Journal of Industrial and Management Optimization

https://www.aimspress.com/journal/jimo

JIMO, 22(4): 2147–2180.
DOI:10.3934/jimo.2026079
Received: 02 January 2026
Revised: 09 March 2026
Accepted: 19 March 2026
Published: 31 March 2026

Research article

Constructing a new robust bilevel fashion product supply chain network with uncertain demand and transportation cost

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Appendix A: Omitted Proofs

Proof of Theorem 1

Proof First, constraint (3.12) can be equivalently represented as $M(CM_w) \le F_1$ with

$$M(CM_w) = \sup_{CM_w \in U^2_{CM}} \left\{ \sum_w \sum_t CM_w * x^t_w * DM_w - \min_{CM'_w \in U^1_{CM}} \phi(CM_w, CM'_w) \right\},$$

where $\phi(CM_w, CM'_w)$ measures the distance between CM_w and CM'_w. The function $\phi(\cdot,\cdot)$ is assumed to be convex, closed, and nonnegative, and $\phi(CM_w, CM_w) = 0$ in the case of $CM_w \in \mathbb{R}_W$. In this paper, $\phi(CM_w, CM'_w) = \alpha(\|CM_w - CM'_w\|_1)$, where α denotes a function that is convex and nonnegative, satisfying the condition $\alpha(0) = 0$.

After introducing auxiliary variables $\pi, \iota, \varrho, \varpi$, we have

$$M(CM_w) = \sup_{CM_w \in U^2_{CM}, CM'_w \in U^1_{CM}, \pi, \iota} \left\{ \sum_w \sum_t CM_w * x^t_w * DM_w - \phi(CM_w, CM'_w) | \pi = CM_w, \iota = CM'_w \right\}.$$

Based on Lagrange duality, we obtain

$$M(CM_w) = \min_{\varrho,\varpi} \sup_{CM_w \in U^2_{CM}, CM'_w \in U^1_{CM}, \pi, \iota} \left\{ \sum_w \sum_t CM_w * x^t_w * DM_w \right. \\ \left. - \phi(\pi,\iota) - \varrho(\iota - CM'_w) - \varpi(\pi - CM_w) \right\}.$$

We divide $M(CM_w)$ into three parts: $M(CM_w) = \min_{\varrho,\varpi}\{m_1(\varpi, x^t_w) + m_2(\varrho,\varpi) + m_3(\varrho)\}$ with

$$m_1(\varpi, x^t_w) = \sup_{CM_w \in U^2_{CM}} \left\{ \sum_w \sum_t CM_w * x^t_w * DM_w + \varpi CM_w \right\},$$
$$m_2(\varrho,\varpi) = \sup_{\pi,\iota} \{-\phi(\pi,\iota) - \varrho\iota - \varpi\pi\},$$
$$m_3(\varrho) = \sup_{CM'_w \in U^1_{CM}} \left\{ \varrho CM'_w \right\}.$$

Rewriting the first part, one has

$$m_1(\varpi, x^t_w) = \sup_{CM_w \in U^2_{CM}} \left\{ \sum_w \sum_t CM_w * x^t_w * DM_w + \varpi CM_w \right\} \\ = \sup_{CM_w} \left\{ \sum_w \sum_t CM_w * x^t_w * DM_w + \varpi CM_w - \delta(CM_w | U^2_{CM}) \right\} \\ = \min_{\vartheta} \left\{ \delta^*(\vartheta | U^2_{CM}) - [f(\vartheta, x^t_w) + \varpi\vartheta]_* \right\}.$$

Using the relationship between U^2_{CM} and Z^2_{CW}, it can be verified that $\delta^*(\vartheta | U^2_{CM}) = CM^0_w \vartheta + \delta^*(\widetilde{CM}_w \vartheta | Z^2_{CM})$. Proceeding, we can also derive the concave conjugate of the second term in m_1, the term associated with variable ϑ,

$$m_1(\varpi, x^t_w) = \min_{\vartheta} \left\{ CM^0_w \vartheta + \delta^*(\widetilde{CM}_w \vartheta | Z^2_{CM}) - f_*(\vartheta - \varpi, x^t_w) \right\}.$$

