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Research article

How does green advertising serve as a quality signal in the gray market?

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Appendix A

Table A.1 Optimal decision and profit for j-type manufacturers and retailers.

	No	Unilateral	Bilateral
$w_1^s(j)$	$\frac{v_{Hj}}{2}$	$\frac{v_{Hj}\left[\theta v_{Hj}(2-\theta)+v_{Lj}(2+\theta)\right]}{4\left(\theta v_{Hj}+v_{Lj}\right)}$	$\frac{v_{Hj}v_L j(3-\theta)\theta\left[3v_{Hj}\theta+v_{Lj}(2+\theta)\right]}{8v_{Hj}v_{Lj}+8\left(v_{Hj}^2+v_{Lj}^2\right)\theta-2\left(v_{Hj}-v_{Lj}\right)^2\theta^2}$
$w_2^s(j)$	$\frac{v_{Lj}}{2}$	$\frac{\theta v_{Hj}v_{Lj}}{\theta v_{Hj}+v_{Lj}}$	$\frac{v_{Hj}v_L j(3-\theta)\theta\left[3v_{Hj}\theta+v_{Lj}(2+\theta)\right]}{8v_{Hj}v_{Lj}+8\left(v_{Hj}^2+v_{Lj}^2\right)\theta-2\left(v_{Hj}-v_{Lj}\right)^2\theta^2}$
$D_H(j)$	$\frac{1}{4}$	$\frac{2-\theta}{2(4-\theta)}$	$\frac{B_{q1}(v_{Hj},v_{Lj})}{2B(v_{Hj},v_{Lj})}$
$D_L(j)$	$\frac{1}{4}$	$\frac{v_{Lj}}{2\left(\theta v_{Hj}+v_{Lj}\right)}$	$\frac{Bq1(v_{Lj},v_{Hj})}{2B(v_{Hj},v_{Lj})}$
$D_{G1}(j)$	*N/A*	*N/A*	$\frac{B_{qg}(v_{Hj},v_{Lj})}{2B(v_{Hj},v_{Lj})}$
$D_{G2}(j)$	*N/A*	$\frac{v_{Hj}(6-\theta)-v_{Lj}(2-\theta)}{4(4-\theta)\left(\theta v_{Hj}+v_{Lj}\right)}$	$\frac{B_{qg}(v_{Lj},v_{Hj})}{2B(v_{Hj},v_{Lj})}$
$p_H^S(j)$	$\frac{3v_{Hj}}{4}$	$\frac{v_{Hj}\left[12-\theta)^2 v_{Lj}+\theta v_{Hj}(2-\theta)(6-\theta)\right]}{4(4-\theta)\left(\theta v_{Hj}+v_{Lj}\right)}$	$\frac{B_{p1}(v_H,v_L)}{2B(v_H,v_L)}$

Continued on next page

	No	Unilateral	Bilateral
$p_{G1}^S(j)$	N/A	N/A	$\frac{B_{pg}(v_{Lj},v_{Hj})}{B(v_{Hj},v_{Lj})}$
$p_L^S(j)$	$\frac{3v_{Lj}}{4}$	$\frac{v_{Lj}\left(2\theta v_{Hj}+v_{Lj}\right)}{2\left(\theta v_{Hj}+v_{Lj}\right)}$	$\frac{B_{p1}(v_{Lj},v_{Hj})}{2B(v_{Hj},v_{Lj})}$
$p_{G2}^S(j)$	N/A	$\frac{\theta v_{Hj}\left[\theta v_{Hj}(6-\theta)+v_{Lj}(14-3\theta)\right]}{4(4-\theta)\left(\theta v_{Hj}+v_{Lj}\right)}$	$\frac{B_{pg}(v_{Hj},v_{Lj})}{B(v_{Hj},v_{Lj})}$

$$
\begin{aligned}
&B_{p1}(v_{Hj},v_{Lj})=v_{Hj}\left[2v_{Hj}^{\ 2}(4-\theta)(2-\theta)\theta+v_{Lj}^{\ 2}(4-\theta)\theta\left(7-\theta^2\right)+v_{Hj}v_{Lj}(3\theta^4-20\theta^3+27\theta^2-2\theta+16)\right]\\
&B_{q1}(v_{Hj},v_{Lj})=2v_{Hj}^{\ 2}(4-\theta)(2-\theta)\theta+v_{Lj}^{\ 2}(4-\theta)(1-\theta)\theta+v_{Hj}v_{Lj}\left[\theta(\theta^2-9\theta-2)+16\right]\\
&B_{pg}(v_{Hj},v_{Lj})=v_{Hj}\theta\left[v_{Hj}^{\ 2}(4-\theta)\theta+v_{Lj}^{\ 2}(4-\theta)^2\theta v_{Hj}v_{Lj}(5-\theta)\left(2+\theta^2\right)\right]\\
&B_{qg}(v_{Hj},v_{Lj})=2v_{Lj}^{\ 2}(4-\theta)\theta-v_{Hj}^{\ 2}(4-\theta)(1-\theta)\theta-v_{Hj}v_{Lj}\left[\theta(3\theta^2-15\theta+2)+4\right]\\
&B(v_{Hj},v_{Lj})=(4-\theta)\left[v_{Hj}^{\ 2}(4-\theta)\theta+v_{Lj}^{\ 2}(4-\theta)\theta+2v_{Hj}v_{Lj}\left(2+\theta^2\right)\right]
\end{aligned}
\tag{A.1}
$$

Table A.1 presents the equilibrium pricing and profits for manufacturers and retailers under three market structures. It is straightforward to check that (i) in the bilateral gray market, $w_1^s(j)=w_2^s(j)$, and $p_H^S(j)$, $p_L^S(j)$, $p_{G1}^S(j)$, $p_{G2}^S(j)$ are symmetric in the v_H and v_L. (ii) $\partial\left(p_H^S(j)-p_{G2}^S(j)\right)/\partial\theta\le 0$. The closer θ is to 0, the greater the difference between $p_H^S(j)$, $p_L^S(j)$ $p_{G1}^S(j)$, $p_{G2}^S(j)$; The closer θ is to 1, the smaller the difference. If $\theta=0$, the selling price is 0, and $p_i^S(j)$, $\pi_{r1}^S(j)$, $\pi_{r2}^S(j)$, $R(j)$ are linear functions of v_i, *i={H,L}*; If $\theta=1$, $p_H^S(j)=p_{G2}^S(j)$ in unilateral gray market, retailer 1's profit is zero, and $p_H^S(j)=p_L^S(j)=p_{G2}^S(j)$ in the bilateral market, both retailers' profits are zero. (iii) $w_1^s(j)$, $w_2^s(j)$ $p_H^S(j)$, $p_L^S(j)$, $p_{G1}^S(j)$, $p_{G2}^S(j)$ and $R(j)$ are concave in v_i, and $\pi_{r1}^S(j)$, $\pi_{r2}^s(j)$ are convex in v_i.

