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Research article

Optimal portfolio choice with capital gains tax under Heston's stochastic volatility model

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Appendix

Proof of Theorem 2.1

Proof. For any $t < u \leq T$, by applying Itô's formula for $V(s, X^\pi(s), m(s))$ on $[t, u]$, we obtain

$$V(u, X^\pi(u), m(u)) = V(t, x, m) + \int_t^u \mathcal{L}^\pi V(s, X^\pi(s), m(s))\, ds + (\mathcal{M}^V(u) - \mathcal{M}^V(t)), \tag{A.1}$$

where $\mathcal{L}^\pi$ is the operator in (2.9) with J replaced by V, and $\mathcal{M}^V$ is a local martingale.

By condition (iii) in Theorem 2.1, we find

$$\begin{aligned} V_t + V_m k(\theta - m) + \frac{1}{2} V_{mm}\sigma^2 m + \Big\{ & V_x[\pi_1(t)(1-\tau)\lambda m + \pi_0(t)(r + \lambda m - r_1) + r_1]x \\ & + \frac{1}{2} V_{xx}[(1-\tau)\pi_1(t) + \pi_0(t)]^2 x^2 m + \lambda_1[V(t, x(1-\alpha\pi_0), m) - V(t, x, m)] \\ & + V_{mx}[(1-\tau)\pi_1(t) + \pi_0(t)]\sigma\rho\, xm \Big\} \leq 0; \end{aligned}$$

i.e., $\mathcal{L}^\pi V(s, X^\pi(s), m(s)) \leq 0$ for any admissible π. Therefore,

$$V(u, X^\pi(u), m(u)) \leq V(t, x, m) + (M^V(u) - M^V(t)). \tag{A.2}$$

Taking the conditional expectation for Eq. (A.2) and using condition (iv), we can obtain

$$\mathrm{E}\big[V(u, X^{\pi}(u), m(u)) \,\big|\, X^{\pi}(t) = x,\ m(t) = m\big] \leq V(t, x, m).$$

Letting $u = T$ and using the condition (ii), we have

$$\mathrm{E}\big[U(X^{\pi}(T)) \,\big|\, X^{\pi}(t) = x,\ m(t) = m\big] \leq V(t, x, m). \tag{A.3}$$

Therefore, we have $J(t, x, m) = \sup_{\pi\in\Pi} \mathrm{E}[U(X^{\pi}(T))| X^{\pi}(t) = x,\ m(t) = m] \leq V(t, x, m)$.

Furthermore, if there exists an admissible strategy $\pi^* \in \Pi$ attaining the maximum in the HJB equation (2.10), then along the controlled process $(X^{\pi^*}(s), m(s))$, we have $\mathcal{L}^{\pi^*} V(s, X^{\pi^*}(s), m(s)) = 0$. Therefore

$$V(T, X^{\pi^*}(T), m(T)) = V(t, x, m) + (\mathcal{M}^V(T) - \mathcal{M}^V(t)).$$

Taking the conditional expectation and using conditions (ii) and (iv) again, we obtain

$$V(t, x, m) = \mathrm{E}\big[V(T, X^{\pi^*}(T), m(T)) \,\big|\, X^{\pi^*}(t) = x,\ m(t) = m\big] = \mathrm{E}\big[U(X^{\pi^*}(T)) \,\big|\, X^{\pi^*}(t) = x,\ m(t) = m\big].$$

Thus we have $V(t, x, m) \leq J(t, x, m)$. Finally, combining with Eq. (A.3) gives $V = J$. □

Proof of Theorem 3.1

Proof. In order to solve the optimal investment problem (2.6), we first conjecture that the value function satisfies some explicit form. Then we adopt the first-order optimality conditions for $\pi_0(t)$ and $\pi_1(t)$. Finally, we substitute them back into (2.10), and solve the corresponding HJB equation for the optimal portfolio weights $(\pi_0^*(t), \pi_1^*(t), \omega^*(t))$.

Note that the utility function is a power function, and we postulate that the value function $J(t, x, m)$ is separable and $J(t, x.m)$ is given by

$$J(t, x, m) = \frac{x^{1-\gamma}}{1-\gamma} \mathrm{e}^{A(t)m+B(t)}. \tag{A.4}$$

Furthermore, the terminal condition $J(T, x, m) = U(x)$ implies that $A(T) = 0, B(T) = 0$.

