



Research article

A correlated Heston's stochastic volatility model: A binomial tree approach

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A. Convergence in distribution of tree approach to Heston model

Fleming and Soner [15] proved weak convergence of discrete Markov chain approximations to one-dimensional Hamilton–Jacobi–Bellman partial differential equations. The present appendix adapts this framework to the binomial tree method studied in the main text and provides a corresponding proof of weak convergence.

Consider the stochastic processes governed by the SDEs

$$\begin{aligned}dX &= f_X(X, Y, t)dt + g_X(X, Y, t)dW_X, \\dY &= f_Y(X, Y, t)dt + g_Y(X, Y, t)dW_Y,\end{aligned}\tag{A.1}$$

where the Brownian motions W_X and W_Y satisfy the correlation condition

$$dW_X dW_Y = \rho dt.$$

Let (X_t, Y_t) be the continuous stochastic process and $(X_t^{(n)}, Y_t^{(n)})$ be the n -step binomial tree approximated stochastic process.

The proof consists of two steps. First, local weak convergence is proved. Then, the weak convergence at expiry is proved using the Lévy continuity theorem.

A.1. Local convergence in distribution

Consider the binomial tree approximation of the continuous stochastic process (X_t, Y_t) , denoted as $(X_t^{(n)}, Y_t^{(n)})$. The change of the process $(\Delta X_t^{(n)}, \Delta Y_t^{(n)})$ can take four possible states at each step:

$$(\Delta X_{u,r}, \Delta Y_r), \quad (\Delta X_{u,l}, \Delta Y_l), \quad (\Delta X_{d,r}, \Delta Y_r), \quad (\Delta X_{d,l}, \Delta Y_l),$$

where u, d refer to up and down steps, and r, l refer to right and left jumps in the Y -component.

The characteristic function of the process $e^{iu\Delta X_{u,r}+iv\Delta Y_r}$ can be expanded using a Taylor series as follows:

$$\begin{aligned} e^{iu\Delta X_{u,r}+iv\Delta Y_r} &= 1 + iu\Delta X_{u,r} + iv\Delta Y_r - \frac{u^2}{2}(\Delta X_{u,r})^2 - \frac{v^2}{2}(\Delta Y_r)^2 \\ &\quad - uv\Delta X_{u,r}\Delta Y_r + o(\Delta t), \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} e^{iu\Delta X_{u,l}+iv\Delta Y_l} &= 1 + iu\Delta X_{u,l} + iv\Delta Y_l - \frac{u^2}{2}(\Delta X_{u,l})^2 - \frac{v^2}{2}(\Delta Y_l)^2 \\ &\quad - uv\Delta X_{u,l}\Delta Y_l + o(\Delta t), \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} e^{iu\Delta X_{d,r}+iv\Delta Y_r} &= 1 + iu\Delta X_{d,r} + iv\Delta Y_r - \frac{u^2}{2}(\Delta X_{d,r})^2 - \frac{v^2}{2}(\Delta Y_r)^2 \\ &\quad - uv\Delta X_{d,r}\Delta Y_r + o(\Delta t), \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} e^{iu\Delta X_{d,l}+iv\Delta Y_l} &= 1 + iu\Delta X_{d,l} + iv\Delta Y_l - \frac{u^2}{2}(\Delta X_{d,l})^2 - \frac{v^2}{2}(\Delta Y_l)^2 \\ &\quad - uv\Delta X_{d,l}\Delta Y_l + o(\Delta t). \end{aligned} \quad (\text{A.5})$$

Thus, the characteristic function of $(X_t^{(n)}, Y_t^{(n)})$ is given by the weighted sum of these expansions:

$$\mathbb{E}[e^{iu\Delta X_t^{(n)}+iv\Delta Y_t^{(n)}}] = p_{u,r}e^{iu\Delta X_{u,r}+iv\Delta Y_r} + p_{u,l}e^{iu\Delta X_{u,l}+iv\Delta Y_l} + p_{d,r}e^{iu\Delta X_{d,r}+iv\Delta Y_r} + p_{d,l}e^{iu\Delta X_{d,l}+iv\Delta Y_l},$$

where the probabilities $p_{u,r}, p_{u,l}, p_{d,r}$, and $p_{d,l}$ sum to 1.