In a no gray market case, the profits of retailer 1, retailer 2, and the manufacturer are as follows:

$$\pi_{r1}^S(j)=\frac{v_{Hj}}{16},\pi_{r2}^S(j)=\frac{v_{Lj}}{16},R(j)=\frac{v_{Hj}}{8}+\frac{v_{Lj}}{8} \tag{A.2}$$

In a unilateral gray market case, the profits of retailer 1, retailer 2, and the manufacturer are as follows:

$$
\begin{aligned}
&\pi_{r1}^S(j)=\frac{v_{Hj}(2-\theta)^2}{4(4-\theta)},\pi_{r2}^S(j)=\frac{\theta v_{Hj}\left[\theta v_{Hj}(6-\theta)^2-v_{Lj}(2-\theta)\right]^2+16v_{Lj}^{\ 3}(4-\theta)^2}{[8(4-\theta)\left(\theta v_{Hj}+v_{Lj}\right)]^2}\\
&R(j)=\frac{v_{Hj}\left[\theta v_{Hj}(2-\theta)^2+v_{Lj}\left(4+12\theta-3\theta^2\right)\right]}{8(4-\theta)\left(\theta v_{Hj}+v_{Lj}\right)}
\end{aligned}
\tag{A.3}
$$

In a bilateral gray market case, the profits of retailer 1, retailer 2, and the manufacturer are as follows:

$$\pi_{r1}^{S}(j)=\pi_{r2}^{S}(j)=\frac{Bp1(v_{Hj},v_{Lj})Bq1(v_{Hj},v_{Lj})+2Bpg(v_{Hj},v_{Lj})Bqg(v_{Hj},v_{Lj})}{4B(v_{Hj},v_{L}j)^{2}}$$

$$\frac{-2w1B(v_{Hj},v_{Lj})\left(Bq1(v_{Hj},v_{Lj})+Bqg(v_{Hj},v_{Lj})\right)}{4B(vHj,vLj)^{2}} \tag{A.4}$$

$$R(j)=\frac{v_{Hj}v_{Lj}\left(v_{Hj}+v_{Lj}\right)\left(-3+\theta\right)^{2}\theta\left(1+2\theta\right)}{2B(v_{Hj},v_{Lj})}$$

The profits for retailer 1 and retailer 2 are identical, where the first term represents the profit each retailer obtains from the authorized channel; the second term represents the profit derived from the gray market channel; and the final term represents the loss incurred due to the erosion of gray products.

Proof of Corollary 3.1

Corollary 3.1 is straightforward from Proposition 1. Price and profit increase with willingness to pay, and willingness to pay increases with μ.

Proof of Lemma 3.1

Neither the *l*-type manufacturer or *h*-type manufacturer will choose $0<g<g^{p*}$ in a pooling equilibrium. Thus, $R\left(\mu_0\right)-k_l(g^{p*}-g)<R\left(\mu_0\right)-k_h(g^{p*}-g)$ because $(g^{p*}-g)>0$. This completes the proof. We can obtain the result when $g^{p*}<g$ and $g^{p*}=0$ are similar.

Proof Proposition 3.1

(i) To capture all pooling equilibrium, we should consider the following inequality: $g=\{g\geq 0\,|\,R\left(\mu_0\right)\geq R\left(1\right),R\left(\mu_0\right)-k_h g\geq R\left(0\right),R\left(\mu_0\right)-k_l g\geq R\left(0\right)\}$, where the first two inequalities restrict both types of manufacturers from choosing *g*=0. We can obtain Proposition 3.1 (i) by solving the last two inequalities.

(ii) Suppose there is a pooling equilibrium $\tilde{g}\in(\frac{R(\mu_0)-R(0)}{k_l},+\infty)$. For the *l*-type manufacture, $there\ is\ R(\mu_0)-k_i\cdot\tilde{g}<R(\mu_0)-k_i\cdot\frac{R(\mu_0)-R(0)}{k_i}=R(0)$. $\tilde{g}$ is not a pooling equilibrium.

Proof of Lemma 3.2

(i) Both the *l*-type manufacturer and *h*-type manufacturers will not choose $0<g<\frac{g_h^{s*}}{2}$ in a separating equilibrium, so we obtain constraints for posterior belief:

$$\begin{aligned} &R\left(\mu\right)-k_h g\leq R\left(1\right)-k_h g_h^{s*}\\ &R\left(\mu\right)-k_l g\leq R\left(0\right)\end{aligned} \tag{A.5}$$

We can obtain $R\left(\mu\right)\leq \min\{R\left(0\right)-k_l g,R(1)-k_h(g_h^{s*}-g)\}$ by solving the above two inequalities.

(ii) It is straightforward to check that when $\hat{g}\leq g<g_h^{s*-}$, $R(1)-k_h(g_h^{s*}-g)<R\left(0\right)-k_l g$. Only the second inequality is considered.

(iii) No rational manufacturer would choose $g_h^{s*+}<g<+\infty$, so there are no constraints on posterior beliefs.

Proof of Proposition 3.2

According to Lemma 3.2, the feasible region of g to separate is $g=\{g\geq 0 \mid R(1)-k_h g\geq R(0)\geq R(1)-k_l g\}$. The first inequality guarantees that the h-type manufacturer will reveal its true type, and the second inequality guarantees that the l-type manufacturer will not adversely select it. Solving the two inequalities, we obtain Proposition 3(i).

Next, we show that Proposition 3.2(i) has captured all possible separating equilibrium through contradiction.

Suppose there is a separating equilibrium of h-type manufacturer $g_0\in\left[0,\frac{R(1)-R(0)}{k_l}\right)$.We have $R(1)-k_l g_0>R(1)-k_l\frac{R(1)-R(0)}{k_l}=R(0)$. l-type manufacturer $\tilde{g}$ is better off than $g=0$. This contradicts to the definition of separating equilibrium.

Similarly, suppose there is a separating equilibrium of h-type manufacturer $g_0\in\left(\frac{R(1)-R(0)}{k_h},+\infty\right)$.We have $R(1)-k_h g_0<R(1)-k_h\frac{R(1)-R(0)}{k_h}=R(0)$. For the h-type manufacturer, $g=0$ is better off than $\tilde{g}$. The market is pooling. A contradiction.

Proof of Proposition 3.3

(i) $\forall g_1^{p*}\in[0,\frac{R(\mu_0)-R(0)}{k_l}]$

There exists a positive and arbitrary small ε_1 such that

$$R(\mu_0)-k_l g_1^{p*}\geq R(\mu_0)-k_l\frac{R(\mu_0)-R(0)}{k_l}=R(0)\geq R(1)-k_l(g_1^{p*}+\varepsilon_1)$$

$R(\mu_0)-k_h g_1^{p*}<R(1)-k_h(g_1^{p*}+\varepsilon_1)$ considering $g_1^{p*}+\varepsilon$ is equilibrium-dominated for the l-type manufacturer but not for the h-type manufacturer. Only the h-type manufacturer has an incentive to deviate the original equilibrium. By Lemma 3.1 (ii), we have $R(\mu)\leq R(\mu_0)-k_h[g_1^{p*}-(g_1^{p*}+\varepsilon)]=R(\mu_0)+k_h\varepsilon_1<R(1)$, indicating that downstream retailers and customers believe that the manufacturer choosing a green advertising level of $g_1^{p*}+\varepsilon_1$ has a positive probability of being l-type. Note that Lemma 3.1 is the necessary condition of pooling equilibrium, so no pooling equilibrium satisfies intuitive criterion.