For the second part, $m_2(\varrho,\varpi)$ can be simplified as

$$m_2(\varrho,\varpi) = \sup_{\pi,\iota} \{-\phi(\pi,\iota) - \varrho\iota - \varpi\pi\} = \phi^{**}(-\varrho, -\varpi),$$

and finally, for the third part, $m_3(\varrho)$, we have

$$m_3(\varrho) = \sup_{CM'_w \in U^1_{CM}} \left\{ \varrho CM'_w \right\} = \delta^*(\varrho | Z^1_{CM}) = CM^0_w \varrho + \delta^*(\widetilde{CM}_w \varrho | Z^1_{CM}).$$

Using the above expressions, inequality $M(CM_w) \leq F_1$ can be equivalently represented as,

$$\min_{\varrho,\varpi} \left\{ \min_{\vartheta} \left\{ CM^0_w \vartheta + \delta^*(\widetilde{CM}_w \vartheta | Z^2_{CM}) - f_*(\vartheta - \varpi, x^t_w) \right\} \right. \\ \left. + \phi^{**}(-\varrho, -\varpi) + CM^0_w \varrho + \delta^*(\widetilde{CM}_w \varrho | Z^1_{CM}) \right\} \leq F_1.$$

Therefore, the inequality $M(CM_w) \leq F_1$ holds if and only if we can find ϱ, ϖ, and ϑ, such that

$$CM_w^0\vartheta + \delta^*(\widetilde{CM}_w\vartheta|Z_{CM}^2) - f_*(\vartheta - \varpi, x_w^t) + \phi^{**}(-\varrho, -\varpi)$$
$$+ CM_w^0\varrho + \delta^*(\widetilde{CM}_w\varrho|Z_{CM}^1) \leq F_1.$$

In addition, by the following result

$$\phi^{**}(-\varrho, -\varpi) = \sup_{\pi,\iota}\{-\phi(\pi, \iota) - \varrho\iota - \varpi\pi\} \geq \sup_{\pi}\{-\phi(\pi, \pi) - (\varrho + \varpi)\pi\}$$
$$= \sup_{\pi}\{-(\varrho + \varpi)\pi\} = \begin{cases} 0, & \varrho = -\varpi, \\ \infty, & \varrho \neq -\varpi, \end{cases}$$

we have $\varrho = -\varpi$ and $\phi^{**}(-\varrho, -\varpi) = 0$.

Note that $f_*(\vartheta - \varpi, x_w^t) = f_*(\vartheta + \varrho, x_w^t)$ due to $\varrho = -\varpi$. Thus, one has

$$f_*(\vartheta + \varrho, x_w^t) = \min_{CM_w}\{CM_w(\vartheta + \varrho) - CM_w * x_w^t * DM_w\}$$
$$= \begin{cases} 0, & \vartheta + \varrho = x_w^t * DM_w, \\ \infty, & \vartheta + \varrho \neq x_w^t * DM_w. \end{cases}$$

Hence, $f_*(\vartheta + \varrho, x_w^t) = 0$ with $\vartheta + \varrho = x_w^t * DM_w$ and $||\varrho||_1 \leq \theta_{CM}$.