Let f_X, g_X, f_Y , and g_Y denote the functions $f_X(X, Y, t), g_X(X, Y, t), f_Y(X, Y, t)$, and $g_Y(X, Y, t)$. Recalling the assumption of the two dimensional binomial tree,

$$\begin{aligned} \mathbb{E}[\Delta X_t] &= p_{u,r}\Delta X_{u,r} + p_{u,l}\Delta X_{u,l} + p_{d,r}\Delta X_{d,r} + p_{d,l}\Delta X_{d,l} \\ &= f_X\Delta t + o(\Delta t), \end{aligned}$$

$$\begin{aligned} \mathbb{E}[\Delta X_t^2] &= p_{u,r}(\Delta X_{u,r})^2 + p_{u,l}(\Delta X_{u,l})^2 + p_{d,r}(\Delta X_{d,r})^2 + p_{d,l}(\Delta X_{d,l})^2 \\ &= g_X^2\Delta t + o(\Delta t), \end{aligned}$$

$$\begin{aligned} \mathbb{E}[\Delta Y_t] &= p_{u,r}\Delta Y_r + p_{u,l}\Delta Y_l + p_{d,r}\Delta Y_r + p_{d,l}\Delta Y_l \\ &= f_Y\Delta t + o(\Delta t), \end{aligned}$$

$$\begin{aligned} \mathbb{E}[\Delta Y_t^2] &= p_{u,r}(\Delta Y_r)^2 + p_{u,l}(\Delta Y_l)^2 + p_{d,r}(\Delta Y_r)^2 + p_{d,l}(\Delta Y_l)^2 \\ &= g_Y^2\Delta t + o(\Delta t), \end{aligned}$$

$$\mathbb{E}[\Delta X_t\Delta Y_t] = p_{u,r}\Delta X_{u,r}\Delta Y_r + p_{u,l}\Delta X_{u,l}\Delta Y_l + p_{d,r}\Delta X_{d,r}\Delta Y_r + p_{d,l}\Delta X_{d,l}\Delta Y_l$$

$$= \rho g_X g_Y \Delta t + o(\Delta t);$$

thus,

$$\mathbb{E}[e^{iu\Delta X_t^{(n)} + iv\Delta Y_t^{(n)}}] = 1 + \left(iuf_X + ivf_Y - \frac{u^2}{2}g_X^2 - \frac{v^2}{2}g_Y^2 - uv\rho g_X g_Y \right) \Delta t + o(\Delta t). \quad (\text{A.6})$$

On the other hand, the Taylor series expansion of the characteristic function of the continuous form is given by

$$\begin{aligned} \mathbb{E}[e^{iudX_t + ivdY_t}] &= \mathbb{E}\left[1 + (iudX_t + ivdY_t) + \frac{1}{2}(iudX_t + ivdY_t)^2 + o(dt)\right] \\ &= 1 + iuf_X dt + ivf_Y dt + \frac{1}{2}(-u^2 g_X^2 dt - v^2 g_Y^2 dt - 2uv\rho g_X g_Y dt) + o(dt) \\ &= 1 + \left(iuf_X + ivf_Y - \frac{u^2}{2}g_X^2 - \frac{v^2}{2}g_Y^2 - uv\rho g_X g_Y \right) dt + o(dt). \end{aligned} \quad (\text{A.7})$$

The characteristic function and the distribution function are uniquely related. When the characteristic function of the binomial tree converges to the continuous form, $(\Delta X_t^{(n)}, \Delta Y_t^{(n)})$ converges in distribution to $(\Delta X_t, \Delta Y_t)$.

A.2. Convergence in distribution at expiry

To establish the convergence in distribution of $(X_T^{(n)}, Y_T^{(n)})$ in the binomial tree model to the continuous-time model, the characteristic function method is employed. Let $\phi_{\Delta X^{(n)}, \Delta Y^{(n)}}(u, v)$ denote the joint characteristic function of the increments $\Delta X^{(n)}$ and $\Delta Y^{(n)}$ in the binomial tree model. Under the assumption of local convergence, the characteristic function of each increment can be approximated as

$$\begin{aligned} \phi_{\Delta X^{(n)}, \Delta Y^{(n)}}(u, v) &= \mathbb{E}\left[e^{iu\Delta X^{(n)} + iv\Delta Y^{(n)}}\right] \\ &\approx 1 + \left(iuf_X + ivf_Y - \frac{u^2}{2}g_X^2 - \frac{v^2}{2}g_Y^2 - uv\rho g_X g_Y \right) \Delta t + o(\Delta t). \end{aligned} \quad (\text{A.8})$$