(ii) $\forall g_{h1}^{s*}\in(\frac{R(1)-R(0)}{k_l},\frac{R(1)-R(0)}{k_h}]$, there exists a positive and arbitrary small ε_2 such that,

$$R(0)\approx R(0)+k_l\varepsilon_2=R(1)-k_l\frac{R(1)-R(0)}{k_l}+k_l\varepsilon_2\geq R(1)-k_l(g_{h1}^{s*}-\varepsilon_2)$$

$$R(1)-k_h g_{h1}^{s*}\leq R(1)-k_h(g_{h1}^{s*}-\varepsilon_2)$$

considering $g_{h1}^{s*}-\varepsilon_2$ is equilibrium-dominated for the l-type manufacturer but not for the h-type manufacturer. Only the h-type manufacturer has the incentive to deviate the original equilibrium. By Lemma 3.2 (ii), we have $R(\mu)\leq R(1)-k_h[g_{h1}^{s*}-(g_{h1}^{s*}-\varepsilon_2)]=R(1)-k_h\varepsilon_2<R(1)$, indicating that downstream retailers and customers believe that the manufacturer choosing a green advertising level

of $g_{h1}^{s*}-\varepsilon_2$ has a positive probability of being l-type . Note that Lemma3.2 is the necessary condition of pooling equilibrium, so $\forall g_{h1}^{s*}$ does not satisfy the intuitive criterion.

Only $g_{h2}^{s*}=\frac{R(1)-R(0)}{k_l}$ satisfies the intuitive criterion, because

$R(0)<R(1)-k_l(g_{h2}^{s*}-\varepsilon_2), R(1)-k_l g_{h2}^{s*}<R(1)-k_l(g_{h2}^{s*}-\varepsilon_2)$

$R(0)>R(1)-k_l(g_{h2}^{s*}+\varepsilon_2), R(1)-k_l g_{h2}^{s*}>R(1)-k_l(g_{h2}^{s*}+\varepsilon_2)$, which is consistent with Lemma 3.2.

Proof of Proposition 3.4

The conclusion is obtained immediately following Proposition 3.3.

Boundary on off-equilibrium belief

Lemma 3.1 establishes a one-to-one correspondence between μ and g and thus characterizes the feasible region supporting all possible pooling equilibrium. We can find that the upper bound of customer belief is a continuous differentiable increasing function of g because $R(\mu)$ is increasing with μ and $\lim_{g\to g^{p*-}} R(\mu_0)-k_l(g^{p*}-g)=\lim_{g\to g^{p*+}} R(\mu_0)-k_h(g^{p*}-g)=R(\mu_0)$. This implies that under the premise of the existence of a pooling equilibrium, it enables a certain degree of ‘irrationality’ in customers’ perception of signals, which may arise from individual green concepts and differing perceptions of product quality. Customers' beliefs about product quality do not necessarily increase with the addition of green advertising; they may decrease or present any possible irregular form within certain intervals. However, the boundary of customers' perception of signals must be a continuous, monotonically increasing function (it is incorrect to assume that beliefs on the right side of the pooling equilibrium remain unchanged). In the pooling equilibrium, the set of customers' beliefs is a convex set.

Similar to the situation in pooling equilibria, Lemma 3.2 establishes a one-to-one correspondence between μ and g and thus characterizes the feasible region supporting all possible pooling equilibrium. However, unlike pooling equilibrium, the upper bound of beliefs may have a cusp $\hat{g}$. We can verify that the upper bound of beliefs is not differentiable at $\hat{g}$ where $\lim_{g\to\hat{g}^-} R(0)+k_l g=\lim_{g\to\hat{g}^+} R(1)-k_h(g_h^{s*}-g)=R[\mu(\hat{g})]=\frac{k_l R(1)-[R(0)+k_l g_h^{s*}]}{k_l-k_h}$ and $\lim_{g\to\hat{g}^-}\frac{R(0)+k_l g-[R(0)+k_l\hat{g}]}{g-\hat{g}}=k_l\neq k_h=\lim_{g\to\hat{g}^-}\frac{R(1)-k_h(g_h^{s*}-g)-[R(1)-k_h(g_h^{s*}-\hat{g})]}{g-\hat{g}}$.

The upper bound of customer belief is a continuous and increasing function of g because $R(\mu)$ is increasing with μ . In a separating equilibrium, the customers' belief set is also convex (the feasible region of all linear programming problems is a convex set).

Although there is no necessary connection between the level of green advertising and customers' quality belief, customers always place higher expectation on the firm advertising more environmentally on the off-equilibrium boundary. A possible reason is that customers might think that increased exposure to green advertisements indicates a company has more funds to enhance product quality, which in turn reflects a trustworthy corporate culture that effectively oversees its manufacturing processes. Therefore, the higher the level of green advertisements of the firm, the higher the belief of customers about the quality of the product produced by the firm. If not, it would at least be a non-decreasing function of g . A reasonable off-equilibrium belief must satisfy two requirements: (i) Be in line with the reality, and (ii) according to Athey (2001), the posterior belief (u) must induce a certain revenue scheme *P(g)* such that equilibrium exists. The revenue scheme $P(g)$ is a perturbation curve that supports equilibrium, and a weakly smaller than k_i passes through (0,R(0)) and the equilibrium points , g^* where g^* is an equilibrium investment for advertising.

Appendix B

Proof of Proposition 5.1

(i) Under the *h*-type supply chain, the decision-making problems faced by retailer 1, retailer 2, and the manufacturer are as follows:

$$\begin{cases} \pi_{r1} = \max\limits_{p_H}\left(p_H - w_1\right)D_H \\ \pi_{r2} = \max\limits_{p_L,p_G}\left(p_L - w_2\right)D_L + \left(p_G - w_2\right)D_G \\ \pi_{M-h} = \max\limits_{w_1,w_2} w_1 D_H + w_2(D_L + D_G) - k_h * g^2 \end{cases} \tag{B.1}$$

The two retailers compete simultaneously on price in the H-market in the second stage, and the retail prices of authorized and gray product are

$$p_{Hh} = \frac{1}{3}[2 + v_{H0} + 2w_1 + w_2 - \left(2 + v_{H0}\right)\theta], p_{Gh} = \frac{1}{3}[1 + w_1 + 2w_2 + v_{H0}\left(-1+\theta\right) - \theta] \tag{B.2}$$

respectively. Retailer 2 is a monopolist in L-market, $p_L = \frac{1}{2}\left(1 + bg + v_{L0} + w_2\right)$. Manufacturers choose the optimal wholesale price and optimal green advertising level given g at first stage

$$w_1 = \frac{1}{2}\left(5 + bg + v_{H0} + v_{L0} - \left(2 + v_{H0}\right)\theta\right), w_2 = \frac{1}{2}\left(3 + bg + v_{L0}\right) \tag{B.3}$$

Given g, the optimal profit of the *h*-type manufacturer can be expressed as $\pi_{M-h}(g;k_h) = \left(\frac{b^2}{8} - k_h\right)g^2 + \frac{1}{4}b\left(3 + v_{L0}\right)g + \frac{35}{24} + \frac{1}{8}v_{L0}\left(6 + v_{L0}\right) + \frac{1}{12}v_{H0}\left(4 + v_{H0}\right)\left(1-\theta\right) - \frac{\theta}{3}$, and according to first-order optimal conditions, the *h*-type manufacturer chooses the optimal level of green advertising $g_h^* = \frac{b\left(3 + v_{L0}\right)}{8k_h - b^2}$.

