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Opinion paper

Dynamic analysis of subsidies for reducing farmers' grain storage losses

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Appendix

Proof of Proposition 1

According to optimal control theory, we denote the optimal profit functions of small-scale farmers and grain enterprises as follows:

$$J_F^D(v,G)=e^{-\rho t}V_F^D(v,G),\qquad J_R^D(v,G)=e^{-\rho t}V_R^D(v,G).$$

Moreover, for any $v\geq 0$ and $G\geq 0$, the following Hamilton–Jacobi–Bellman (HJB) equation are satisfied:

$$\rho V_F^D(v,G)=\max\left[w(a-\alpha p(t))(\lambda v(t)+\gamma G(t))-\frac{b}{2}u(t)^2+V_{Fv}(\beta u(t)-kv(t))+V_{FG}(\theta A(t)-\omega G(t))\right],\quad \text{(A.1)}$$

$$\rho V_R^D(v,G)=\max\left[(p-w)(a-\alpha p(t))(\lambda v(t)+\gamma G(t))-\frac{\delta}{2}A(t)^2+V_{Rv}(\beta u(t)-kv(t))+V_{RG}(\theta A(t)-\omega G(t))\right].\quad \text{(A.2)}$$

By taking the first-order derivatives of the right-hand side of (A.2) with respect to p and A, and setting them equal to zero, we can obtain:

$$p=\frac{a}{2\alpha}+\frac{w}{2},\qquad A=\frac{\theta V_{RG}}{\delta}.$$

We substitute them into Equation (A.1), then take the first-order derivatives of the right-hand side with respect to w and u, and set these derivatives equal to zero. From this, we can obtain:

$$w=\frac{a}{2\alpha},\qquad u=\frac{\beta V_{Fv}}{b}.$$

Substituting the above results into (A.1) and (A.2) and simplifying, we obtain:

$$\rho V_F^D(v,G) = \frac{\lambda a^2}{8\alpha} - kV_{Fv}v(t) + \frac{\gamma a^2}{8\alpha} - \omega V_{FG}G(t) + \frac{\beta^2 V_{Fv}^2}{2b} + \frac{\theta^2 V_{FG}^2}{\delta}, \tag{A.3}$$

$$\rho V_R^D(v,G) = \frac{\lambda a^2}{16\alpha} - kV_{Rv}v(t) + \frac{\gamma a^2}{16\alpha} - \omega V_{RG}G(t) + \frac{\theta^2 V_{RG}^2}{2\delta} + \frac{\beta^2 V_{Rv}^2}{b}. \tag{A.4}$$

Based on the structure of the aforementioned equations, assume the linear expressions of $V_F^D(v,G)$ and $V_R^D(v,G)$ with respect to v and G are as follows:

$$\begin{cases} V_F^D(v,G) = s_1^D v + s_2^D G + s_3^D, \\ V_R^D(v,G) = r_1^D v + r_2^D G + r_3^D, \end{cases}$$

where both s_i^D and r_i^D are constants, $i = 1,2,3$.

Replace $V_F, V_R, V_{Fv}, V_{Rv}, V_{FG}, V_{RG}$ in (A.3) and (A.4) with s_i^D and r_i^D, and after rearrangement, we can obtain:

$$V_{Fv} = \frac{\lambda a^2}{8\alpha(\rho+k)}, \qquad V_{FG} = \frac{\gamma a^2}{8\alpha(\rho+\omega)},$$

$$V_{Rv} = \frac{\lambda a^2}{16\alpha(\rho+k)}, \qquad V_{RG} = \frac{\gamma a^2}{16\alpha(\rho+\omega)}.$$

Thus, we can obtain:

$$p^{D*} = \frac{3a}{4\alpha}, \qquad w^{D*} = \frac{a}{2\alpha}, \qquad u^{D*} = \frac{\beta\lambda a^2}{8\alpha b(\rho+k)}, \qquad A^{D*} = \frac{\gamma\theta a^2}{16\alpha\delta(\rho+\omega)}.$$