For the terminal time T , the joint characteristic function of (X_T, Y_T) in the binomial tree model is given by the product of the characteristic functions of the increments

$$\phi_{X_T^{(n)}, Y_T^{(n)}}(u, v) = \prod_{i=1}^n \phi_{\Delta X_i^{(n)}, \Delta Y_i^{(n)}}(u, v).$$

Taking the logarithm and summing over all increments yields

$$\begin{aligned}
\ln \phi_{X_T^{(n)}, Y_T^{(n)}}(u, v) &= \sum_{i=1}^n \ln \left[1 + \left(iuf_X(X_i^{(n)}, Y_i^{(n)}, t_i) + ivf_Y(X_i^{(n)}, Y_i^{(n)}, t_i) - \frac{u^2}{2}g_X^2(X_i^{(n)}, Y_i^{(n)}, t_i) \right. \right. \\
&\quad \left. \left. - \frac{v^2}{2}g_Y^2(X_i^{(n)}, Y_i^{(n)}, t_i) - uv\rho g_X(X_i^{(n)}, Y_i^{(n)}, t_i)g_Y(X_i^{(n)}, Y_i^{(n)}, t_i) \right) \Delta t \right] + o(n\Delta t) \\
&\approx \sum_{i=1}^n \left[\left(iuf_X(X_i^{(n)}, Y_i^{(n)}, t_i) + ivf_Y(X_i^{(n)}, Y_i^{(n)}, t_i) - \frac{u^2}{2}g_X^2(X_i^{(n)}, Y_i^{(n)}, t_i) \right. \right. \\
&\quad \left. \left. - \frac{v^2}{2}g_Y^2(X_i^{(n)}, Y_i^{(n)}, t_i) - uv\rho g_X(X_i^{(n)}, Y_i^{(n)}, t_i)g_Y(X_i^{(n)}, Y_i^{(n)}, t_i) \right) \Delta t + o(\Delta t) \right] + o(n\Delta t).
\end{aligned} \tag{A.9}$$

Note that $\ln(1 + \epsilon) = \epsilon + o(\epsilon)$, and $\Delta t = T/n$; thus, $o(n\Delta t) = o(T) = o(1)$. As $n \rightarrow \infty$ and $\Delta t \rightarrow 0$, the sum converges to an integral:

$$\begin{aligned}
\ln \phi_{X_T^{(n)}, Y_T^{(n)}}(u, v) &\rightarrow iu \int_0^T f_X(X_t, Y_t, t) dt + iv \int_0^T f_Y(X_t, Y_t, t) dt \\
&\quad - \frac{1}{2} \int_0^T (u^2 g_X^2(X_t, Y_t, t) + v^2 g_Y^2(X_t, Y_t, t) \\
&\quad + 2uv\rho g_X(X_t, Y_t, t)g_Y(X_t, Y_t, t)) dt.
\end{aligned} \tag{A.10}$$

Exponentiation of both sides yields precisely the joint characteristic function of (X_T, Y_T) in the continuous-time model. By Lévy's continuity theorem, the convergence of characteristic functions implies the convergence in distribution of (X_T, Y_T) in the binomial tree model to the continuous-time model.

A.3. Convergence in option pricing

Given $(S^{(n)}, \nu^{(n)}) \xrightarrow{d} (S, \nu)$, let $S_T^{(n)}$ be the terminal asset price nodes of the tree model, $F_n(x)$ be the distribution of $S_T^{(n)}$, and $F(x)$ be the distribution of S_T .

By moment matching, we have $\mathbb{E}[S_T^{(n)}] = S_0 e^{rT}$ such that $\lim_{n \rightarrow \infty} \mathbb{E}[S_T^{(n)}] = S_0 e^{rT} < \infty$. For a European call option, the option pricing is given by

$$v(S, 0) = e^{-rT} \mathbb{E}[\max\{S_T - K, 0\}] = e^{-rT} \int_K^\infty (x - K) dF(x).$$

The convergence of call option pricing from tree approach is given by

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left| e^{-rT} \int_K^\infty (x - K) dF_n(x) \right| &= \lim_{n \rightarrow \infty} \left| e^{-rT} \left[\int_0^\infty (x - K) dF_n(x) - \int_0^K (x - K) dF_n(x) \right] \right| \\
&\leq \lim_{n \rightarrow \infty} e^{-rT} \left[\left| \int_0^\infty x dF_n(x) - K \right| + \left| \int_0^K (x - K) dF_n(x) \right| \right] \\
&= \lim_{n \rightarrow \infty} e^{-rT} \left[\left| S_0 e^{rT} - K \right| + \left| \int_0^K (x - K) dF_n(x) \right| \right] < \infty.
\end{aligned}$$

The convergence of put option pricing from tree approach is given by

$$\lim_{n \rightarrow \infty} \left| e^{-rT} \int_0^K (K-x) dF_n(x) \right| < \infty$$

because it is a finite integral of a finite function.