In the *l*-type supply chain, we consider manufacturer-preferred equilibrium. The price of the product in both markets is take-it-or-leave-it for all customers. Retailer 1 and Retailer 2 are in Bertrand competition on the H market. A rational retailer 2 would set $p_G = w_2$ (because this is the marginal cost of retailer 2 selling the gray product). We assume customers always choose authorized products when both products are indifferent to them, so rational retailer 1 will set $p_{Hl} = w_{2l} + (1-\theta)v_{H0}$ to maximize their profit.

Next, we derive the manufacturer's optimal wholesale price for retailer 2. The decision problem retailer 2 faces in the L-market is $\max\limits_{p_{Ll}} \pi_{r2} = \begin{cases} p_{Ll} - w_{2l}, p_{Ll} \le v_{L0} + bg \\ 0, p_{Ll} > v_{L0} + bg \end{cases}$, the optimal retail price for retailer 2 is $p_{Ll} = v_{L0} + bg$, and the optimal profit is $\pi_{r2} = v_{L0} + bg - w_2$.

We first consider the optimal decision facing an *l*-type manufacturer in L-market l. For a given green advertising level g, their decision problem is:

$$\pi_{M-l} = \max_{w_2} + w_2 D_L - k_l g^2$$
$$s.t.\, \pi_{r2} = v_{L0} + \beta g - w_2 \geq 0 \tag{B.4}$$

Solving the optimization problem of constraints, we obtain $w_2 = p_{Ll}$. Retailer 1 faces the same decision problem as retailer 2. We can easily get $p_{Hl} = v_{L0} + bg + (1-\theta)v_{H0}$ by repeating the above analysis. A rational *l*-type manufacture should not only prevent retailer 1 from selling gray products in reverse, but also ensure that retailer 1's profit is not negative. Thus, the wholesale price satisfies $w_2 \leq w_1 \leq (1-\theta)v_{H0} + w_2$, and the manufacture set $w_{1l} = v_{L0} + bg + (1-\theta)v_{H0}$ to maximize their profit.

For a given g, the profit of the *l*-type manufacturer is $\pi_{M-l}(g;k_l) = 2(v_{L0} + bg) + (1-\theta)v_{H0} - k_l g^2$. The optimal level of green advertisements is $g_l^* = \dfrac{b}{k_l}$.

Proof of Lemma 5.1

Lemma 5.1 follows directly by the conclusion of proposition 5.1.

Poof of Proposition 5.2

The intuition behind this proof is to identify a range of green advertising levels that is too costly for the low-quality manufacturer to mimic, but profitable enough for the high-quality manufacturer to undertake, thereby achieving separation.

To achieve the separation equilibrium, we can get the optimization problem for the *h*-type manufacturer,

$$\max_{g} \pi_h(g)$$
$$s.t.\, \pi_{hl}^* \leq \pi_h(g), \pi_{ll}^* \geq \pi_{lh}(g)$$

The first constraint guarantees that the *h*-type manufacturer reveals their true type. Denote $\bar{g}$ as the larger root of $\pi_{hl}^* = \pi_h(g)$, and $\underline{g}$ is the smaller one. From the first constraint, we obtain $\underline{g} \leq g \leq \bar{g}$.

$$\underline{g} = g_h^* - \frac{\sqrt{\varphi(k_h)}}{3(8k_h - b^2)}, \bar{g} = g_h^* + \frac{\sqrt{\varphi(k_h)}}{3(8k_h - b^2)} \tag{B.5}$$

The second constraint prevents *l*-type adverse selection behavior. Denote $\bar{g}'$ as the larger root of $\pi_{ll}^* = \pi_{lh}(g)$, and $\underline{g}'$ is the smaller one. From the second constraint, we obtain $g \leq \underline{g}'$ or $g \geq \bar{g}'$

$$\underline{g}' = g_{lh}^* - \frac{\sqrt{\varphi(k_l)}}{3(8k_l - b^2)}, \bar{g}' = g_{lh}^* + \frac{\sqrt{\varphi(k_l)}}{3(8k_l - b^2)} \tag{B.6}$$

$$\varphi(k_h) = \frac{72b^4}{k_h} + 24k_h\left[35 - 3(10 - v_{L0})v_{L0} - 2(8 - v_{H0})v_{H0}(1-\theta) - 8\theta\right] \tag{B.7}$$

Thus, the candidate sets satisfying the above two constraint of Θ are $[\underline{g}, \underline{g}']$ and $[\bar{g}', \bar{g}]$. Next, we prove there cannot exist any separate equilibrium in $[\underline{g}, \underline{g}']$ by proving $\underline{g}' \leq \underline{g}$, i.e., if $\Theta = [\underline{g}, \underline{g}']$, then Θ is an empty set, which helps us characterize the signal set of green advertisement Θ.

We first prove that $[\underline{g}, \underline{g}']$ cannot be Θ, and then prove that $[\bar{g}', \bar{g}]$ is the unique Θ. Given the definition of $g_{lh}^*, g_l^*, \pi_{lh}(g), \pi_l(g)$, it is straightforward to see $\pi_{lh}^* > \pi_{lh}(g_l^*) > \pi_l^*$. Given that $\pi_{lh}(g)$ is unimodal in *g*, there exists a unique $\underline{g}' < g_l^*$ such that $\pi_{lh}(\underline{g}') = \pi_l^*$, and a unique $\bar{g}' > g_{lh}^*$ such that $\pi_{lh}(\bar{g}') = \pi_l^*$. Similarly, we can show that there exist unique $\underline{g} < g_{hl}^*$ and $\bar{g} > g_h^*$ such that

$\pi_h(\underline{g}) = \pi_{hl}^* = \pi_h(\bar{g})$. We show $\underline{g}' \le \underline{g}$ by contradiction. Suppose $\underline{g} \le \underline{g}'$

$$\begin{aligned}&\pi_h(\underline{g}) - \pi_{lh}(\underline{g}')\\ &= \int_0^{\underline{g}} 2(\frac{b^2}{8} - k_h)\ t + \frac{1}{4}b\left(3+v_{L0}\right)dt - \int_0^{\underline{g}'} 2(\frac{b^2}{8} - k_l)t + \frac{1}{4}b\left(3+v_{L0}\right)dt \\ &= \underline{g}^2(k_l - k_h) - \int_{\underline{g}}^{\underline{g}'} 2(\frac{b^2}{8} - k_l)t + \frac{1}{4}b\left(3+v_{L0}\right)dt < \underline{g}^2(k_l - k_h)\end{aligned} \tag{B.8}$$

This inequality holds because $\underline{g}' < g_l^* < g_{lh}^*$. Further, we have

$$\begin{aligned}&\pi_{hl}^* - \pi_l^* = \int_0^{g_{hl}^*} b - 2k_h t dt - \int_0^{g_l^*} b - 2k_l t dt \\ &= (g_l^*)^2(k_l - k_h) + \int_{g_l^*}^{g_{hl}^*} b - 2k_h t dt > (g_l^*)^2(k_l - k_h)\end{aligned} \tag{B.9}$$

This inequality holds because $g_l^* < g_{hl}^*$.

