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Research article

The green incentive wave: Dynamic optimization of referral rewards and green advertising in sustainable energy diffusion

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Appendix: Proofs of propositions

Proof of Proposition 1. We apply Pontryagin's maximum principle. The costate variable $\lambda(t)$ satisfies the adjoint equation $\lambda' = -\frac{\partial H}{\partial q} = \theta r^2(1-2q) - (\mu+\lambda)[(\tau+\theta r)(1-2q) - \eta - \sigma g]$. Differentiating the Hamiltonian with respect to each control and setting the result equal to zero yields the pair of first-order necessary conditions $\frac{\partial H}{\partial g} = \sigma(\mu+\lambda)(1-q) - cg = 0$, $\frac{\partial H}{\partial r} = \theta q(1-q)(\mu+\lambda-2r) = 0$. That these conditions identify a maximum rather than a minimum or saddle point is guaranteed by the second-order partial derivatives $\frac{\partial^2 H}{\partial g^2} = -c < 0$, $\frac{\partial^2 H}{\partial r^2} = -2\theta q(1-q) \le 0$ (since $0 \le q \le 1$), and $\frac{\partial^2 H}{\partial g \partial r} = \frac{\partial^2 H}{\partial r \partial g} = 0$, which jointly render the Hessian of the Hamiltonian negative definite at the critical point. Solving the first-order conditions for the controls gives $g^* = \frac{\sigma(\mu+\lambda)(1-q)}{c}$, $r^* = \frac{\mu+\lambda}{2}$. It remains to characterize the optimal adoption path. Substituting g^* and r^* back into the stationarity condition $\frac{\partial H}{\partial q} = 0$ and rearranging, we arrive at $q^*(t) = \frac{4c(\tau-\eta) + (\mu+\lambda(t))\left(c\theta - 4\sigma^2\right)}{8c\tau + (\mu+\lambda(t))\left(2c\theta - 4\sigma^2\right)}$. A final verification confirms that this is indeed the adoption path that maximizes H. Computing $\frac{\partial^2 H}{\partial q^2} = -2\tau(\mu+\lambda) - \frac{\theta(\mu+\lambda)^2}{2}$, we observe that this expression is non-positive for all admissible parameter values, so $q^*(t)$ does correspond to a global maximum of the Hamiltonian in q.

Proof of Proposition 2. Recall from Proposition 1 that optimal advertising is given by $g^*(t) = \sigma(\mu+\lambda(t))(1-q(t))/c$. Differentiating this expression with respect to time by the product rule produces $g'(t) = \sigma[\lambda'(t)(1-q(t)) - (\mu+\lambda(t))q'(t)]/c$. We now substitute the state and costate dynamics and simplify. After collecting terms, the derivative reduces to $g'(t) = -\sigma(1-q(t))(\mu+\lambda(t))\left[\frac{\theta(\mu+\lambda(t))}{4} + \tau(1-q(t))\right]/c$. The sign of $g'(t)$ is determined entirely by the sign of the bracketed expression on the right-hand side, since all multiplicative factors outside the bracket are strictly positive: $\sigma > 0$ and $c > 0$ by assumption, $1-q(t) \ge 0$ because q is a cumulative adoption fraction bounded between zero and one, and $\mu+\lambda(t) > 0$ as an immediate consequence of the optimality condition $r^*(t) = (\mu+\lambda(t))/2 > 0$. Turning to the bracket itself, both summands are non-negative: $\theta(\mu+\lambda(t))/4 > 0$ because $\theta > 0$ and $\mu+\lambda(t) > 0$, and $\tau(1-q(t)) \ge 0$ because $\tau > 0$ and $1-q(t) \ge 0$. The bracket is therefore strictly positive whenever the market is not fully saturated. It follows that $g'(t) \le 0$ throughout the planning horizon, establishing the claimed monotonic decline of optimal advertising.

Proof of Proposition 3. The proof proceeds in three steps. We first show that the referral reward is decreasing at $t=0$, then that it is increasing at $t=T$, and finally invoke continuity to establish the existence of a turning point t^*.

