



Research article

Systemic resilience in heterogeneous supply chains: optimal targeted interventions against risk contagion

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Appendix

Proof of Lemma 2.2

Proof. By summing the four compartmental equations in system (2.1), the evolution of the total enterprise population $N_k(t) = S_k(t) + E_k(t) + I_k(t) + R_k(t)$ for each degree class k is governed by the following linear differential equation:

$$\frac{dN_k(t)}{dt} = b - dN_k(t) - \mu I_k(t). \quad (\text{A.1})$$

Since the parameters μ, d and the state variable $I_k(t)$ are non-negative, the growth of the population is bounded by the following differential inequality:

$$\frac{dN_k(t)}{dt} \leq b - dN_k(t). \quad (\text{A.2})$$

To invoke the comparison principle, we consider the associated linear auxiliary equation

$$\frac{d\bar{N}_k(t)}{dt} = b - d\bar{N}_k(t), \quad \text{with } \bar{N}_k(0) = N_k(0). \quad (\text{A.3})$$

The closed-form solution to (A.3) is readily obtained as:

$$\bar{N}_k(t) = \frac{b}{d} + \left(N_k(0) - \frac{b}{d} \right) e^{-dt}. \quad (\text{A.4})$$

According to the standard comparison theorem for ordinary differential equations, the trajectory $N_k(t)$ satisfies $N_k(t) \leq \bar{N}_k(t)$ for all $t \geq 0$. It follows from (A.4) that if $N_k(0) \leq b/d$, then $\bar{N}_k(t)$ is monotonically non-increasing (for $N_k(0) > b/d$) or bounded by b/d (for $N_k(0) \leq b/d$). Specifically, the condition $N_k(0) \leq b/d$ implies $\bar{N}_k(t) \leq b/d$ for all $t \geq 0$, which yields

$$N_k(t) \leq \frac{b}{d}, \quad \forall t \geq 0. \quad (\text{A.5})$$

Furthermore, by examining the vector field of (2.1) on the boundaries of the non-negative orthant \mathbb{R}_+^{4n} , it is clear that the trajectories cannot cross the axes, ensuring the non-negativity of the state variables. Therefore, the region

$$\Omega = \left\{ (S_1, E_1, I_1, R_1, \dots, S_n, E_n, I_n, R_n) \in \mathbb{R}_+^{4n} : \sum_{X \in \{S, E, I, R\}} X_k(t) \leq \frac{b}{d}, \quad k = 1, \dots, n \right\}$$

is positively invariant under the flow induced by system (2.1). This completes the proof of Lemma 2.2. \square

Proof of Theorem 2.3

Proof. The existence and uniqueness of the risk-endemic equilibrium $\mathbf{E}^+ = (S_k^+, E_k^+, I_k^+, R_k^+)$ are established by analyzing the self-consistent equation for the steady-state contagion pressure Θ^* . In a heterogeneous network, Θ^* represents the connectivity-weighted average of distressed enterprises, defined as

$$\Theta^* = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{I_k^+}{N_k^+}, \quad \text{where} \quad N_k^+ = S_k^+ + E_k^+ + I_k^+ + R_k^+.$$

A non-trivial endemic state exists if and only if there resides a unique positive solution $\Theta^* \in (0, 1]$ such that system (2.1) is in a steady state. Setting the governing equations to zero, we obtain the stationary compartmental algebraic system

$$b - \alpha k S_k \Theta^* + \delta R_k - d S_k = 0, \quad (\text{A.6})$$

$$\alpha k S_k \Theta^* - (\beta + d) E_k = 0, \quad (\text{A.7})$$

$$\beta E_k - (\omega + d + \mu) I_k = 0, \quad (\text{A.8})$$

$$\omega I_k - (\delta + d) R_k = 0. \quad (\text{A.9})$$

By solving system (A.6)–(A.9) sequentially in terms of the distressed population I_k , we derive the following proportional relationships:

$$R_k = \frac{\omega}{\delta + d} I_k, \quad E_k = \frac{\omega + d + \mu}{\beta} I_k, \quad S_k = \frac{(\beta + d)(\omega + d + \mu)}{\alpha \beta k \Theta^*} I_k.$$

Summing these components, the total degree- k enterprise population N_k is expressed as

$$N_k = I_k \left[\frac{(\beta + d)(\omega + d + \mu)}{\alpha\beta k\Theta^*} + \frac{\omega + d + \mu}{\beta} + 1 + \frac{\omega}{\delta + d} \right].$$