As a consequence, one has

$$\begin{cases} CM_w^0\vartheta + \delta^*(\widetilde{CM}_w\vartheta|Z_{CM}^2) + CM_w^0\varrho + \delta^*(\widetilde{CM}_w\varrho|Z_{CM}^1) \leq F_1, \\ \vartheta + \varrho = x_w^t * DM_w, \\ ||\varrho||_1 \leq \theta_{CM}. \end{cases}$$

Under uncertainty sets (U_{CM}^1, U_{CM}^2) with perturbation sets (Z_{CM}^1, Z_{CM}^2), we have

$$\begin{cases} \delta^*(\widetilde{CM}_w\varrho|Z_{CM}^1) = \kappa'_{CM}\sqrt{(\sigma'_{CM}m_{CM})^2} + \mu'_{CM}|\sigma'_{CM}l_{CM}| + \sigma'_{CM}|u_{CM}|, \\ m_{CM} + l_{CM} + u_{CM} = -\sum_w\sum_t \eta_w^t * \widetilde{CM}_w, \end{cases}$$

and

$$\begin{cases} \delta^*(\widetilde{CM}_w\vartheta|Z_{CM}^2) = \mu_{CM}|\sigma_{CM}c_{CM}| + \kappa_{CM}\sqrt{(\sigma_{CM}d_{CM})^2}, \\ c_{CM} + d_{CM} = -\sum_w\sum_t(x_w^t * DM_w - \eta_w^t) * \widetilde{CM}_w. \end{cases}$$

Based on the above analysis, we derive the equivalent reformulation (4.3). The proof of theorem is complete. □

Proof of Theorem 2

Proof We start by reformulating constraint (3.13) in the equivalent form $M(CW_{wr}) \leq F_2$ under the condition that

$$M(CW_{wr}) = \sup_{CW_{wr}\in U_{CW}^2}\Bigg\{\sum_w\sum_r\sum_t CW_{wr} * y_{wr}^t * DW_{wr}$$
$$- \min_{CW'_{wr}\in U_{CW}^1} \phi(CW_{wr}, CW'_{wr})\Bigg\}.$$

After introducing auxiliary variables $\pi, \iota, \varsigma, \varphi$, we obtain

$$M(CW_{wr}) = \sup_{CW_{wr}\in U^2_{CW}, CM'_w\in U^1_{CW}, \pi, \iota} \left\{ \sum_w \sum_r \sum_t CW_{wr} * y^t_{wr} * DW_{wr} - \phi(CW_{wr}, CW'_{wr}) | \pi = CW_{wr}, \iota = CW'_{wr} \right\}.$$

Based on Lagrange duality, one has

$$M(CW_{wr}) = \min_{\varsigma,\varphi} \sup_{CW_{wr}\in U^2_{CW}, CW'_{wr}\in U^1_{CW}, \pi, \iota} \left\{ \sum_w \sum_r \sum_t CW_{wr} * y^t_{wr} * DW_{wr} - \phi(\pi, \iota) - \varsigma(\iota - CW'_{wr}) - \varpi(\pi - CW_{wr}) \right\}.$$

We next divide $M(CW_{wr})$ into three parts: $M(CW_{wr}) = \min_{\varsigma,\varphi} \left\{ m_1(\varphi, y^t_{wr}) + m_2(\varsigma, \varphi) + m_3(\varsigma) \right\}$ with

$$m_1(\varphi, y^t_{wr}) = \sup_{CW_{wr}\in U^2_{CW}} \left\{ \sum_w \sum_r \sum_t CW_{wr} * y^t_{wr} * DW_{wr} + \varphi CW_{wr} \right\},$$
$$m_2(\varsigma, \varphi) = \sup_{\pi,\iota} \{ -\phi(\pi, \iota) - \varsigma\iota - \varphi\pi \},$$
$$m_3(\varsigma) = \sup_{CW'_{wr}\in U^1_{CW}} \left\{ \varsigma CW'_{wr} \right\}.$$

Rewriting the first part $m_1(\varphi, y^t_{wr})$, we obtain:

$$m_1(\varphi, y^t_{wr}) = \sup_{CW_{wr}} \left\{ \sum_w \sum_r \sum_t CW_{wr} * y^t_{wr} * DW_{wr} + \varphi CW_{wr} - \delta(CW_{wr} | U^2_{CW}) \right\}$$
$$= \min_{\vartheta} \left\{ \delta^*(\vartheta | U^2_{CW}) - [f(\vartheta, y^t_{wr}) + \varphi\vartheta]_* \right\}.$$