Recall that $\pi_{lh}(g)$, $\pi_h(g)$ are unimodal and concave in g, so there exists a unique $\underline{g}' < g_l^*$ such that $\pi_{lh}(\underline{g}') = \pi_l^*$ and a unique $\underline{g} < g_{hl}^*$ such that $\pi_h(\underline{g}) = \pi_{hl}^*$. We obtain $\pi_h(\underline{g}) - \pi_{lh}(\underline{g}') = \pi_{hl}^* - \pi_l^*$, and $< (g_l^*)^2(k_l - k_h)$ because $\underline{g} \le \underline{g}' < g_l^*$. This contradiction comes from the assumption $\underline{g} \le \underline{g}'$. Thus, $\underline{g}' \le \underline{g}$. There cannot be any g_h^{s*} in $[\underline{g}, \underline{g}']$. Thus, $[\underline{g}, \underline{g}']$ cannot be Θ

Next, we prove that Θ can be characterized as $[\bar{g}', \bar{g}]$. Suppose $\bar{g}' > \bar{g}$.

$$\begin{aligned}&\pi_h(\bar{g}) - \pi_{lh}(\bar{g}')\\ &= \int_0^{\bar{g}} 2(\frac{b^2}{8} - k_h)\ t + \frac{1}{4}b\left(3+v_{L0}\right)dt - \int_0^{\bar{g}'} 2(\frac{b^2}{8} - k_l)t + \frac{1}{4}b\left(3+v_{L0}\right)dt \\ &= (\bar{g}')^2(k_l - k_h) - \int_{\bar{g}}^{\bar{g}'} 2(\frac{b^2}{8} - k_h)t + \frac{1}{4}b\left(3+v_{L0}\right)dt > (\bar{g}')^2(k_l - k_h)\end{aligned} \tag{B.10}$$

This inequation holds because $g_h^* < \bar{g} < \bar{g}'$.

Similarly,

$$\begin{aligned}&\pi_{hl}^* - \pi_l^* \\ &= \int_0^{g_{hl}^*} 2b - 2k_h t dt - \int_0^{g_l^*} 2b - 2k_l t dt \\ &= (g_{hl}^*)^2(k_l - k_h) + \int_{g_l^*}^{g_{hl}^*} 2b - 2k_l t dt < (g_{hl}^*)^2(k_l - k_h)\end{aligned} \tag{B.11}$$

This inequation holds because $g_l^* < g_{hl}^* < \bar{g}$ and $g_{hl}^* < \bar{g}$. Recall that $\pi_{lh}(g), \pi_{lh}(g)$ are unimodal and concave in g, so there exists a unique $\bar{g}' > g_l^*$ such that $\pi_{lh}(\bar{g}') = \pi_l^*$ and a unique $\bar{g} > g_{hl}^*$ such that $\pi_h(\bar{g}) = \pi_{hl}^*$. We obtain $\pi_h(\bar{g}) - \pi_{lh}(\bar{g}')$ = $\pi_{hl}^* - \pi_l^*$, and $(g_{hl}^*)^2(k_l - k_h) < (\bar{g}')^2(k_l - k_h)$ because $\bar{g}' > \bar{g} > g_{hl}^*$. This contradiction comes from the assumption $\bar{g}' > \bar{g}$. Thus, $\bar{g}' < \bar{g}$ and Θ can be characterized as $[\bar{g}', \bar{g}]$.

To separate themselves from l-type and maximize profit, h-type will choose a green advertising level in Θ. There are two scenarios. The first one is that $g_h^* \in \Theta$, i.e., h-type can get the profit the same as the symmetric information case. We call this the costlessly separating equilibrium outcome. The second one is $g_h^* \notin \Theta$, where h-type will distort their signal level upward to make an effort for separating from l-type and thus deviate from their fitst-best choice. We call this the costly separating equilibrium outcome. The above two scenarios can be characterized in detail as follows: If $\bar{g}' > g_h^*$, then $\pi_h(g)$ decreases for $g \in [\bar{g}', \bar{g}]$. In this case, the h-type manufacturer gets more profit with a smaller g_h^{s*}, and the decision constraint for h-type to separate is $g_h^{s*} \ge \bar{g}'$. Thus, the h-type manufacturer can set $g_h^{s*} = \bar{g}'$ to get the most profitable separating equilibrium outcome. Since $\pi_h(\bar{g}') < \pi_h^*$, the difference between them is the cost of separating. Finally, to achieve the separating equilibrium outcome of the market, the h-type manufacturer pays extra costs, while the l-type manufacturer does not pay any costs. If $\bar{g}' \le g_h^*$, then $\pi_h(g)$ increases for $g \in [\bar{g}', g_h^*]$ and decreases for $g \in [g_h^*, \bar{g}]$. In this case, the h-type manufacturer gets their optimal profit by choosing g_h^* without any extra cost. Thus, we can set $g_h^{s*} = g_h^*$.

Proof of Proposition 5.3

It is straightforward to verify (1) by solving $\bar{g}' \le g_h^* \le \bar{g}$, which is the N&S condition for costlessly separating equilibrium. Notice that the second inequality does not impose any constraints on k_l and k_h. We focus on the first inequality.

To prove (2) and (3), we rewrite the first inequality:

$$\begin{aligned}
& g_h^* \ge g_{lh}^* + \frac{\sqrt{\varphi(k_l)}}{3(8k_l - b^2)} \\
& \Leftrightarrow \frac{3(8k_l - b^2)}{8a_h - b^2} b(3 + v_{L0}) \ge 3b(3 + v_{L0}) + \sqrt{\varphi(k_l)} \\
& \Leftrightarrow 3b(3 + v_{L0})\left(\frac{8k_l - b^2}{8k_h - b^2} - \frac{8k_h - b^2}{8k_h - b^2}\right) \ge \sqrt{\varphi(k_l)} \\
& \Leftrightarrow 8(k_l - k_h) \ge \frac{\sqrt{\varphi(k_l)}}{g_h^*}
\end{aligned} \tag{B.12}$$

h-type suppliers will lower their green advertising level of equilibrium as k_h increases, so the right-hand side increases, while the left-hand side decreases, and (B.12) are more difficult to meet. When k_h increases until it exceeds the threshold, the manufacturer can achieve only costly-separating equilibrium. On the contrary, the above formula is increasingly easy to satisfy. When k_l increases, the right-hand side decreases and the left-hand side increases, and (B.12) is increasingly easy satisfied because $\varphi''(k_l) = 144b^4 / k_l^3 > 0$, $\lim_{a_l \to \infty} \varphi'(k_l) = 72[7 - (10 - v_{L0}) v_{L0} + 2\theta] \le 0$, if $v_{L0} \le 5 - \sqrt{2}\sqrt{9 - \theta}$.

Proposition B.1 In a gray market where the customers of the L-market have a green preference, there is no pooling equilibrium g^{p*}.

