accompanying set of transversally conditions are $\lim_{t \rightarrow \infty} \omega'_1(t)s(t)e^{-\tau t} = 0$, $\lim_{t \rightarrow \infty} \omega'_2(t)c(t)e^{-\tau t} = 0$, and $\lim_{t \rightarrow \infty} \omega'_3(t)r(t)e^{-\tau t} = 0$. Now, solving differential equation (A.15) w.r.t. $\omega'_3(t)$, and using transversally condition $\lim_{t \rightarrow \infty} \omega'_3(t)r(t)e^{-\tau t} = 0$ gives $\omega'_3(t) = \frac{(n+1)\mu q(t)}{\tau + \rho}$. From above differential equations (A.11)-(A.15), and use output level (3.9) and $\omega'_3(t) = \frac{(n+1)\mu q(t)}{\tau + \rho}$, then given the output level (3.30), that is, $q(t) = \frac{a - \mu[s(t) - r(t)] - c(t)}{2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s(t)}$, one can obtain, respectively,

$$\dot{k}(t) = (\tau - \sigma)k(t) - \frac{(2n+1)[a_2\varphi m\vartheta + (n-1)]}{2\alpha} \left\{ \frac{a - \mu[s(t) - r(t)] - c(t)}{2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s(t)} \right\}^2 - \frac{(n+1)\mu\tau}{\alpha(\tau + \rho)} \left\{ \frac{a - \mu[s(t) - r(t)] - c(t)}{2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s(t)} \right\} \quad (\text{A.16})$$

$$\dot{h}(t) = (\tau - \delta)h(t) - \frac{n\{a - \mu[s(t) - r(t)] - c(t)\}}{\beta[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s(t)]} \quad (\text{A.17})$$

Combining differential equations (2.1)-(2.3) and (A.16)-(A.17) gives following control system:

$$\begin{cases} \dot{k}(t) = (\tau - \sigma)k(t) - \frac{(2n+1)[a_2\varphi m\vartheta + (n-1)]}{2\alpha} \left\{ \frac{a - \mu[s(t) - r(t)] - c(t)}{2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s(t)} \right\}^2 \\ \quad - \frac{(n+1)\mu\tau}{\alpha(\tau + \rho)} \left\{ \frac{a - \mu[s(t) - r(t)] - c(t)}{2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s(t)} \right\} \\ \dot{h}(t) = (\tau - \delta)h(t) - \frac{n\{a - \mu[s(t) - r(t)] - c(t)\}}{\beta[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s(t)]} \\ \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \\ \dot{r}(t) = \rho[s(t) - r(t)] \end{cases} \quad (\text{A.18})$$

Solving the system (A.18) under the steady state conditions $\dot{k}(t) = \dot{h}(t) = \dot{s}(t) = \dot{c}(t) = \dot{r}(t) = 0$, and denoting the steady state equilibrium by the superscript "g'", we have $k^{g'} = \sigma s^{g'}$, $h^{g'} = \delta c^{g'}$, $s^{g'} = r^{g'} \frac{(2n+1)[a_2\varphi m\vartheta + (n-1)](\tau - \delta)^2 \delta^2}{2\sigma\alpha(\tau - \sigma)} (c^{g'})^2 + \frac{(n+1)(\tau - \delta)\mu\tau\delta}{\sigma\alpha(\tau - \sigma)(\tau + \rho)} c^{g'}$, $c^{g'} = -\frac{V_2}{3V_1} + \sqrt{-\frac{E}{2} + \sqrt{\left(\frac{E}{2}\right)^2 + \left(\frac{R}{3}\right)^3}} + \sqrt{-\frac{E}{2} - \sqrt{\left(\frac{E}{2}\right)^2 + \left(\frac{R}{3}\right)^3}}$, where $E = \frac{3V_1V_3 - V_2^2}{3V_1^2}$, $R = \frac{2V_2^3 - 9V_1V_2V_3 + 27V_1^2V_4}{27V_1^3}$, in which $V_1 = \beta(\tau + \rho)(2n + 1)(2a_2\varphi m\vartheta + n - 1)[a_2\varphi m\vartheta + n - 1](\tau - \delta)^3\delta^3$, $V_2 = 2\beta\mu\tau(2a_2\varphi m\vartheta + n - 1)(n + 1)(\tau - \delta)^2\delta^2$, $V_3 = 2\sigma\alpha(\tau + \rho)(\tau - \sigma)[2(a_1 - a_2\varphi m\nu)(\tau - \delta)\beta\delta + n]$, $V_4 = -2na\sigma\alpha(\tau + \rho)(\tau - \sigma)$.