Substitute

$$u^{D*} = \frac{\beta\lambda a^2}{8\alpha b(\rho+k)}, \qquad A^{D*} = \frac{\gamma\theta a^2}{16\alpha\delta(\rho+\omega)}$$

into

$$v'(t) = \beta u(t) - kv(t), \qquad G'(t) = \theta A(t) - \omega G(t),$$

and we can obtain:

$$v^{D*}(t) = \left[v_0 - \frac{\beta^2\lambda a^2}{8\alpha bk(\rho+k)}\right]e^{-kt} + \frac{\beta^2\lambda a^2}{8\alpha bk(\rho+k)},$$

$$G^{D*}(t) = \left[G_0 - \frac{\theta^2\gamma a^2}{16\alpha\delta\omega(\rho+\omega)}\right]e^{-\omega t} + \frac{\theta^2\gamma a^2}{16\alpha\delta\omega(\rho+\omega)}.$$

Proof of Proposition 2

Under the grain-price subsidy model, the profit functions of farmers and grain enterprises are as follows:

$$\max\ \pi_F^M = \int_0^{T_1} e^{-\rho t}[wq(t) - C_u(t)]dt + \int_{T_1}^{T} e^{-\rho t}[wq(t) - C_u(t)]dt,$$

$$\max\ \pi_R^M = \int_0^{T_1} e^{-\rho t}[(1+\tau_1)p - w]q(t) - C_A(t)\,dt + \int_{T_1}^{T} e^{-\rho t}[(p-w)q(t) - C_A(t)]dt.$$

To facilitate calculations, the above formula can be divided into two phases, $[0, T_1]$ and $(T_1, T]$, and the optimal solutions for the two phases can be calculated separately.

The Hamiltonian functions for farmers and grain enterprises are constructed, respectively, based on optimal control theory:

$$\rho H_F^M(v,G) = \max\left[w(a-\alpha p(t))(\lambda v(t)+\gamma G(t))-\frac{b}{2}u(t)^2+H_{Fv}(\beta u(t)-kv(t))+H_{FG}(\theta A(t)-\omega G(t))\right], \quad \text{(A.5)}$$

$$\rho H_R^M(v,G) = \max\left[[(1+\tau_1)p-w](a-\alpha p(t))(\lambda v(t)+\gamma G(t))-\frac{\delta}{2}A(t)^2+H_{Rv}(\beta u(t)-kv(t))+H_{RG}(\theta A(t)-\omega G(t))\right]. \quad \text{(A.6)}$$

By taking the first-order derivatives of the right-hand side of (A.6) with respect to p and A, respectively, and setting them equal to zero, we can obtain:

$$p = \frac{a+\alpha w}{\alpha(2+\tau_1)}, \qquad A = \frac{\theta H_{RG_1}}{\delta}.$$

Substituting it into equation (A.5), then taking the first-order derivatives of the right-hand side with respect to w and u, respectively, and setting them equal to zero, we can obtain:

$$w = \frac{(1+\tau_1)a}{2\alpha}, \qquad u = \frac{\beta H_{Fv_1}}{b}.$$

Substituting the above results into (A.5) and (A.6) and rearranging, we get:

$$\rho H_F^M(v,G) = \left[\frac{\lambda(1+\tau_1)a^2}{4\alpha(2+\tau_1)} - kH_{Fv_1}\right]v(t) + \left[\frac{\gamma(1+\tau_1)a^2}{4\alpha(2+\tau_1)} - \omega H_{FG_1}\right]G(t) + \frac{\beta^2 H_{Fv_1}^2}{2b} + \frac{\theta^2 H_{FG_1} H_{RG_1}}{\delta}, \quad \text{(A.7)}$$

$$\rho H_R^M(v,G) = \left[\frac{\lambda(1+\tau_1)^2a^2}{4\alpha(2+\tau_1)^2} - kH_{Rv_1}\right]v(t) + \left[\frac{\gamma(1+\tau_1)^2a^2}{4\alpha(2+\tau_1)^2} - \omega H_{RG_1}\right]G(t) + \frac{\theta^2 H_{RG_1}^2}{2\delta} + \frac{\beta^2 H_{Rv_1} H_{Fv_1}}{b}. \quad \text{(A.8)}$$

Based on the structure of the above equations, it is assumed that the linear expressions of $H_F^M(v,G)$ and $H_R^M(v,G)$ with respect to v and G are as follows:

$$\begin{cases} H_F^M(v,G) = s_1^M v + s_2^M G + s_3^M, \\ H_R^M(v,G) = r_1^M v + r_2^M G + r_3^M, \end{cases}$$

where both s_i^M and r_i^M are constants, $i = 1, 2, 3$.