Proof of Proposition B.1

There is a non-empty signaling set about green advertising $\Theta', \Theta' \in R$ such that the following belief structure exists:

$$\mu(g)=\begin{cases}\mu_0, g\in\Theta' \\ 1, otherwise\end{cases} \tag{B.13}$$

we use the following pooling belief structure, which is widely used in signaling transmission literature such as Guo and Jiang (2016) and Sun et al. (2019):

$g < g^{p*}, \mu(g)=\mu; g \ge g^{p*}, \mu(g)=1$. In a pooling equilibrium, both types of suppliers choose the same g^{p*}, so neither retailer 1 nor retailer 2 can update their beliefs. The decision problem facing retailer 1 is to maximize $\pi_{jp}^{R1}=\mu\pi_{jh}^{R1}+(1-\mu)\pi_{jl}^{R1}=\mu\left(p_{Hh}-w_1\right)D_H$, and the second equality holds by Proposition 5.1. Similarly, the decision problem facing retailer 2 is to maximize $\pi_{jp}^{R2}=\mu\pi_{jh}^{R2}+(1-\mu)\pi_{jl}^{R2}=\left(p_L-w_2\right)D_L+\left(p_G-w_2\right)D_G$. The optimal decision for both retailers are the same as the optimal decision in the *h*-type supply chain. Therefore, the optimal wholesale price of the *h*-type manufacturer and the *l*-type manufacturer in pooling equilibrium are the same. Thus, we can obtain $\pi_{lp}=\pi_{lh}, \pi_{hp}=\pi_{hh}$. Based on our pooling belief structure and the manufacturer's profit maximization principle, pooling equilibrium cannot be in the interval where the profit declines. $\max\limits_{g\le g^{p*}}\pi_{jp}(\mathrm{g})=\pi_{jp}(g^{p*})$ ensures that the manufacturer obtains the local maximum benefit in the pooling equilibrium. Thus, $g^{p*}\le\min\{g_{lh}^*, g_h^*\}=g_{lh}^*$. For *h*-type manufacturers, the profit of pooling must be greater than that of revealing themselves, that is, $\max\limits_{g}\pi_{hp}(g\le g^{p*})\ge\max\limits_{g}\pi_h(g>g^{p*})$.

For the *l*-type manufacturer, the profit of pooling must be greater than that of pretending to be *h*-type, which implies $g^{p*}\ge\max\{g_{lh}^*, g_h^*\}=g_h^*$. Thus, pooling equilibrium does not exist.

Proof of Proposition 6.1

To focus on the green-gray effect, we assume $v_{Hh}=1, v_{Hl}=v_{lH}=l, v_{ll}=0$ (if we do not set it this way, results would be held qualitatively, but the expressions would be more complex). Following Section 3, the demand and profit in the h-type supply chain can be derived as:

$$\begin{cases} D_H=1-(p_H-p_G)(1+g) \\ D_L=1-\dfrac{p_L}{l} \\ D_G=(p_H-p_G)(1+g)-\left(\dfrac{1}{g}+1\right)p_G \end{cases}$$

$$\begin{cases} \pi_{r1}=\max\limits_{p_H,}\left(p_H-w_1\right)D_H \\ \pi_{r2}=\max\limits_{p_L,p_G}\left(p_L-w_2\right)D_L+\left(p_{G2}-w_2\right)D_G \\ \pi_M=\max\limits_{w_1,w_2}w_1D_j+w_2(D_L+D_G) \end{cases}$$

$$w_1=\frac{2g+(1+g)(2+3g)l}{2g(2+g)+4(1+g)^2 l}, w_2=\frac{g(4+3g)l}{2g(2+g)+4(1+g)^2 l}$$

$$p_H=\frac{4g(3+2g)+(1+g)(12+g(22+9g))l}{2(4+3g)\left(g(2+g)+2(1+g)^2 l\right)}, p_G=\frac{g\left(2g(3+2g)+(1+g)^2(14+9g)l\right)}{2(1+g)(4+3g)\left(g(2+g)+2(1+g)^2 l\right)}$$

$$p_L=\frac{1}{2}\left(l+\frac{g(4+3g)l}{2g(2+g)+4(1+g)^2 l}\right), \pi_{M-h}(g)=-\frac{1}{2}g+\frac{8g(1+g)+(8+g(52+g(70+27g)))l}{8(4+3g)\left(g(2+g)+2(1+g)^2 l\right)}$$

Similarly, the profit of the l-type manufacturer is

$$\pi_{M-l}(g)=-\frac{1}{2}g^2+\frac{(2+3g)l^2}{4(4+3g)}$$

$\pi_{M-l}'(g)=-g^2+\dfrac{3l}{2(4+3g)^2}$， $\pi_{M-l}'(g)$ is continuous in g if $g\geq 0$

$0<\pi_{M-l}'(0)<\dfrac{3}{32}, \pi_{M-l}'(\dfrac{1}{3})<0$， there exist $0<g_l^*<\dfrac{1}{3}$ that maximize $\pi_{M-l}(g)$.

Proof of Proposition 6.2

It is straightforward to check that there does not exist a belief off equilibrium path in which the manufacturer choice is inconsistent and $\max_g \pi_{lp}(g)>\pi_l^*$.

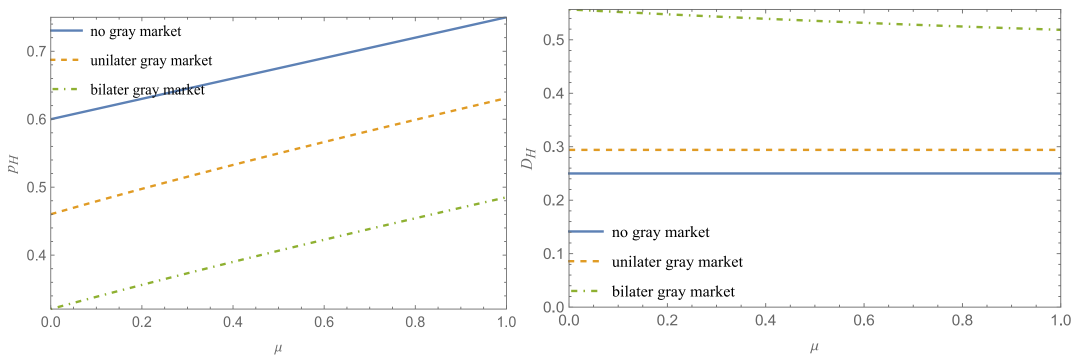
Appendix C

Lemma 3.3 Cho & Kreps (1987)

(i) A strategy profile s' for player i of θ is dominated by an equilibrium strategy profile s if $R(s_i,s_{-i},\theta)>\max_{\mu} R(s',s_{-i}(\mu),\theta)$, where μ is possible beliefs and $s_{-i}(\mu)$ is the best response function of other players.

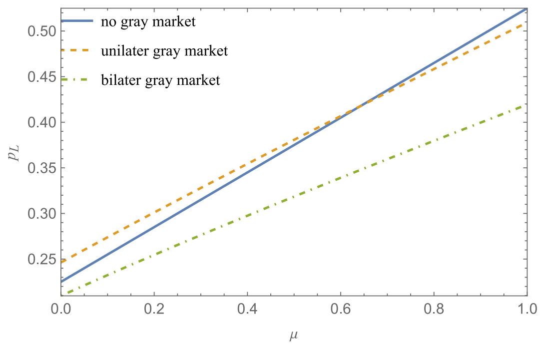
(ii) For any strategy profile s not on the equilibrium path, if s is equilibrium dominated for θ_1 but not for θ_2, beliefs must only place weight on those types that are not equilibrium dominated.