The Jacobian matrix $J^{g'}$ associated with the canonical system (A.18) is given by

$$J^{g'} = \frac{\partial(\dot{k}, \dot{h}, \dot{s}, \dot{c}, \dot{r})}{\partial(k, h, s, c, r)} = \begin{bmatrix} \tau - \sigma & 0 & D_1 & D_2 & -\mu D_2 \\ 0 & \tau - \delta & D_3 & D_4 & -\mu D_3 \\ -1 & 0 & \sigma & 0 & 0 \\ 0 & -1 & 0 & \delta & 0 \\ 0 & 0 & \rho & 0 & -\rho \end{bmatrix}$$

where $D_1 = \frac{(2n+1)[a_2\varphi m\vartheta + (n-1)]}{\alpha} \left\{ \frac{\mu[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s^{g'} + (2a_2\varphi m\vartheta + n - 1)(a - c^{g'})]}{\beta[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s^{g'}]^2} \right\} \frac{(a - c^{g'})}{2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s^{g'}} - \frac{(n+1)\mu\tau}{\alpha(\tau + \rho)}$, $D_2 = \frac{\mu\{(2n+1)[a_2\varphi m\vartheta + (n-1)](a - c^{g'}) + (n+1)\tau\}}{\alpha(\tau + \rho)[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s^{g'}]}$, $D_3 = \frac{n\{\mu[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s^{g'}] + (2a_2\varphi m\vartheta + n - 1)(a - c)\}}{\beta[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s^{g'}]^2} > 0$, $D_4 = \frac{\mu n}{\beta[2(a_1 - a_2\varphi m\nu) + (2a_2\varphi m\vartheta + n - 1)s^{g'}]}$.

From above Jacobian matrix $J^{g'}$, we have $T(J^{g'}) = 2\tau - \rho$, $Det(J^{g'}) = \rho\{D_2D_3(1 - \mu) - [D_4 + (\tau - \delta)\delta][D_1 - \mu D_2 + (\tau - \sigma)\sigma]\} < 0$, from which we can obtain that the steady-state equilibrium $\{k^{g'}, h^{g'}, s^{g'}, c^{g'}, r^{g'}\}$ is stable (unstable) in the saddle-point sense iff $2\tau - \rho > (<) 0$. \square