To simplify the notation, let $\mathcal{A} = (\beta + d)(\omega + d + \mu)$ and $\mathcal{B} = \frac{\omega + d + \mu}{\beta} + 1 + \frac{\omega}{\delta + d}$. The prevalence ratio within the degree- k stratum, $\phi_k(\Theta^*) = I_k/N_k$, is then characterized by

$$\phi_k(\Theta^*) = \frac{\alpha\beta k\Theta^*}{\mathcal{A} + \alpha\beta k\mathcal{B}\Theta^*}.$$

Substituting $\phi_k(\Theta^*)$ into the definition of Θ^* , we obtain the self-consistent characteristic equation

$$\Theta^* = \frac{1}{\langle k \rangle} \sum_{k=1}^n k P_k \frac{\alpha\beta k\Theta^*}{\mathcal{A} + \alpha\beta k\mathcal{B}\Theta^*}.$$

We exclude the trivial solution $\Theta^* = 0$, which corresponds to the distress-free equilibrium \mathbf{E}^0 . For a non-trivial equilibrium ($\Theta^* > 0$), we divide both sides by Θ^* to define the function $f(\Theta^*)$:

$$f(\Theta^*) := \frac{\alpha\beta}{\langle k \rangle} \sum_{k=1}^n \frac{k^2 P_k}{\mathcal{A} + \alpha\beta k\mathcal{B}\Theta^*} = 1.$$

We analyze the analytical properties of $f(\Theta^*)$ on the interval $[0, \infty)$:

1. Since the denominator $\mathcal{A} + \alpha\beta k\mathcal{B}\Theta^*$ is a strictly increasing linear function of Θ^* for all $k \geq 1$, $f(\Theta^*)$ is a continuous and strictly decreasing function on $[0, \infty)$.
2. At the limit of vanishing distress, the function yields

$$f(0) = \frac{\alpha\beta \sum_{k=1}^n k^2 P_k}{\langle k \rangle \mathcal{A}} = \frac{\alpha\beta \langle k^2 \rangle}{\langle k \rangle (\beta + d)(\omega + d + \mu)} = \mathcal{R}_0.$$

As the contagion pressure becomes extreme, the function asymptotically approaches zero:
 $\lim_{\Theta^* \rightarrow \infty} f(\Theta^*) = 0.$

By the intermediate value theorem, the characteristic equation $f(\Theta^*) = 1$ possesses a unique positive solution $\Theta^* \in (0, \infty)$ if and only if $f(0) > 1$, which is precisely the condition $\mathcal{R}_0 > 1$. Given the unique determination of Θ^* , the steady-state values $(S_k^+, E_k^+, I_k^+, R_k^+)$ are uniquely recovered for each degree class k . This concludes the proof. \square

Proof of Theorem 3.1

Proof. To evaluate the local stability of the distress-free equilibrium \mathbf{E}^0 , we examine the linearized dynamics of the distressed subsystem consisting of the exposed (E_k) and infected (I_k) compartments. The Jacobian of this $2n$ -dimensional subsystem at \mathbf{E}^0 is given by

$$\mathcal{J} := \mathcal{M} - \mathcal{N},$$

where $\mathcal{M} \geq 0$ is the transmission (new infection) matrix and \mathcal{N} is the compartmental transition matrix. By properties (C1)–(C5), \mathcal{N} is a non-singular M-matrix: its off-diagonal entries are non-positive, and its eigenvalues have strictly positive real parts due to exit and recovery rates (d, ω, μ) , which ensure diagonal dominance.

The stability of $\dot{\mathbf{x}} = \mathcal{J} \mathbf{x}$ is determined by the spectral abscissa $s(\mathcal{J}) = \max\{\operatorname{Re}(\lambda) : \lambda \in \sigma(\mathcal{J})\}$. For a non-singular M-matrix \mathcal{N} and a non-negative matrix \mathcal{M} , the Perron–Frobenius theory and M-matrix theory [42] give the standard equivalence

$$s(\mathcal{M} - \mathcal{N}) < 0 \iff \rho(\mathcal{M} \mathcal{N}^{-1}) < 1.$$

By definition, the basic reproduction number is $\mathcal{R}_0 = \rho(\mathcal{M} \mathcal{N}^{-1})$. Therefore, if $\mathcal{R}_0 < 1$, then $s(\mathcal{J}) < 0$; all eigenvalues of \mathcal{J} lie in the open left half-plane, so \mathbf{E}^0 is locally asymptotically stable. If $\mathcal{R}_0 > 1$, then $s(\mathcal{J}) > 0$; at least one eigenvalue has a positive real part, so \mathbf{E}^0 is unstable.

This completes the proof. □



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