Replacing $H_{Fv_1}, H_{FG_1}, H_{Rv_1}$, and H_{RG_1} in (A.7) and (A.8) with s_i^M and r_i^M, and rearranging, we can obtain:

$$H_{Fv_1} = \rho s_1^M = \frac{\lambda(1+\tau_1)a^2}{4\alpha(2+\tau_1)} - ks_1^M, \qquad H_{FG_1} = \rho s_2^M = \frac{\gamma(1+\tau_1)a^2}{4\alpha(2+\tau_1)} - \omega s_2^M,$$

$$H_{Rv_1} = \rho r_1^M = \frac{\lambda(1+\tau_1)^2a^2}{4\alpha(2+\tau_1)^2} - kr_1^M, \qquad H_{RG_1} = \rho r_2^M = \frac{\gamma(1+\tau_1)^2a^2}{4\alpha(2+\tau_1)^2} - \omega r_2^M.$$

So

$$H_{Fv_1} = \frac{\lambda(1+\tau_1)a^2}{4\alpha(2+\tau_1)(\rho+k)}, \qquad H_{FG_1} = \frac{\gamma(1+\tau_1)a^2}{4\alpha(2+\tau_1)(\rho+\omega)},$$

$$H_{Rv_1} = \frac{\lambda(1+\tau_1)^2a^2}{4\alpha(2+\tau_1)^2(\rho+k)}, \qquad H_{RG_1} = \frac{\gamma(1+\tau_1)^2a^2}{4\alpha(2+\tau_1)^2(\rho+\omega)}.$$

Thus, we can obtain:

$$p_1^{M*} = \frac{(3+2\tau_1)a}{2\alpha(2+\tau_1)}, \qquad w_1^{M*} = \frac{(1+\tau_1)a}{2\alpha},$$

$$A_1^{M*} = \frac{\gamma\theta(1+\tau_1)^2a^2}{4\alpha\delta(2+\tau_1)^2(\rho+\omega)}, \qquad u_1^{M*} = \frac{\beta\lambda(1+\tau_1)a^2}{4\alpha b(2+\tau_1)(\rho+k)}.$$

Substituting u_1^{M*} and A_1^{M*} into $v'(t) = \beta u(t) - kv(t)$ and $G'(t) = \theta A(t) - \omega G(t)$, respectively, we can obtain:

$$v_1^{M*}(t) = \left[v_0 - \frac{\lambda\beta^2(1+\tau_1)a^2}{4\alpha bk(2+\tau_1)(\rho+k)}\right]e^{-kt} + \frac{\lambda\beta^2(1+\tau_1)a^2}{4\alpha bk(2+\tau_1)(\rho+k)},$$

$$G_1^{M*}(t) = \left[G_0 - \frac{\lambda\theta^2(1+\tau_1)^2a^2}{4\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}\right]e^{-\omega t} + \frac{\lambda\theta^2(1+\tau_1)^2a^2}{4\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}.$$

Similarly, it can be obtained that in Phase $(T_1, T]$:

$$p_2^{M*} = \frac{3a}{4\alpha}, \qquad w_2^{M*} = \frac{a}{2\alpha}, \qquad A_2^{M*} = \frac{\gamma\theta a^2}{16\alpha\delta(\rho+\omega)}, \qquad u_2^{M*} = \frac{\beta\lambda a^2}{8\alpha b(\rho+k)},$$

$$v_2^{M*}(t) = \frac{\beta^2\lambda a^2}{8\alpha bk(2+\tau_1)(\rho+k)}e^{-k(t-T_1)} + \left[v_0 - \frac{\lambda\beta^2(2+\tau_1)a^2}{4\alpha bk(2+\tau_1)(\rho+k)}\right]e^{-kt} + \frac{\beta^2\lambda a^2}{8\alpha bk(\rho+k)},$$