By exploiting the relationship between U^2_{CW} and Z^2_{CW}, we first verify that: $\delta^*(\vartheta | U^2_{CW}) = CW^0_{wr}\vartheta + \delta^*(\widetilde{CW}wr\vartheta | ZCW^2)$. Building on this, the concave conjugate for the second term in m_1 (which involves ϑ) can be derived, yielding: $m_1(\varphi, y^t_{wr}) = \min_{\vartheta}\{CW^0_{wr}\vartheta + \delta^*(\widetilde{CW}_{wr}\vartheta | Z^2_{CW}) - f_*(\vartheta - \varphi, y^t_{wr})\}$. Furthermore, m_2 can be simplified as $m_2(\varsigma, \varphi) = \phi^{**}(-\varsigma, -\varphi)$, and, finally, $m_3(\varsigma) = \delta^*(\varsigma | Z^1_{CW}) = CW^0_{wr}\varsigma + \delta^*(\widetilde{CW}_{wr}\varsigma | Z^1_{CW})\}$.

Based on the above analysis, inequality $M(CW_{wr}) \le F_2$ can be equivalently represented as:

$$\min_{\varsigma,\varphi} \left\{ \min_{\vartheta} \left\{ CW^0_{wr}\vartheta + \delta^*(\widetilde{CW}_{wr}\vartheta | Z^2_{CW}) - f_*(\vartheta - \varphi, y^t_{wr}) \right\} + \phi^{**}(-\varsigma, -\varphi) + CW^0_{wr}\varsigma + \delta^*(\widetilde{CW}_{wr}\varsigma | Z^1_{CW}) \right\} \le F_2.$$

Thus, the satisfaction of $M(CW_{wr}) \le F_2$ requires, and is guaranteed by, the existence of ς, φ, and ϑ, for which

$$CW^0_{wr}\vartheta + \delta^*(\widetilde{CW}_{wr}\vartheta | Z^2_{CW}) - f_*(\vartheta - \varphi, y^t_{wr}) + \phi^{**}(-\varsigma, -\varphi) + CW^0_{wr}\varsigma + \delta^*(\widetilde{CW}_{wr}\varsigma | Z^1_{CW}) \le F_2.$$

Furthermore, by the following result:

$$\begin{aligned}\phi^{**}(-\varsigma,-\varphi) &= \sup_{\pi,\iota}\{-\phi(\pi,\iota)-\varsigma\iota-\varphi\pi\} \geq \sup_{\pi}\{-\phi(\pi,\pi)-(\varsigma+\varphi)\pi\} \\ &= \sup_{\pi}\{-(\varsigma+\varphi)\pi\} = \begin{cases} 0, & \varsigma=-\varphi, \\ \infty, & \varsigma\neq-\varphi,\end{cases}\end{aligned}$$

we obtain $\varsigma = -\varphi$ and $\phi^{**}(-\varsigma,-\varphi) = 0$.

Note that $f_*(\vartheta-\varphi, y^t_{wr}) = f_*(\vartheta+\varsigma, y^t_{wr})$ due to $\varsigma = -\varphi$. Thus, one has

$$\begin{aligned} f_*(\vartheta+\varsigma, y^t_{wr}) &= \min_{CW_{wr}}\{CW_{wr}(\vartheta+\varsigma) - CW_{wr} * y^t_{wr} * DW_{wr}\} \\ &= \begin{cases} 0, & \vartheta+\varsigma = CW_{wr} * y^t_{wr} * DW_{wr}, \\ \infty, & \vartheta+\varsigma \neq CW_{wr} * y^t_{wr} * DW_{wr}.\end{cases}\end{aligned}$$