Appendix D

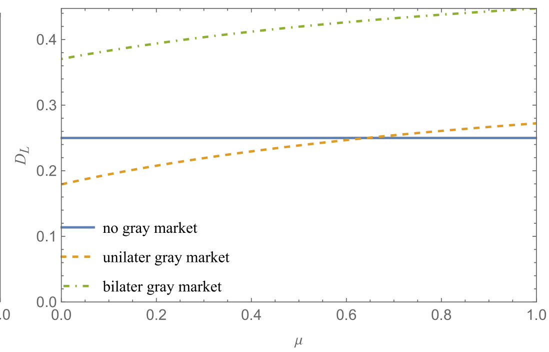


(a) Price of H-product (b) Demand of H-product

Figure D1. Authorized product in the H market.

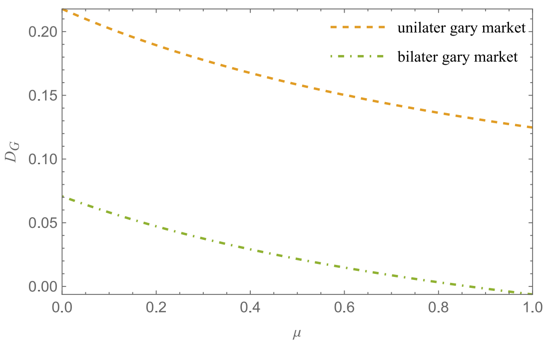


(a) Price of L-product

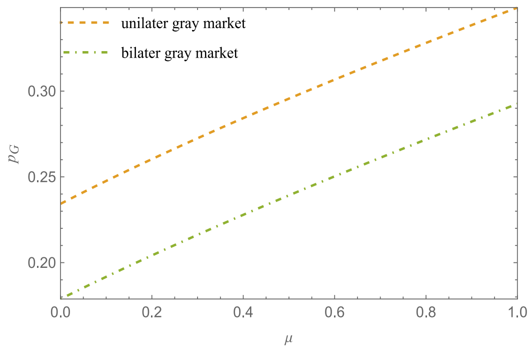


(b) Demand of L-product

Figure D2 Authorized product in the L market.

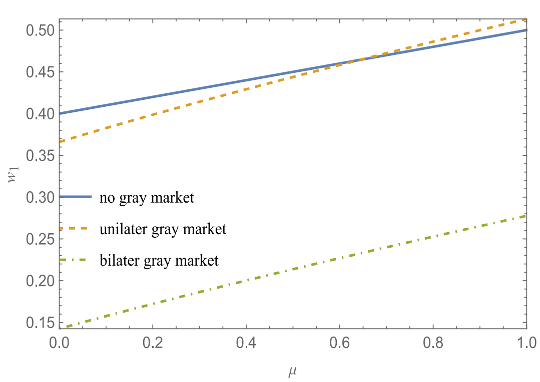


(a) Price of gray product

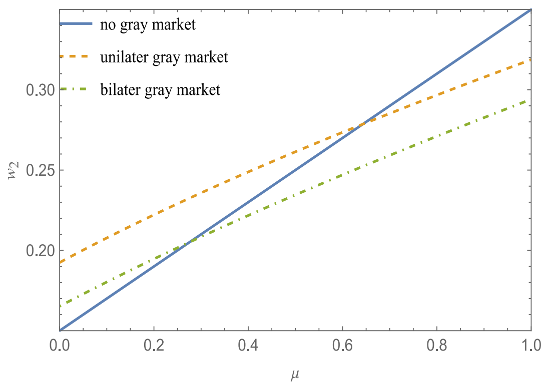


(b) Price of gray product

Figure D3 Gray product in the H market.

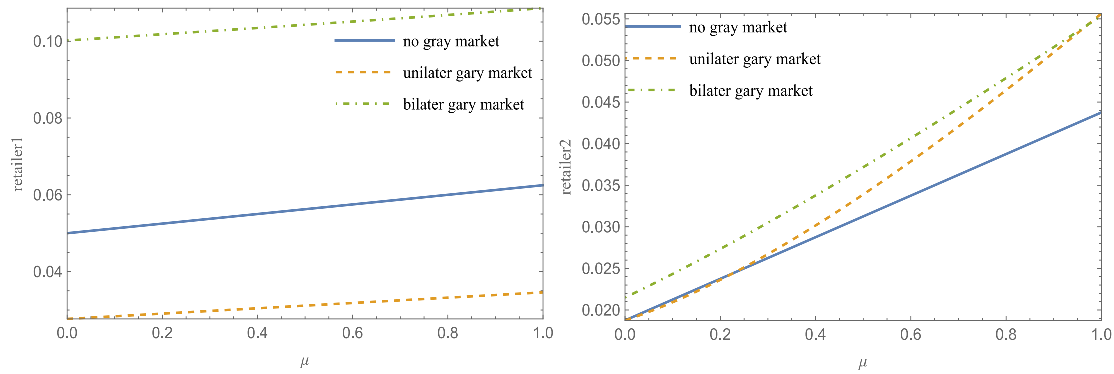


(a) Wholesale price for retailer 1



(b) Wholesale price for retailer 2

Figure D4. Wholesale price.



(a) Retailer 1's profit (b) Retailer2's profit

Figure D5. Retailers' profit.

Optimal decision

The most apparent characteristic of retailer 1 is that it has the strongest pricing power. With the entry of retailer 2, retailer 1 will lower its prices to compete for the H-market share. When retailer 1 is able to sell unauthorized products, it will further lower the price of authorized products in the H-market, shifting its 'profit focus' to the L-market. This leads to a phenomenon where the demand for retailer 1's authorized product (H product) moves in the opposite direction of its price. The increasing demand for the H product in the bilateral gray market indicates a significant shift of 'profit focus' to the L-market. This impact is substantial because the price elasticity of demand for authorized products is very high. Consequently, retailer 1 does not lose any profit by lowering the equilibrium price.

Figure D5 shows that unilateral gray market cannot change the wholesale price and selling price drastically, whereas bilateral market can. This suggests that if both retailers can sell unauthorized products in each other's market, demand will increase, and the price will fall to a degree far beyond the unilateral case.

For retailer 2, there is no change in the *l*-type supply chain between unilateral and bilateral markets. It is very interesting that retailer 2 will lower the price of unauthorized products (G2 products) in response to retailer 1's incursion into the L-market. The reason can be due to retailer 2 lowering the price of a G2 product in exchange for less competition with retailer 1 at the L-market. There is an intersection between the demand for no gray market and unilateral gray market, corresponding to the selling price. This phenomenon can be attributed to a shift in 'profit focus', where retailer 2 aims to capture a larger share of the H-market and slightly raises price in the L-market to screen out lower-value customers.