$$G_2^{M*}(t) = \frac{\lambda\theta^2(3\tau_1^2+4\tau_1)a^2}{16\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}e^{-\omega(t-T_1)} + \left[G_0 - \frac{\lambda\theta^2(1+\tau_1)^2a^2}{4\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}\right]e^{-\omega t} + \frac{\gamma\theta^2a^2}{16\alpha\delta\omega(\rho+\omega)},$$

$$w^{M*} = \begin{cases} \dfrac{(1+\tau_1)a}{2\alpha}, & t \in [0, T_1], \\ \dfrac{a}{2\alpha}, & t \in (T_1, T], \end{cases} \qquad p^{M*} = \begin{cases} \dfrac{(3+2\tau_1)a}{2\alpha(2+\tau_1)}, & t \in [0, T_1], \\ \dfrac{3a}{4\alpha}, & t \in (T_1, T], \end{cases}$$

$$u^{M*} = \begin{cases} \dfrac{\lambda\beta(1+\tau_1)a^2}{4\alpha b(2+\tau_1)(\rho+k)}, & t \in [0, T_1], \\ \dfrac{\lambda\beta a^2}{8\alpha b(\rho+k)}, & t \in (T_1, T], \end{cases} \qquad A^{M*} = \begin{cases} \dfrac{\gamma\theta(1+\tau_1)^2a^2}{4\alpha\delta(2+\tau_1)^2(\rho+\omega)}, & t \in [0, T_1], \\ \dfrac{\gamma\theta a^2}{16\alpha\delta(\rho+\omega)}, & t \in (T_1, T]. \end{cases}$$

At this point, the quality trajectory of grain and the goodwill trajectory of grain enterprises are, respectively:

$$v(t)^{M*} = \begin{cases} \left[v_0 - \dfrac{\lambda\beta^2(2+\tau_1)a^2}{4\alpha bk(2+\tau_1)(\rho+k)}\right]e^{-kt} + \dfrac{\lambda\beta^2(2+\tau_1)a^2}{4\alpha bk(2+\tau_1)(\rho+k)}, & t \in [0, T_1], \\ \dfrac{\beta^2\lambda a^2}{8\alpha bk(2+\tau_1)(\rho+k)}e^{-k(t-T_1)} + \left[v_0 - \dfrac{\lambda\beta^2(2+\tau_1)a^2}{4\alpha bk(2+\tau_1)(\rho+k)}\right]e^{-kt} + \dfrac{\beta^2\lambda a^2}{8\alpha bk(\rho+k)}, & t \in (T_1, T], \end{cases}$$

$$G(t)^{M*}=\begin{cases}\left[G_0-\dfrac{\lambda\theta^2(1+\tau_1)^2a^2}{4\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}\right]e^{-\omega t}+\dfrac{\lambda\theta^2(1+\tau_1)^2a^2}{4\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}, & t\in[0,T_1],\\ \dfrac{\lambda\theta^2(3\tau_1^2+4\tau_1)a^2}{16\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}e^{-\omega(t-T_1)}+\left[G_0-\dfrac{\lambda\theta^2(1+\tau_1)^2a^2}{4\alpha\delta\omega(2+\tau_1)^2(\rho+\omega)}\right]e^{-\omega t}+\dfrac{\gamma\theta^2a^2}{16\alpha\delta\omega(\rho+\omega)}, & t\in(T_1,T].\end{cases}$$

The proof completed.

Proof of Corollary 2

Because

$$u^{M*}=\begin{cases}\dfrac{\lambda\beta(1+\tau_1)a^2}{4\alpha b(2+\tau_1)(\rho+k)}, & t\in[0,T_1],\\ \dfrac{\lambda\beta a^2}{8\alpha b(\rho+k)}, & t\in(T_1,T],\end{cases}$$

we have

$$\frac{\partial u^{M*}}{\partial\lambda}=\begin{cases}\dfrac{\beta(1+\tau_1)a^2}{4\alpha b(2+\tau_1)(\rho+k)}>0, & t\in[0,T_1],\\ \dfrac{\beta a^2}{8\alpha b(\rho+k)}>0, & t\in(T_1,T],\end{cases}\qquad \frac{\partial u^{M*}}{\partial k}=\begin{cases}-\dfrac{\lambda\beta(1+\tau_1)a^2}{4\alpha b(2+\tau_1)(\rho+k)^2}<0, & t\in[0,T_1],\\ -\dfrac{\lambda\beta a^2}{8\alpha b(\rho+k)^2}<0, & t\in(T_1,T].\end{cases}$$