A.4. Asymptotic error induced by the censorship mechanism

Appendix A.2 establishes weak convergence of the tree approximation under the moment-matching construction in the absence of censorship. The following analysis focuses on characterizing the set of states for which the transition probabilities remain within their admissible range so that no truncation is required. This characterization specifies when the proposed tree construction operates without activating the censorship mechanism.

Because the transition probabilities of the ω -tree and the ν -tree correspond one-to-one under the transformation $\omega_t = 2\sqrt{\nu_t}/\xi$, it suffices to analyze the ω -tree formulation. The purpose of this section is to identify conditions under which censorship does not occur and to describe how these conditions depend on the discretization parameter. Note that the transition probability $q_{i,j}$ admits the corresponding representation in the ω -tree formulation, namely

$$q_{i,j} = \frac{\mathbb{E}[\omega_{i+1} \mid \omega_i = \omega_{i,j}] - \omega_{i+1,j-1}}{\omega_{i+1,j+1} - \omega_{i+1,j-1}}, \quad (\text{A.11})$$

where $\omega_{i+1,j\pm 1} = \omega_{i,j} \pm \sqrt{\Delta t}$. Under the Euler–Maruyama discretization,

$$\mathbb{E}[\omega_{i+1} \mid \omega_i = \omega_{i,j}] = \omega_{i,j} + m(\omega_{i,j}) \Delta t + o(\Delta t) \quad (\text{A.12})$$

such that the condition $q_{i,j} \in (0, 1)$ is equivalent to

$$|m(\omega_{i,j})| \Delta t < \sqrt{\Delta t}. \quad (\text{A.13})$$

In particular, the case $q_{i,j} > 1$ is equivalent to

$$m(\omega_{i,j}) \Delta t > \sqrt{\Delta t} \quad (\text{A.14})$$

as $\Delta t \rightarrow 0$. For simplicity, we write $\omega_j := \omega_{i,j} = \omega_0 + j\sqrt{\Delta t}$ for all j in the following proofs.

Lemma 4. *Let $\omega_t^{(n)}$ denote the n -step tree approximation of ω_t . If $\omega_0 \in \bigcup_{k=1}^{\infty} (k\sqrt{\Delta t}, (k+1/2)\sqrt{\Delta t}]$, and $2\kappa\theta > \xi^2$, then*

$$\mathbb{P}(\omega_t^{(n)} < 0) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (\text{A.15})$$

Proof. Define $\ell := \inf\{k \in \mathbb{Z} : \omega_k > 0\}$. Then, $0 < \omega_\ell \leq \sqrt{\Delta t}/2$. By the definition of the drift term,

$$m(\omega_\ell) = \left(\frac{2\kappa\theta}{\xi^2} - \frac{1}{2} \right) \frac{1}{\omega_\ell} - \frac{\kappa}{2} \omega_\ell.$$

As $n \rightarrow \infty$, thus $\Delta t \rightarrow 0$,

$$m(\omega_\ell) = \left(\frac{2\kappa\theta}{\xi^2} - \frac{1}{2} \right) \frac{1}{\omega_\ell} + o(1).$$

Because $0 < \omega_\ell \leq \sqrt{\Delta t}/2$ such that

$$\frac{1}{\omega_\ell} \geq \frac{2}{\sqrt{\Delta t}},$$

and by $2\kappa\theta > \xi^2$, we thus obtain

$$\begin{aligned} m(\omega_\ell) &\geq \left(\frac{2\kappa\theta}{\xi^2} - \frac{1}{2} \right) \frac{2}{\sqrt{\Delta t}} + o(1) \\ &> \frac{1}{\sqrt{\Delta t}} + o(1). \end{aligned}$$

Multiplying by Δt gives

$$m(\omega_\ell)\Delta t > \sqrt{\Delta t} + o(\sqrt{\Delta t}).$$

Therefore, for sufficiently small Δt ,

$$m(\omega_\ell)\Delta t > \sqrt{\Delta t}.$$

From the characterization of the transition probability, this implies that the upward transition probability from ω_ℓ converges to 1 as $\Delta t \rightarrow 0$. \square