Appendix 12. Supplementary numerical results

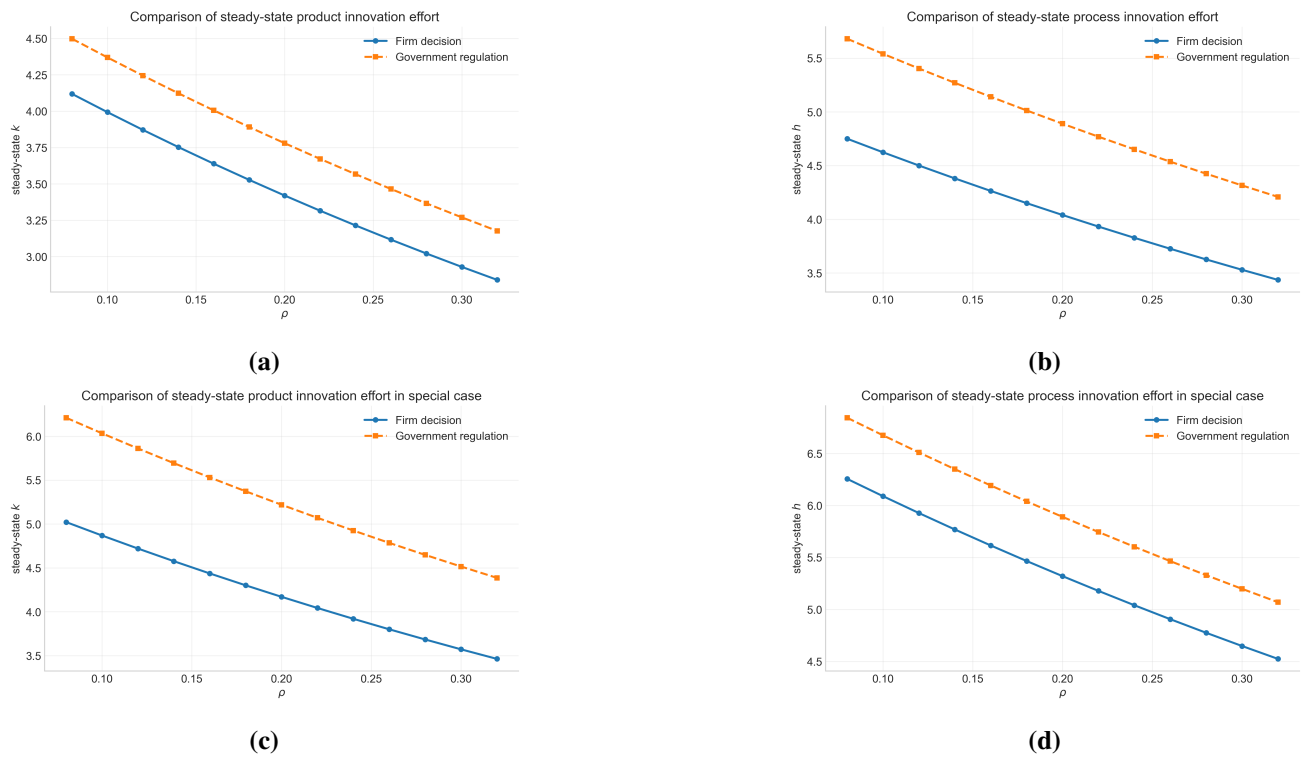


Figure A.1. Comparisons of steady-state innovation levels under firm decision-making and government regulation in the neighborhood of baseline value of consumer memory parameter, ρ .

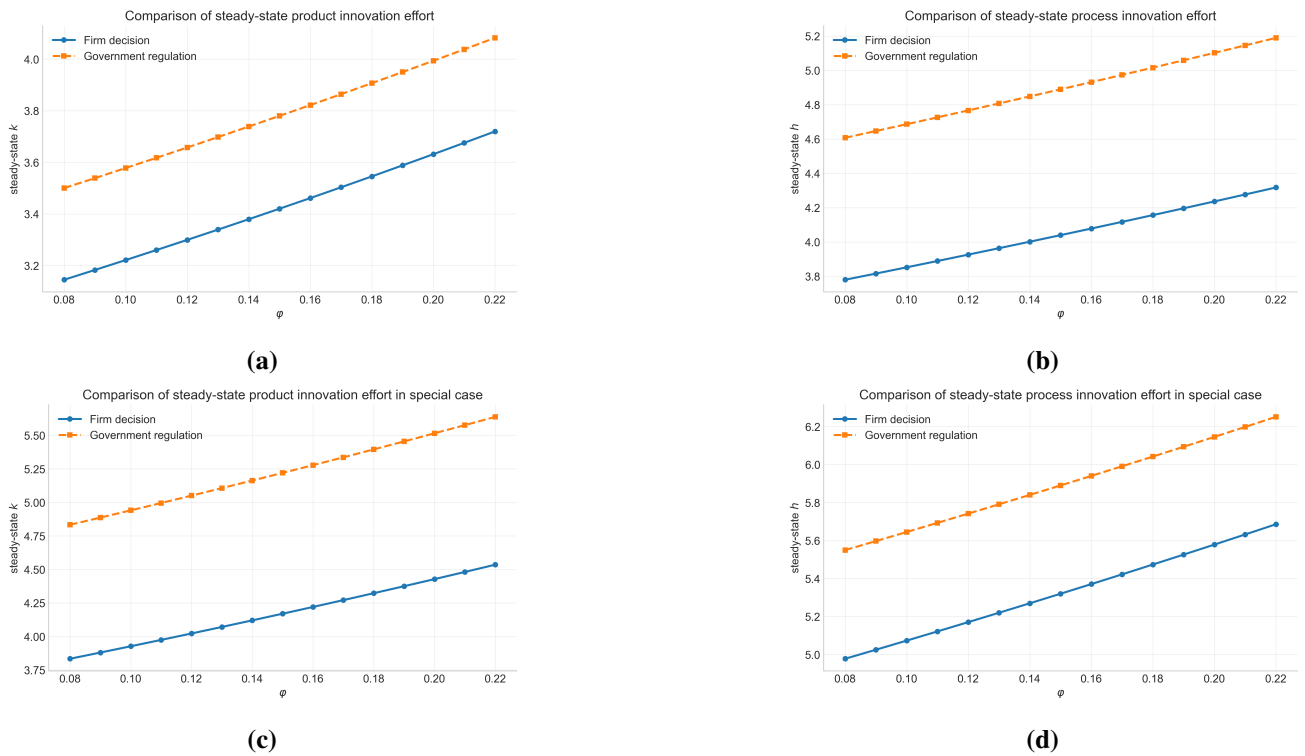


Figure A.2. Comparisons of steady-state innovation levels under firm decision-making and government regulation in the neighborhood of baseline value of network externality intensity, φ .

