



Research article

Digital innovation and product innovation in a monopoly exhibiting network externality: A dynamic analysis

Shengbiao Ma¹, Huiquan Li^{2,*} and Ran Jiang^{3,*}

- ¹ Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai 200052, P.R.China
- ² School of Finance and Economics, Qinghai University, Xining 810016, Qinghai, P. R. China
- ³ School of International Business, Shanghai University of International Business and Economics, Shanghai, 201620, P. R. China

* Correspondence: Email: 2025990002@qhu.edu.cn; jiangran@suibe.edu.cn.

Supplementary

Appendix

A. Proof of Proposition 3 :

(i) The reaction of the effort rate in product innovation to a change in digital innovation is $\frac{\partial k(t)}{\partial h(t)} = \frac{\partial k(t)}{\partial \theta(t)} \frac{\partial \theta(t)}{\partial h(t)}$. Using the kinematic equation (2.2), one can obtain $\frac{\partial \theta(t)}{\partial h(t)} = \frac{\partial \int_0^t \dot{\theta}(s) ds}{\partial h(t)} = t > 0$. From the kinematic equation (3.16), one can obtain $\frac{\partial k(t)}{\partial \theta(t)} = -\frac{b_2 b_3 (\rho + \delta)}{b_1 \beta_1 [2(\rho + \delta) - \alpha b_4]} < 0$, so we have $\frac{\partial k(t)}{\partial h(t)} < 0$. (ii) Similarly, one can write $\frac{\partial h(t)}{\partial k(t)} = \frac{\partial h(t)}{\partial q(t)} \frac{\partial q(t)}{\partial k(t)}$. Using the kinematic equation (2.1), we have $\frac{\partial q(t)}{\partial k(t)} = \frac{\partial \int_0^t \dot{q}(s) ds}{\partial k(t)} = t > 0$. According to the kinematic equation (3.17), one can obtain $\frac{\partial h(t)}{\partial q(t)} = -\frac{b_2 b_3 (\rho + \delta)}{b_1 \beta_2 [2(\rho + \delta) - \alpha b_4]} < 0$, so we have $\frac{\partial h(t)}{\partial k(t)} < 0$.

B. Proof of Proposition 4 :

From equations (3.19) and (3.20), we have $\frac{\partial k^m(q, \theta, Q)}{\partial q(t)} / \frac{\partial h^m(q, \theta, Q)}{\partial q(t)} = \frac{b_2 \beta_2 (\rho + \sigma_2)}{\beta_1 b_3 (\rho + \sigma_1)} > 0$; $\frac{\partial k^m(q, \theta, Q)}{\partial \theta(t)} / \frac{\partial h^m(q, \theta, Q)}{\partial \theta(t)} = \frac{b_2 \beta_2 (\rho + \sigma_2)}{\beta_1 b_3 (\rho + \sigma_1)} > 0$; $\frac{\partial k^m(q, \theta, Q)}{\partial Q(t)} / \frac{\partial h^m(q, \theta, Q)}{\partial Q(t)} = \frac{b_2 \beta_2 (\rho + \sigma_2)}{\beta_1 b_3 (\rho + \sigma_1)} > 0$. From which one can obtain that the following

relationship always exists $\frac{\partial k^m(q,\theta,Q)}{\partial q(t)} / \frac{\partial h^m(q,\theta,Q)}{\partial q(t)} = \frac{\partial k^m(q,\theta,Q)}{\partial \theta(t)} / \frac{\partial h^m(q,\theta,Q)}{\partial \theta(t)} = \frac{\partial k^m(q,\theta,Q)}{\partial Q(t)} / \frac{\partial h^m(q,\theta,Q)}{\partial Q(t)} > 0$.

C. Proof of Proposition 5 :

Solving the system (3.20) under stationarity conditions $\dot{k}(t) = \dot{h}(t) = \dot{q}(t) = \dot{\theta}(t) = \dot{Q}(t) = 0$, one can obtain the steady-state equilibrium $\{k^m, h^m, q^m, \theta^m, Q^m\}$, where $k^m = \sigma_1 q^m$, $h^m = \sigma_2 \theta^m$, $q^m = \frac{a(\rho+\sigma_2)\sigma_2 M_1}{[(\rho+\sigma_2)\sigma_2 - b_3 M_2][(\rho+\sigma_1)\sigma_1 - b_2 M_1] - M_2 M_1 b_2 b_3}$, $\theta^m = \frac{[(\rho+\sigma_1)\sigma_1 - b_2 M_1]q^m - a M_1}{M_1 b_3}$, $Q^m = \frac{\alpha(\rho+\delta)(a+b_2 q^m + b_3 \theta^m)}{2\delta(\rho+\delta) - \alpha b_4(2\delta+\rho)}$, $M_1 = \frac{\delta b_2(\rho+\delta)}{b_1 \beta_1 [2\delta(\rho+\delta) - \alpha b_4(2\delta+\rho)]}$, $M_2 = \frac{\delta b_3(\rho+\delta)}{b_1 \beta_2 [2\delta(\rho+\delta) - \alpha b_4(2\delta+\rho)]}$.