Consider a fixed initial value $\omega_0 > 0$ and because $\Delta t = T/n$, suppose that $k\sqrt{\Delta t} < \omega_0 \leq (k + 1/2)\sqrt{\Delta t}$. Substituting $\Delta t = T/n$ yields

$$k\sqrt{\frac{T}{n}} < \omega_0 \leq \left(k + \frac{1}{2}\right)\sqrt{\frac{T}{n}},$$

which is equivalent to

$$\frac{k^2 T}{\omega_0^2} < n \leq \frac{(k + \frac{1}{2})^2 T}{\omega_0^2}.$$

Define

$$a_k := \frac{k^2 T}{\omega_0^2}, \quad b_k := \frac{(k + \frac{1}{2})^2 T}{\omega_0^2}.$$

Then,

$$b_k - a_k = \frac{(k + \frac{1}{2})^2 - k^2}{\omega_0^2} T = \frac{k + \frac{1}{4}}{\omega_0^2} T.$$

Because $b_k - a_k$ grows linearly in k , there exists k_0 such that

$$b_k - a_k > 1 \quad \text{for all } k \geq k_0.$$

Hence, for sufficiently large k , the interval $(a_k, b_k]$ contains at least one natural number. Define

$$n_k := \lfloor b_k \rfloor.$$

Then, for all $k \geq k_0$, one has

$$a_k < n_k \leq b_k$$

so that the corresponding time step $\Delta t_k = T/n_k$ satisfies

$$k\sqrt{\Delta t_k} < \omega_0 \leq \left(k + \frac{1}{2}\right)\sqrt{\Delta t_k}.$$

Therefore, there exists a subsequence $\{n_k\}$ of \mathbb{N} that can be constructed such that the condition of Lemma 4 holds along this subsequence. Nevertheless, even when the positivity condition is not strictly guaranteed outside this subsequence, the numerical experiments demonstrate stable convergence behavior of the proposed censored scheme, as shown in Section 4.2.

Lemma 5. *Let*

$$\ell := \inf\{k \in \mathbb{N} : \omega_k > 0\}$$

and define the integer

$$K := \left\lceil \frac{2\kappa\theta}{\xi^2} - \frac{1}{2} \right\rceil + 1.$$

If $2\kappa\theta > \xi^2$, then there exists $\Delta t_0 > 0$ such that for all $\Delta t < \Delta t_0$ and all $j > \ell + K$,

$$|m(\omega_j)| \Delta t < \sqrt{\Delta t}. \quad (\text{A.16})$$

In particular, the transition probabilities are not censored for all $j > \ell + K$.

Proof. For all $j > \ell + K$, it holds that

$$\omega_j \geq \omega_\ell + K \sqrt{\Delta t} \geq K \sqrt{\Delta t}.$$

Using the definition of the drift term,

$$m(\omega_j) = \left(\frac{2\kappa\theta}{\xi^2} - \frac{1}{2} \right) \frac{1}{\omega_j} - \frac{\kappa}{2} \omega_j,$$

which implies

$$|m(\omega_j)\Delta t| \leq \left| \left(\frac{2\kappa\theta}{\xi^2} - \frac{1}{2} \right) \frac{\Delta t}{\omega_j} \right| + o(\sqrt{\Delta t}).$$

Because $\omega_j \geq K \sqrt{\Delta t}$,

$$|m(\omega_j)\Delta t| \leq \left| \left(\frac{2\kappa\theta}{\xi^2} - \frac{1}{2} \right) \frac{\sqrt{\Delta t}}{K} \right| + o(\sqrt{\Delta t}).$$

By the definition of K , it follows that

$$|m(\omega_j)\Delta t| < \sqrt{\Delta t}$$

as $\Delta t \rightarrow 0$. □

Appendix A.2 establishes weak convergence of the uncensored tree approximation under the moment-matching construction. Lemma 4 shows that positivity of the volatility tree can be ensured along a subsequence, and Lemma 5 shows that censorship, if it occurs, is confined to a boundary neighborhood of width proportional to $\sqrt{\Delta t}$.

Consequently, the region in which censorship may be activated forms a boundary layer whose Lebesgue measure vanishes as $\Delta t \rightarrow 0$. Although numerical approximation errors may still arise due to censorship at finite discretization levels, such effects are treated as part of the finite-step numerical error rather than as a theoretical obstruction to the implementation of the proposed scheme. The asymptotic convergence result for the uncensored tree is not claimed to fully characterize the

censored construction. Instead, the overall performance of the proposed methodology is assessed primarily through the numerical results reported in Section 4.4, where convergence behavior and pricing accuracy are demonstrated to be stable across discretization levels.



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