Appendix 13. The benchmark modelling without reference effects

The problem faced by the multi-product monopolist is given by

$$\begin{aligned} \Pi = & \max_{q_i(t), k(t), h(t)} \int_0^{+\infty} e^{-\tau t} \left\{ \sum_{i=1}^n [a - a_1 q_i(t) + a_2 \varphi (v - \vartheta s(t)) Q_i(t) - s(t) \sum_{j \neq i} q_j(t) - c(t)] q_i(t) - \frac{\alpha}{2} k^2(t) - \frac{\beta}{2} h^2(t) \right\} dt \\ s.t. & \begin{cases} \dot{s}(t) = -k(t) + \sigma s(t) \\ \dot{c}(t) = -h(t) + \delta c(t) \end{cases} \end{aligned} \quad (\text{A.19})$$

The corresponding Hamiltonian function writes

$$\begin{aligned} H = & \sum_{i=1}^n [a - a_1 q_i(t) + a_2 \varphi (v - \vartheta s(t)) Q_i(t) - s(t) \sum_{j \neq i} q_j(t) - c(t)] q_i(t) - \frac{\alpha}{2} k^2(t) - \frac{\beta}{2} h^2(t) \\ & + \omega_1(t) [-k(t) + \sigma s(t)] + \omega_2(t) [-h(t) + \delta c(t)] \end{aligned} \quad (\text{A.20})$$

Imposing symmetric condition $q_i(t) = q_j(t) = q(t)$, the first-order conditions and co-state equations are:

$$\begin{aligned} \frac{\partial H}{\partial q(t)} = 0 & \Rightarrow a - c(t) - 2\{a_1 + (n-1)s(t) - a_2 \varphi n [v - \vartheta s(t)]\} q(t) = 0 \\ \Rightarrow q^*(t) & = \frac{a - c(t)}{2\{a_1 + (n-1)s(t) - a_2 \varphi n [v - \vartheta s(t)]\}} \end{aligned} \quad (\text{A.21})$$

$$\frac{\partial H}{\partial k(t)} = 0 \Rightarrow -\alpha k(t) - \omega_1(t) = 0 \Rightarrow k^*(t) = -\frac{\omega_1(t)}{\alpha} \quad (\text{A.22})$$

$$\frac{\partial H}{\partial h(t)} = 0 \Rightarrow -\beta h(t) - \omega_2(t) = 0 \Rightarrow h^*(t) = -\frac{\omega_2(t)}{\beta} \quad (\text{A.23})$$

$$\dot{\omega}_1(t) = \tau\omega_1(t) - \frac{\partial H}{\partial s(t)} = (\tau - \sigma)\omega_1(t) - nq^2(t)[a_2\varphi n\vartheta - (n - 1)] \quad (\text{A.24})$$

$$\dot{\omega}_2(t) = \tau\omega_2(t) - \frac{\partial H}{\partial c(t)} = (\tau - \delta)\omega_2(t) + nq(t) \quad (\text{A.25})$$

Using the optimal strategies and imposing stationary conditions on co-state dynamics, one can acquire the following algebra system determining the steady-state equilibrium:

$$\begin{cases} q^E = \frac{\beta\delta(\tau-\delta)}{n}c^E \\ q^E = \frac{a-c^E}{2[a_1+(n-1)s^E-a_2\varphi n\{v-\vartheta s^E\}]} \\ \alpha\sigma(\tau-\sigma)s^E = n(q^E)^2[n-1-a_2\varphi n\nu] \\ k^E = \sigma s^E \\ h^E = \delta c^E \end{cases} \quad (\text{A.26})$$

which is irrelevant with the reference effects regarding the perceptions of horizontal differentiation. Compared with the benchmark model without reference effects, the steady-state solution in the baseline model depends on the parameters governing the reference adjustment process. This confirms that the reference effects influence steady-state outcomes through the dynamic system, even though the reference gap vanishes in steady state.



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