Let $J^m = \frac{\partial(\dot{k}, \dot{h}, \dot{q}, \dot{\theta}, \dot{Q})}{\partial(k, h, q, \theta, Q)}$ denote the Jacobian matrix of the dynamic control system (3.20) evaluated at the steady-state equilibrium $\{k^m, h^m, q^m, \theta^m, Q^m\}$, namely,

$$J^m = \frac{\partial(\dot{k}, \dot{h}, \dot{q}, \dot{\theta}, \dot{Q})}{\partial(k, h, q, \theta, Q)} = \begin{bmatrix} \rho + \sigma_1 & 0 & -\frac{b_2^2 M}{\beta_1} & -\frac{b_2 b_3 M}{\beta_1} & -\frac{b_2 b_4 M}{\beta_1} \\ 0 & \rho + \sigma_2 & -\frac{b_2 b_3 M}{\beta_2} & -\frac{b_3^2 M}{\beta_2} & -\frac{b_3 b_4 M}{\beta_2} \\ 1 & 0 & \sigma_1 & 0 & 0 \\ 0 & 1 & 0 & \sigma_2 & 0 \\ 0 & 0 & \frac{\alpha b_2(\rho+\delta)}{2(\rho+\delta) - \alpha b_4} & \frac{\alpha b_3(\rho+\delta)}{2(\rho+\delta) - \alpha b_4} & -Y \end{bmatrix}$$

where $M = \frac{(\rho+\delta)}{b_1 [2(\rho+\delta) - \alpha b_4]}$, and Y is given in Appendix D.

To establish the saddle point property, it is necessary to show that the Jacobian matrix J^m of dynamic control system (3.20) has exactly one negative real eigenvalue. This condition is satisfied when $\text{Det}(J^m) < 0$ and $T(J^m) > 0$, meaning that the determinant of J^m is negative while its trace is positive.

From the Jacobian matrix J^m , we can obtain $T(J^m) = \frac{[2(\rho+\sigma_1+\sigma_2)-\delta][2(\rho+\delta)-\alpha b_4]+\alpha b_4(\delta+\rho)}{2(\rho+\delta)-\alpha b_4}$, $\text{Det}(J^m) = -(\rho + \sigma_1)(\rho + \sigma_2)\{\sigma_1 \sigma_2 Y + Y M [\frac{\sigma_2 b_2^2}{\beta_1(\rho+\sigma_1)} + \frac{\sigma_1 b_3^2}{\beta_2(\rho+\sigma_2)}] + \frac{\alpha(\rho+\delta)b_4 M}{2(\rho+\delta)-\alpha b_4} [\frac{\sigma_1 b_3^2}{\beta_2(\rho+\sigma_2)} + \frac{\sigma_2 b_2^2}{\beta_1(\rho+\sigma_1)}]\} < 0$. Consequently, it appears that the steady-state equilibrium $\{k^m, h^m, q^m, \theta^m, Q^m\}$ is a unique saddle-point equilibrium iff $\sigma_1 + \sigma_2 > \bar{\sigma}$, where $\bar{\sigma} = \frac{2\delta[(\rho+\delta)-\alpha b_4]-\rho[4(\rho+\delta)-\alpha b_4]}{2[2(\rho+\delta)-\alpha b_4]}$.