Step 1: Sign of $r'(0)$. Because $r^*(t) = (\mu+\lambda(t))/2$, the time derivative satisfies $r'(t) = \lambda'(t)/2$, so the direction of change of the optimal reward is governed entirely by the sign of $\lambda'(t)$. Substituting the optimal controls g^* and r^* into the costate equation and simplifying yields $\lambda'(t) = (\mu+\lambda(t))\left[\eta - \tau + 2\tau q(t) + \left(\frac{\sigma^2}{c} - \frac{\theta}{4}\right)(\mu+\lambda(t))^2(1-q(t)) + \frac{\theta q(t)(\mu+\lambda(t))}{4}\right]$. Evaluating at $t=0$ with the initial condition $q(0)=0$ gives $\lambda'(0)=(\mu+\lambda(0))\left[\eta-\tau+\left(\frac{\sigma^2}{c}-\frac{\theta}{4}\right)(\mu+\lambda(0))^2\right]$. The expression for $q^*(t)$ in Proposition 1, evaluated at $t=0$, implies the identity $\tau-\eta+(\mu+\lambda(0))\left(\frac{\theta}{4}-\frac{\sigma^2}{c}\right)=0$. Substituting this relation back into $\lambda'(0)$ and rearranging produces $\lambda'(0)=\left(\frac{\sigma^2}{c}-\frac{\theta}{4}\right)\left[(\mu+\lambda(0))^3-(\mu+\lambda(0))^2\right]$. We sign this expression as follows: The economic constraint that the referral reward must remain below the unit margin, $r^*(t)<\mu\le 1$, forces $\mu+\lambda(t)<1$ for all t. When $0<\mu+\lambda(0)<1$, the cubic power is strictly less than the square, so the second factor is negative. Meanwhile, the identity derived above entails $\frac{\theta}{4}<\frac{\sigma^2}{c}$, making the first factor positive. The product of a positive and a negative term is negative, hence $\lambda'(0)<0$ and consequently $r'(0)<0$.

Step 2: Sign of $r'(T)$. At the terminal time, the transversality condition $\lambda(T)=0$ reduces the costate derivative to $\lambda'(T)=\mu\left[q(T)\left(2\tau+\frac{\theta\mu}{4}\right)-(\tau-\eta)-\left(\frac{\theta}{4}-\frac{\sigma^2}{c}\right)\mu^2(1-q(T))\right]$. Setting $\lambda(T)=0$ in the expression for optimal cumulative adoption delivers $q(T)=\dfrac{\tau-\eta+\mu\left(\frac{\theta}{4}-\frac{\sigma^2}{c}\right)}{2\tau+\frac{\theta\mu}{2}-\frac{\sigma^2\mu}{c}}$. For this terminal adoption level to be economically meaningful, it must satisfy $0<q(T)<1$. Under the parameter regime identified in Step 1, both the numerator $\tau-\eta+\mu\left(\frac{\theta}{4}-\frac{\sigma^2}{c}\right)$ and the denominator $2\tau+\frac{\theta\mu}{2}-\frac{\sigma^2\mu}{c}$ are negative, so their ratio is positive, and $q(T)$ lies in the admissible range. Substituting this closed-form $q(T)$ into the expression for $\lambda'(T)$ and carrying out the algebra confirms that $\lambda'(T)>0$, which in turn implies $r'(T)>0$.

Step 3: Existence of the turning point. Steps 1 and 2 have established that the continuous function $\lambda'(t)$ is strictly negative at $t=0$ and strictly positive at $t=T$. By the intermediate value theorem, there exists at least one point $t^*\in(0,T)$ at which $\lambda'(t^*)=0$, equivalently $r'(t^*)=0$. On the interval

$[0,t^*)$, the costate derivative is negative, and therefore, $r'(t)<0$, meaning that the referral reward is declining; on the interval $(t^*,T]$, the costate derivative is positive, and therefore, $r'(t)>0$, meaning the referral reward is rising. The optimal reward path thus traces the claimed U-shaped trajectory with its trough at t^*.