Similarly, we can obtain:

$$\frac{\partial A^{M*}}{\partial\gamma}>0,\quad \frac{\partial\pi_i^{M*}}{\partial\lambda}>0,\quad \frac{\partial\pi_i^{M*}}{\partial\gamma}>0,\quad \frac{\partial u^{M*}}{\partial b}<0,\quad \frac{\partial\pi_i^{M*}}{\partial k}<0,\quad \frac{\partial\pi_i^{M*}}{\partial\omega}<0,\quad i=F,R.$$

When $t\in[0,T_1]$, we have $w^{M*}=\dfrac{(1+\tau_1)a}{2\alpha}$, so

$$\frac{\partial w^{M*}}{\partial\tau_1}=\frac{a}{2\alpha}>0.$$

Similarly, it can be obtained that when $t\in[0,T_1]$, we have

$$\frac{\partial p^{M*}}{\partial\tau_1}>0,\quad \frac{\partial A^{M*}}{\partial\tau_1}>0,\quad \frac{\partial u^{M*}}{\partial\tau_1}>0,\quad \frac{\partial\pi_i^{M*}}{\partial\tau_1}>0,\quad \frac{\partial\pi_i^{M*}}{\partial T_1}>0,\quad i=F,R.$$

The proof completed.

Proof of Corollary 3

When $t\in[0,T_1]$, we have

$$w^{M*}-w^{N*}=\frac{(1+\tau_1)a}{2\alpha}-\frac{a}{2\alpha}=\frac{\tau_1a}{2\alpha}>0,$$

so $w^{M*}>w^{N*}$, and

$$u^{M*}-u^{N*}=\frac{\lambda\beta(1+\tau_1)a^2}{8\alpha b(\rho+k)}-\frac{\lambda\beta a^2}{8\alpha b(\rho+k)}=\frac{\lambda\beta\tau_1a^2}{8\alpha b(\rho+k)}>0.$$

Thus, we can obtain $u^{M*} > u^{N*}$.

Meanwhile, since

$$A^{M*} - A^{N*} = \frac{\gamma\theta(1+\tau_1)a^2}{16\alpha\delta(\rho+\omega)} - \frac{\gamma\theta a^2}{16\alpha\delta(\rho+\omega)} = \frac{\gamma\theta\tau_1 a^2}{16\alpha\delta(\rho+\omega)} > 0,$$

then $A^{M*} > A^{N*}$.

$$v(t)^{M*} - v(t)^{N*} = \begin{cases} \frac{\lambda\beta^2 a^2\tau_1}{8\alpha bk(\rho+k)}\left(1-e^{-kt}\right), & t \in [0, T_1], \\ \frac{\lambda\beta^2 a^2\tau_1}{8\alpha bk(\rho+k)}e^{kT_1}, & t \in (T_1, T], \end{cases}$$

$$G(t)^{M*} - G(t)^{N*} = \begin{cases} \frac{\lambda\theta^2\tau_1 a^2}{16\alpha\delta\omega(\rho+\omega)}\left(1-e^{-\omega t}\right), & t \in [0, T_1], \\ \frac{\lambda\theta^2\tau_1 a^2}{16\alpha\delta\omega(\rho+\omega)}e^{\omega T_1}, & t \in (T_1, T]. \end{cases}$$

Thus, $v(t)^{M*} - v(t)^{N*} > 0$ and $G(t)^{M*} - G(t)^{N*} > 0$, that is, $v(t)^{M*} > v(t)^{N*}$ and $G(t)^{M*} > G(t)^{N*}$.

The proof completed.

Proofs of Propositions 3 and 4

These are the same as the proofs of Propositions 1 and 2 above, and they will not be repeated here.