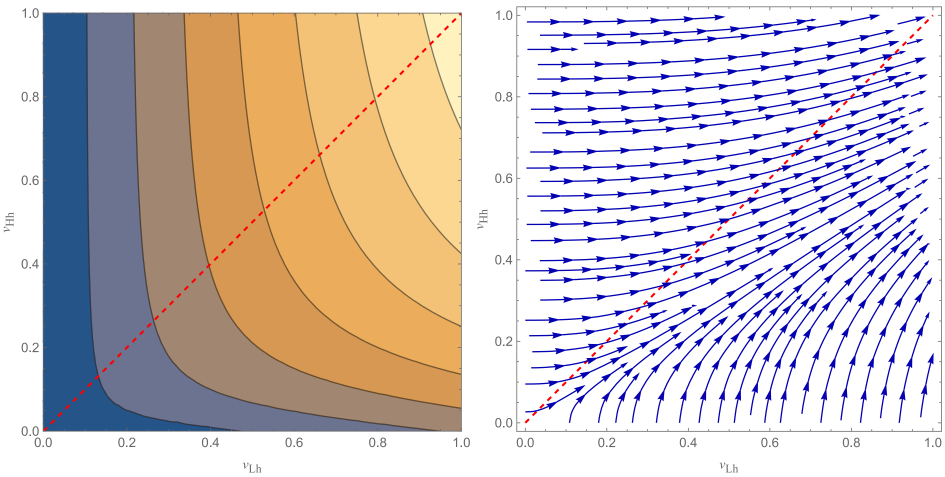
Optimal profit of retailer 1

From Figure D2 and Figure D6(a), We can get the logic of retailer 1's action. Retailer 1 benefits the most in the bilateral market and shifts part of their "profit focus" to the L-market, which enables them to lower the selling price in the H-market and significantly boost customer demand for the H product, leading to substantial profit improvements in both markets. Conversely, retailer 1 suffers the greatest losses in the unilateral gray market, where price competition from a gray product erodes its pricing power. Although the demand for a H product increases, it does not offset the losses incurred from reduced prices. We can categorize retailer 1's pricing strategy into two scenarios: voluntary and coerced price reductions. In the unilateral gray market, the incursion of gray product forces retailer 1 to lower the price, resulting in decreased profit. In contrast, retailer 1's price reduction is a strategic move to enhance profitability in the bilateral gray market.

Optimal profit of retailer 2

From Figure D3, Figure D4, and Figure D6(b), we can get the logic of retailer 2's action. It may seem counterintuitive, as retailer 2's profit remains consistent across unilateral and bilateral gray markets in the least-cost separating equilibrium for high-quality products. However, for low-quality products, retailer 2 distinctly favors the bilateral gray market, a preference that holds in any other equilibrium as well. The incremental profit for retailer 2 in the bilateral gray market originates from the L- market, where authorized products exhibit significant price elasticity of demand. The inflow of a G1 product into the L-market reduces the price of the L product, but the demand offsets the drop in revenue of a G2 product. In the separating equilibrium of a unilateral market, high-quality products have higher price elasticity, the only scenario where a 'profit focus' shift might occur. In the least-cost separating equilibrium, retailer 2 lacks the incentive to sell low-quality gray products, as customers are aware of the quality and offer only a low price.

Appendix E



(a) Profit contour (b) Profit gradient

Figure E1. The direction of profit increase.

Appendix F

The influence of the gray market on supply chain

In Section 3, our work is relatively close to Cao & Zhang (2020). The difference is that they consider the discount factor for gray products to be 1. The gray market improves supply chain performance in three ways: (i) Mitigation of double marginal effects. In our model, the manufacturer faces the demand of high and low markets, and gray products make the marginal revenue curve of the two markets closer to the demand curve, alleviating the double marginal effect. Moreover, the larger the θ, the closer the marginal revenue curve and demand curve will be. For example, in a unilateral market, the intrusion of a gray product sold by retailer 2 reduces the profit of retailer 1, but mitigates the double marginal effect of the high-end market. In this process, retailer 2's role as a "medium" integrates supply chain resources, making decision-making more centralized. In a bilateral market, the double marginal effect of both markets is mitigated. (ii) Enhancement of cannibalization effects. The invasion of gray products has strengthened the price competition in the high-end market, forcing retailer 1 to reduce their retail prices, which on the one hand reduces the profit margin in the high-end market, and on the other hand increases the demand. In a bilateral market, price competition extends to the low-end market, and demand in both markets increases. (iii) Filling residual demand. Cannibalization creates price competition that satisfies the demand for lower customer value without alienating the demand for higher customer value.

The influence of green advertisements on the supply chain

There is no way to know product quality because retailers and customers are in the same information set (Li, et al., 2023). The *h*-type manufacturer has an incentive to differentiate themselves from the *l*-type. In Section 3, we found a reasonable equilibrium that the *h*-type manufacturer invests a certain amount of money to green advertisements, while the *l*-type manufacturer does not send advertisements. There are three major insights: (i) Green advertising can be seen as a tool to address incomplete information. Compared with complete information, only the *h*-type supply chain performance suffers in a reasonable equilibrium, and its loss is equal to the manufacturer's investment in green advertisements. (ii) Even so, when $k_h / k_l < R(1) - R(\mu_0) / R(1) - R(0)$, high-quality supply chains have incentive to differentiate themselves because the customer value of high-quality products is sufficiently high, and signaling cost is sufficiently low compared to their counterparts. (iii) The free circulation of gray products stimulates market vitality, improves supply chain performance, and widens the profit gap. The *h*-type manufacturer, in order to maintain their profit margin, will invest more in green advertisements.

The influence of green advertisements on the gray market

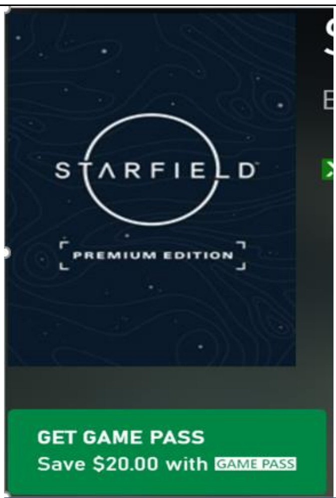
The *l*-type manufacturer captures the profits of both retailers. Furthermore, we prove that there is a separating equilibrium and discuss the conditions under which manufacturer distorts their own green advertising levels in order to achieve separation. In short, when the gap of cost coefficient is large, the *h*-type manufacturer needs to deviate from their optimal level to choose a higher one. This shows that the cost coefficient determines the welfare of the *h*-type manufacturer, but has nothing to do with the existence of pooling equilibrium.

Next, we compare the differences in the refined equilibrium between the two models under asymmetric information. When customers do not have a green preference, the harm of information asymmetry to the gray supply chain is limited, with only the *h*-type manufacturer needing to incur a certain cost for green advertising to distinguish themselves from the *l*-type manufacturer. When customers have a green preference, only a separating equilibrium exists, and the harm of information asymmetry to the *l*-type supply chain is significant. Not only are all the profits of the two retailers taken by the manufacturer, but the manufacturer's own profit also decreases. The managerial implications of comparing these two situations are that when a company needs to enter a new market, it is crucial to research in advance the potential customers' preferences for green products in the new market. Otherwise, it should not enter the new market rashly.

Appendix G

Microsoft does not restrict the cross-region use of redemption codes, resulting in a significant price difference between authorized and gray market games. For example, Table G1(a) shows a video game called “Starfield”, which is priced at $99.99 on the official website, whereas Table G1(b) depicts that it is sold for only 275 RMB, about $38, on the gray market in China.

Table G1. The price of a gray product and an authorized product.

	Game name	Price
(a) Authorized Channel		
(b) Gray market		

Source：https://www.xbox.com/ and (alicdn.com)