Hence, $f_*(\vartheta+\varsigma, y^t_{wr}) = 0$ with $\vartheta+\varsigma = y^t_{wr} * DW_{wr}$ and $\|\varsigma\|_1 \leq \theta_{CW}$, which lead to

$$\begin{cases} CW^0_{wr}\vartheta + \delta^*(\widetilde{CW}_{wr}\vartheta|Z^2_{CW}) + CW^0_{wr}\varsigma + \delta^*(\widetilde{CW}_{wr}\varsigma|Z^1_{CW}) \leq F_2 \\ \vartheta+\varsigma = y^t_{wr} * DW_{wr} \\ \|\varsigma\|_1 \leq \theta_{CW}. \end{cases}$$

Based on uncertainty sets (U^1_{CW}, U^2_{CW}) with perturbation sets (Z^1_{CW}, Z^2_{CW}), we can obtain

$$\begin{cases} \delta^*(\widetilde{CW}_{wr}\varsigma|Z^1_{CW}) = \kappa'_{CW}\sqrt{(\sigma'_{CW}m_{CW})^2} + \mu'_{CW}|\sigma'_{CW}l_{CW}| + \sigma'_{CW}|u_{CW}|, \\ m_{CW} + l_{CW} + u_{CW} = -\sum_w\sum_r\sum_t \eta^t_{wr} * \widetilde{CW}_{wr}, \end{cases}$$

and

$$\begin{cases} \delta^*(\widetilde{CW}_{wr}\vartheta|Z^2_{CW}) = \mu_{CW}|\sigma_{CW}c_{CW}| + \kappa_{CW}\sqrt{(\sigma_{CW}d_{CW})^2}, \\ c_{CW} + d_{CW} = -\sum_w\sum_r\sum_t (y^t_{wr} * DW_{wr} - \eta^t_{wr}) * \widetilde{CW}_{wr}. \end{cases}$$

According to the above analysis, we can obtain equivalent reformulation (4.4). The proof of the theorem is complete. □

Proof of Theorem 3

Proof First, constraints (3.14) can be equivalently represented as $M(D^t_i) \leq \sum_r a^t_{ri} + \sum_r s^t_{ri}$ with

$$M(D^t_i) = \sup_{D^t_i\in U_D}\left\{D^t_i - \min_{\overline{D}^t_i\in\overline{U}_D}\phi(D^t_i, \overline{D}^t_i)\right\},$$

where $\phi(D^t_i, \overline{D}^t_i)$ measures the distance between D^t_i and $\overline{D}^t_i$. The distance function $\phi(D^t_i, D^t_i)$ is defined as $\phi(D^t_i, \overline{D}^t_i) = \alpha(\|D^t_i - \overline{D}^t_i\|_1)$, where α is a convex and nonnegative function with $\alpha(0) = 0$. Here, $\phi(D^t_i, \overline{D}i^t)$

serves to quantify the distance between Di^t and $\overline{D}i^t$, and is given by $\phi(Di^t, \overline{D}i^t) = \alpha(|Di^t - \overline{D}_i^t|_1)$. In this expression, α denotes a convex, nonnegative function, fulfilling $\alpha(0) = 0$.

After introducing auxiliary variables π, ι, χ, τ, we obtain

$$M(D_i^t) = \sup_{D_i^t \in U_D, \overline{D}_i^t \in \overline{U}_D, \pi, \iota} \left\{ D_i^t - \min_{\overline{D}_i^t \in \overline{U}_D} \phi(D_i^t, \overline{D}_i^t) | \pi = D_i^t, \iota = D_i^{t'} \right\}.$$

Based on the Lagrange duality, one has

$$M(D_i^t) = \min_{\chi,\tau} \sup_{D_i^t \in U_D, \overline{D}_i^t \in \overline{U}_D, \pi, \iota} \left\{ D_i^t - \min_{\overline{D}_i^t \in \overline{U}_D} \phi(D_i^t, \overline{D}_i^t) - \chi(\iota - D_i^{t'}) - \tau(\pi - D_i^t) \right\}.$$