D. Proof of Proposition 6 :

The government's Hamiltonian function of optimization problem (4.4) is

$$H = \frac{(\rho + \delta)[3(\rho + \delta) - 2\alpha b_4][a + b_2 q(t) + b_3 \theta(t) + b_4 Q(t) - b_1 c]^2}{2b_1 [2(\rho + \delta) - \alpha b_4]^2} - \frac{\beta_1}{2} k^2(t) - \frac{\beta_2}{2} h^2(t) + \omega_1(t)[k(t) - \sigma_1 q(t)] + \omega_2(t)[h(t) - \sigma_2 \theta(t)] + \omega_3(t) \left\{ \frac{\alpha(\rho + \delta)[a - b_1 c + b_2 q(t) + b_3 \theta(t)] - [2\delta(\rho + \delta) - \alpha b_4(2\delta + \rho)]Q(t)}{2(\rho + \delta) - \alpha b_4} \right\} \quad (\text{D.1})$$

From Hamiltonian function (D.1), first-order conditions and co-state equations are

$$\frac{\partial H}{\partial k(t)} = -\beta_1 k(t) + \omega_1(t) = 0 \quad (\text{D.2})$$

$$\frac{\partial H}{\partial h(t)} = -\beta_2 h(t) + \omega_2(t) = 0 \quad (\text{D.3})$$

$$\begin{aligned} \dot{\omega}_1(t) &= \rho\omega_1(t) - \frac{\partial H}{\partial q(t)} = (\rho + \sigma_1)\omega_1(t) - \frac{b_2(\rho + \delta)[3(\rho + \delta) - 2ab_4][a - b_1c + b_2q(t) + b_3\theta(t) + b_4Q(t)]}{b_1[2(\rho + \delta) - ab_4]^2} \\ &\quad - \frac{ab_2(\rho + \delta)}{2(\rho + \delta) - ab_4}\omega_3(t) \end{aligned} \quad (D.4)$$

$$\begin{aligned} \dot{\omega}_2(t) &= \rho\omega_2(t) - \frac{\partial H}{\partial \theta(t)} = (\rho + \sigma_2)\omega_2(t) - \frac{b_3(\rho + \delta)[3(\rho + \delta) - 2ab_4][a - b_1c + b_2q(t) + b_3\theta(t) + b_4Q(t)]}{b_1[2(\rho + \delta) - ab_4]^2} \\ &\quad - \frac{b_3\alpha(\rho + \delta)}{2(\rho + \delta) - ab_4}\omega_3(t) \end{aligned} \quad (D.5)$$

$$\begin{aligned} \dot{\omega}_3(t) &= \rho\omega_3(t) - \frac{\partial H}{\partial Q(t)} \\ &= \frac{2(\rho + \delta)(\rho + \delta - ab_4)}{2(\rho + \delta) - ab_4}\omega_3(t) - \frac{b_4(\rho + \delta)[3(\rho + \delta) - 2ab_4][a - b_1c + b_2q(t) + b_3\theta(t) + b_4Q(t)]}{b_1[2(\rho + \delta) - ab_4]^2} \end{aligned} \quad (D.6)$$

where the transversality conditions are $\lim_{t \rightarrow \infty} \omega_1(t)q(t)e^{-\rho t} = 0$, $\lim_{t \rightarrow \infty} \omega_2(t)\theta(t)e^{-\rho t} = 0$, and $\lim_{t \rightarrow \infty} \omega_3(t)Q(t)e^{-\rho t} = 0$. Now, solving the differential equation (D.6) w.r.t. $\omega_3(t)$, and using the transversality condition $\lim_{t \rightarrow \infty} \omega_3(t)Q(t)e^{-\rho t} = 0$ gives $\omega_3(t) = \frac{b_4[3(\rho + \delta) - 2ab_4][a - b_1c + b_2q(t) + b_3\theta(t) + b_4Q(t)]}{2b_1(\rho + \delta - ab_4)[2(\rho + \delta) - ab_4]}$. From differential equations (D.2)–(D.6), and using $\omega_3(t) = \frac{b_4[3(\rho + \delta) - 2ab_4][a - b_1c + b_2q(t) + b_3\theta(t) + b_4Q(t)]}{2b_1(\rho + \delta - ab_4)[2(\rho + \delta) - ab_4]}$, we have

$$\dot{k}(t) = (\rho + \sigma_1)k(t) - \frac{b_2(\rho + \delta)[3(\rho + \delta) - 2ab_4][a + b_2q(t) + b_3\theta(t) + b_4Q(t) - b_1c]}{2\beta_1b_1(\rho + \delta - ab_4)[2(\rho + \delta) - ab_4]} \quad (D.7)$$

$$\dot{h}(t) = (\rho + \sigma_2)h(t) - \frac{b_3(\rho + \delta)[3(\rho + \delta) - 2ab_4][a + b_2q(t) + b_3\theta(t) + b_4Q(t) - b_1c]}{2\beta_2b_1(\rho + \delta - ab_4)[2(\rho + \delta) - ab_4]} \quad (D.8)$$