Proof of Corollary 5

Because

$$w^{S*} - w^{N*} = \frac{a}{2\alpha} - \frac{a}{2\alpha} = 0,$$

$w^{S*} = w^{N*}$. Similarly, it can be obtained that $p^{S*} = p^{N*}$ and $A^{S*} = A^{N*}$.

Because

$$v(t)^{S*} - v(t)^{N*} = \frac{\beta^2\lambda a^2\tau_2}{8\alpha bk(1-\tau_2)(\rho+k)}\left(1-e^{-kt}\right) > 0,$$

$$\pi_F^{S*} - \pi_F^{N*} = \frac{\beta^2\lambda^2 a^4}{128b\alpha^2(1-\tau_2)(\rho+k)^2} - \frac{\beta^2\lambda^2 a^4}{128b\alpha^2(\rho+k)^2} > 0,$$

so $v(t)^{S*} > v(t)^{N*}$ and $\pi_F^{S*} > \pi_F^{N*}$. Similarly, it can be obtained that $\pi_R^{S*} > \pi_R^{N*}$.

Proof of Corollary 6

Because

$$w^{M*} - w^{S*} = \begin{cases} \frac{\tau_1 a}{2\alpha} > 0, & t \in [0, T_1], \\ 0, & t \in (T_1, T], \end{cases}$$

then $w^{M*} \geq w^{S*}$. Similarly, we have $p^{M*} \geq p^{S*}$ and $A^{M*} \geq A^{S*}$.

We can easily obtain:

$$u^{S*} - u^{M*} = \frac{\beta^2\lambda a^2}{8\alpha bk(\rho+k)} \cdot \frac{(1+\tau_1)\tau_2 - \tau_1}{1-\tau_2}.$$

According to Figure A1, we know that $u^{S*} \geq u^{M*}$.

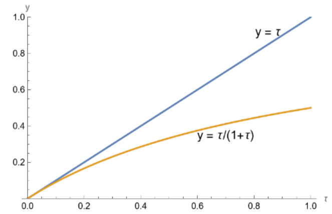

Figure A1. Legend of the figure.

So, we can also get: $v(t)^{S*} > v(t)^{M*}$.

Because

$$\left[\frac{\lambda a^2}{8\alpha(\rho+k)}v_0 + \frac{\gamma a^2}{8\alpha(\rho+\omega)T}G_0\right]\frac{(1+\tau_1)(T+T_1)}{T} > \frac{\lambda a^2}{8\alpha(\rho+k)}v_0 + \frac{\gamma a^2}{8\alpha(\rho+\omega)T}G_0,$$

$$\left[\frac{\beta^2\lambda^2 a^4}{128b\alpha^2(\rho+k)^2} + \frac{\theta^2\gamma^2 a^4}{128\delta\alpha^2(\rho+\omega)^2}\right]\frac{(1+\tau_1)^2(T+T_1)}{T} > \frac{\beta^2\lambda^2 a^4}{128b\alpha^2(1-\tau_2)(\rho+k)^2} + \frac{\theta^2\gamma^2 a^4}{128\delta\alpha^2(\rho+\omega)^2},$$

then $\pi_F^{M*} > \pi_F^{S*}$.

Proof of Corollary 7

When $\tau_2 = \tau_3$, we have

$$\frac{u^{H*}}{u^{S*}} = \begin{cases} 1+\tau_4, & t \in [0, T_2], \\ 1, & t \in (T_2, T]. \end{cases}$$

$$v(t)^{H*} - v(t)^{S*} = \begin{cases} \dfrac{\lambda\beta^2\tau_4 a^2}{8\alpha bk(1-\tau_3)(\rho+k)}\left(1-e^{-kt}\right), & t \in [0, T_2], \\ \dfrac{\lambda\beta^2\tau_4 a^2}{8\alpha bk(1-\tau_3)(\rho+k)}e^{-kt} + \dfrac{\beta^2\lambda a^2\tau_4}{8\alpha bk(1-\tau_3)(\rho+k)}e^{-k(t-T_2)}, & t \in (T_2, T], \end{cases}$$

so $u^{H*} \geq u^{S*}$, $v(t)^{H*} > v(t)^{S*}$.

Similarly, it can be obtained that $\pi_R^{H*} > \pi_R^{M*}$.