We represent $M(D_i^t)$ as three parts: $M(D_i^t) = \min_{\chi,\tau} \{m_1(\tau) + m_2(\chi,\tau) + m_3(\chi)\}$, where

$$\begin{aligned} m_1(\tau) &= \sup_{D_i^t \in U_D} \{D_i^t + \tau D_i^t\}, \\ m_2(\chi,\tau) &= \sup_{\pi,\iota} \{-\phi(\pi,\iota) - \chi\iota - \tau\pi\}, \\ m_3(\chi) &= \sup_{\overline{D}_i^t \in \overline{U}_D} \left\{\chi D_i^{t'}\right\}. \end{aligned}$$

We first deal with $M(D_i^t)$. Rewriting the first part yields $m_1(\tau) = (1+\tau)\left(D_i^{t0} + \mu'_D|\sigma'_D\widetilde{D}_i^t|\right)$, $m_2(\chi,\tau) = \phi^{**}(-\chi,-\tau)$, and $m_3(\chi) = \chi\left(D_i^{t0} + \mu_D|\sigma_D\widetilde{D}_i^t|\right)$.

Based on the above analysis, inequality $M(D_i^t) \leq \sum_r a_{ri}^t + \sum_r s_{ri}^t$ can be equivalently represented as:

$$\begin{aligned} &\min_{\chi,\tau} \left\{(1+\tau)\left(D_i^{t0} + \mu'_D|\sigma'_D\widetilde{D}_i^t|\right) + \phi^{**}(-\chi,-\tau) + CM_w^0\chi \right. \\ &\left. + \delta^*\left(\widetilde{CM}_w\chi|Z_{CM}^1\right)\right\} \leq \sum_r a_{ri}^t + \sum_r s_{ri}^t. \end{aligned}$$

As a result, $M(D_i^t) \leq \sum_r a_{ri}^t + \sum_r s_{ri}^t$ if and only if there exist χ, τ, and ϑ, such that

$$\begin{aligned} &(1+\tau)\left(D_i^{t0} + \mu'_D|\sigma'_D\widetilde{D}_i^t|\right) + \phi^{**}(-\chi,-\tau) + CM_w^0\chi \\ &+ \delta^*\left(\widetilde{CM}_w\chi|Z_{CM}^1\right)\} \leq \sum_r a_{ri}^t + \sum_r s_{ri}^t. \end{aligned}$$

Furthermore, by

$$\begin{aligned} \phi^{**}(-\chi,-\tau) &= \sup_{\pi,\iota} \{-\phi(\pi,\iota) - \chi\iota - \tau\pi\} \geq \sup_{\pi} \{-\phi(\pi,\pi) - (\chi+\tau)\pi\} \\ &= \sup_{\pi} \{-(\chi+\tau)\pi\} = \begin{cases} 0, & \chi = -\tau, \\ \infty, & \chi \neq -\tau, \end{cases} \end{aligned}$$

we obtain $\chi = -\tau$ and $\phi^{**}(-\chi,-\tau) = 0$.

Hence, $\chi = -\tau$ with $\|\chi\|_1 \leq \theta_D$ and based on uncertainty sets $\overline{U}_D, U_D$ with perturbation sets $\overline{Z}_D, Z_D$, we obtain the following reformulation,

$$\begin{cases} D_i^{t0} + \mu'_D|\sigma'_D\widetilde{D}_i^t * \eta_D| + \mu_D|\sigma_D\widetilde{D}_i^t * (1-\eta_D)| \leq \sum_r a_{ri}^t + \sum_r s_{ri}^t, \ \forall\, i, t, \\ |\eta_D| \leq \theta_D. \end{cases}$$

Furthermore, we can obtain equivalent system (4.5). The proof of the theorem is complete. □