Combining the dynamic constraints (2.1), (2.2), and (3.14) as well as the dynamic equations (D.7) and (D.8), we have the following dynamic control system:

$$\left\{ \begin{aligned} \dot{k}(t) &= (\rho + \sigma_1)k(t) - \frac{b_2(\rho + \delta)[3(\rho + \delta) - 2ab_4]}{2\beta_1b_1(\rho + \delta - ab_4)[2(\rho + \delta) - ab_4]} [a + b_2q(t) + b_3\theta(t) + b_4Q(t) - b_1c] \\ \dot{h}(t) &= (\rho + \sigma_2)h(t) - \frac{b_3(\rho + \delta)[3(\rho + \delta) - 2ab_4]}{2\beta_2b_1(\rho + \delta - ab_4)[2(\rho + \delta) - ab_4]} [a + b_2q(t) + b_3\theta(t) + b_4Q(t) - b_1c] \\ \dot{q}(t) &= k(t) - \sigma_1q(t) \\ \dot{\theta}(t) &= h(t) - \sigma_2\theta(t) \\ \dot{Q}(t) &= \frac{\alpha(\rho + \delta)[a + b_2q(t) + b_3\theta(t) - b_1c] - [2\delta(\rho + \delta) - ab_4(2\delta + \rho)]Q(t)}{2(\rho + \delta) - ab_4} \end{aligned} \right. \quad (D.9)$$

Solving the system (D.9) under stationarity conditions $\dot{k}(t) = \dot{h}(t) = \dot{q}(t) = \dot{\theta}(t) = \dot{Q}(t) = 0$, and denoting the steady state equilibrium by superscript “g”, we have $k^g = \sigma_1q^g$, $h^g = \sigma_2\theta^g$, $q^g = \frac{(\rho + \sigma_2)\sigma_2Y_1\beta_2 + b_3X[a - b_1c - (b_3 + b_4X_2)Y_2 + b_4X_3]}{(\rho + \sigma_2)\sigma_2Y_1\beta_2 - b_3X[b_2 + b_4X_1 + (b_3 + b_4X_2)Y_1]}$, $\theta^g = Y_1q^g - Y_2$, $Q^m = \frac{\alpha(\rho + \delta)(a - b_1c + b_2q^g + b_3\theta^g)}{2\delta(\rho + \delta) - ab_4(2\delta + \rho)}$, $X = \frac{(\rho + \delta)[3(\rho + \delta) - 2ab_4]}{2b_1(\rho + \delta - ab_4)[2(\rho + \delta) - ab_4]}$, $Y = \frac{[2\delta(\rho + \delta) - ab_4(2\delta + \rho)]}{2(\rho + \delta) - ab_4}$, $X_1 = \frac{\alpha(\rho + \delta)b_2}{Y[2(\rho + \delta) - ab_4]}$, $X_2 = \frac{\alpha(\rho + \delta)b_3}{Y[2(\rho + \delta) - ab_4]}$, $X_3 = \frac{\alpha(\rho + \delta)(a - b_1c)}{Y[2(\rho + \delta) - ab_4]}$, $Y_1 = \frac{\beta_1(\rho + \sigma_1)\sigma_1}{b_2X(b_3 + b_4X_2)} - \frac{(b_2 + b_4X_1)}{(b_3 + b_4X_2)}$, $Y_2 = \frac{(a - b_1c + b_4X_3)}{(b_3 + b_4X_2)}$.

Let J^s denote the Jacobian matrix associated with the canonical system (D.9), that is,

$$J^s = \frac{\partial(\dot{k}, \dot{h}, \dot{q}, \dot{\theta}, \dot{Q})}{\partial(k, h, q, \theta, Q)} = \begin{bmatrix} \rho + \sigma_1 & 0 & -\frac{b_2^2 X}{\beta_1} & -\frac{b_2 b_3 X}{\beta_1} & -\frac{b_2 b_4 X}{\beta_1} \\ 0 & \rho + \sigma_2 & -\frac{b_2 b_3 X}{\beta_2} & -\frac{b_3^2 X}{\beta_2} & -\frac{b_3 b_4 X}{\beta_2} \\ 1 & 0 & \sigma_1 & 0 & 0 \\ 0 & 1 & 0 & \sigma_2 & 0 \\ 0 & 0 & \frac{ab_2(\rho+\delta)}{2(\rho+\delta)-ab_4} & \frac{ab_3(\rho+\delta)}{2(\rho+\delta)-ab_4} & -Y \end{bmatrix}$$

From the Jacobian matrix J^s , we have $T(J^s) = \frac{[2(\rho+\sigma_1+\sigma_2)-\delta][2(\rho+\delta)-ab_4]+ab_4(\delta+\rho)}{2(\rho+\delta)-ab_4}$, $Det(J^s) = -(\rho + \sigma_1)(\rho + \sigma_2)\{\sigma_1\sigma_2 Y + YX[\frac{\sigma_2 b_2^2}{\beta_1(\rho+\sigma_1)} + \frac{\sigma_1 b_3^2}{\beta_2(\rho+\sigma_2)}] + \frac{\alpha(\rho+\delta)b_4 X}{2(\rho+\delta)-ab_4}[\frac{\sigma_1 b_3^2}{\beta_2(\rho+\sigma_2)} + \frac{\sigma_2 b_2^2}{\beta_1(\rho+\sigma_1)}]\} < 0$. It appears that the steady-state equilibrium $\{k^s, h^s, q^s, \theta^s, Q^s\}$ is a unique saddle-point equilibrium iff $\sigma_1 + \sigma_2 > \bar{\sigma}$, where $\bar{\sigma} = \frac{2\delta[(\rho+\delta)-ab_4]-\rho[4(\rho+\delta)-ab_4]}{2[2(\rho+\delta)-ab_4]}$.

E. Sensitivity analysis:

During the analysis, key parameters a , α , and δ are systematically varied by -40%, -20%, 0, 20%, and 40% relative to their baseline values to examine the resulting dynamic trajectories. We focus on how these parameter changes affect the efforts devoted to product innovation and digital innovation under monopolist decision-making and government regulation. Figures 3–5 show that the time paths of both innovation efforts remain qualitatively unchanged across parameter values. This consistency indicates that the major results are robust to parameter variations within the range considered, thereby reinforcing the reliability of the conclusions reported in Table 1.

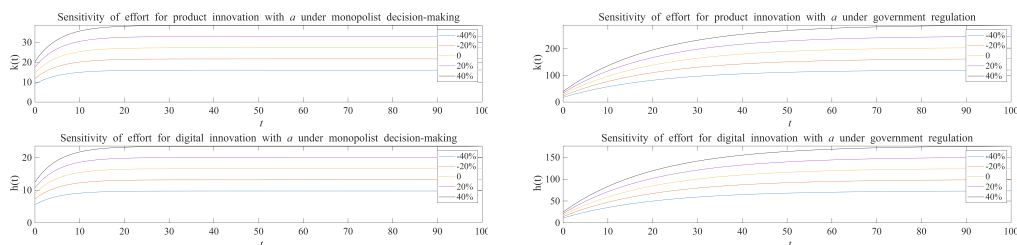


Figure 3. Sensitivity of efforts for product innovation and digital innovation with α .

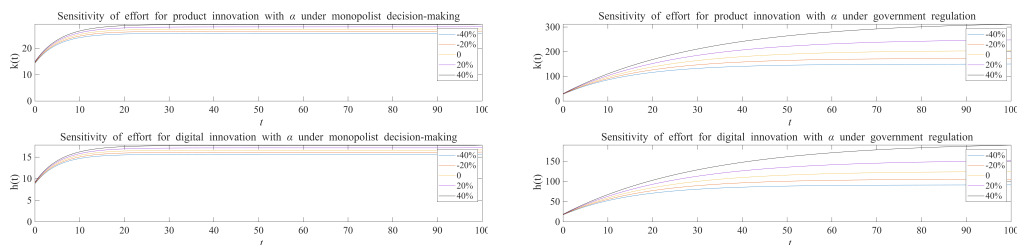


Figure 4. Sensitivity of efforts for product innovation and digital innovation with α .

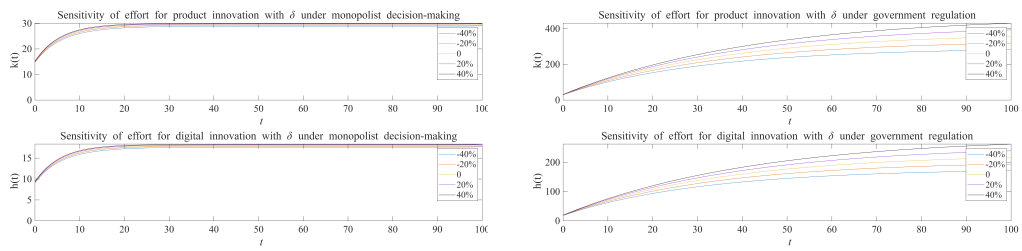


Figure 5. Sensitivity of efforts for product innovation and digital innovation with δ .



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