Using the first-order optimality condition for HJB Eq. (2.10), we obtain that the maximal points $\pi_0^*(t)$ and $\pi_1^*(t)$ are the solutions of the following two equations:

$$J_x(r - r_1)x + \lambda_1 \frac{\partial J(t, x(1 - \alpha\pi_0^*(t)), m)}{\partial \pi_0^*(t)} + \Big[J_x\lambda + J_{xx}[(1-\tau)\pi_1^*(t) + \pi_0^*(t)]x + J_{mx}\sigma\rho\Big]xm = 0 \tag{A.5}$$

and

$$J_x\lambda + J_{xx}[(1-\tau)\pi_1^*(t) + \pi_0^*(t)]x + J_{mx}\sigma\rho = 0. \tag{A.6}$$

Combine Eq. (A.5) and Eq. (A.6), we can get

$$J_x(r - r_1)x + \lambda_1 \frac{\partial J(t, x(1 - \alpha\pi_0^*(t)), m)}{\partial \pi_0^*(t)} = 0. \tag{A.7}$$

Since

$$J_x = x^{-\gamma} e^{A(t)m+B(t)} \tag{A.8}$$

and

$$\frac{\partial J(t, x(1-\alpha\pi_0^*(t)), m)}{\partial \pi_0^*(t)} = -\alpha x^{1-\gamma}(1-\alpha\pi_0^*(t))^{-\gamma} e^{A(t)m+B(t)}, \tag{A.9}$$

we have

$$(r - r_1) - \alpha\lambda_1(1-\alpha\pi_0^*(t))^{-\gamma} = 0. \tag{A.10}$$

Note that $r_1 = (1-\tau)r$, and we can obtain the following result by solving Eq. (A.10):

$$\pi_0^*(t) = \frac{1}{\alpha}\left[1 - \left(\frac{\tau r}{\lambda_1\alpha}\right)^{-\frac{1}{\gamma}}\right]. \tag{A.11}$$

Furthermore, if follows from (A.6) that

$$(1-\tau)\pi_1(t) + \pi_0(t) = \frac{-J_x\lambda - J_{mx}\sigma\rho}{J_{xx}x}. \tag{A.12}$$

By (A.4), we can obtain

$$\begin{cases} J_x = J(t,x,m)\frac{1-\gamma}{x}, \\ J_m = A(t)J(t,x,m), \\ J_{mx} = J(t,x,m)\frac{(1-\gamma)A(t)}{x}, \\ J_{mm} = A(t)^2 J(t,x,m), \\ J_{xx} = J(t,x,m)\frac{\gamma(\gamma-1)}{x^2}. \end{cases} \tag{A.13}$$

Substituting (A.13) into Eq. (A.12), we have

$$\pi_1^*(t) = \frac{1}{\gamma(1-\tau)}\left[\lambda + A(t)\sigma\rho\right] - \frac{\pi_0^*(t)}{1-\tau}. \tag{A.14}$$

In addition, we can get Eq. (3.3) by using $\omega^*(t) = 1 - \pi_0^*(t) - \pi_1^*(t)$.

In the following, we derive the expressions of $A(t)$ and $B(t)$ in (A.4). Plugging Eqs. (A.11), (A.13), and (A.14) into Eq. (2.10) gives

$$\begin{aligned} B_t + A(t)k\theta + M + m\Bigg\{A_t + A^2(t)\left[\frac{\sigma^2}{2}\left(1 + \frac{(1-\gamma)\rho^2}{\gamma}\right)\right] \\ + A(t)\left[-k + \frac{(1-\gamma)\lambda\sigma\rho}{\gamma}\right] + \frac{\lambda^2(1-\gamma)}{2\gamma}\Bigg\} = 0, \end{aligned} \tag{A.15}$$

where A_t and B_t are the first-order derivatives of $A(t)$ and $B(t)$ with respect to time t, respectively.

Next, we separate the variables in Eq. (A.15) to get the following differential equations:

$$B_t + A(t)k\theta + M = 0 \tag{A.16}$$

and

$$A_t + A^2(t)\left[\frac{\sigma^2}{2}\left(1+\frac{(1-\gamma)\rho^2}{\gamma}\right)\right] + A(t)\left[-k+\frac{(1-\gamma)\lambda\sigma\rho}{\gamma}\right] + \frac{\lambda^2(1-\gamma)}{2\gamma} = 0. \tag{A.17}$$

We first solve Eq. (A.16). Since $B(T) = 0$, it is easy find that $B(t) = k\theta\int_t^T A(s)\mathrm{d}s + M(T-t)$. For convenience of calculation, we let $a = -\frac{\sigma^2}{2}\left(1+\frac{(1-\gamma)\rho^2}{\gamma}\right)$, $b = k - \frac{(1-\gamma)\lambda\sigma\rho}{\gamma}$, $c = -\frac{\lambda^2(1-\gamma)}{2\gamma}$. Then Eq. (A.17) can be written as:

$$A^2(t)a + A(t)b + c = A_t. \tag{A.18}$$

The left side of Eq. (A.18) is a quadratic equation, and we can define the discriminant of this equation as $\Delta_1 = b^2 - 4ac$. In this paper, we suppose that the risk aversion coefficient $\gamma > 1$. Then, we can obtain $c = -\frac{\lambda^2(1-\gamma)}{2\gamma} > 0$ and $a = -\frac{\sigma^2}{2}\left(1+\frac{(1-\gamma)\rho^2}{\gamma}\right) < 0$. This implies that $\Delta_1 = b^2 - 4ac > 0$. Hence, the equation $ax^2 + bx + c = 0$ has two solutions, which can be denoted as $v_1 = \frac{-b+\sqrt{\Delta_1}}{2a}$, $v_2 = \frac{-b-\sqrt{\Delta_1}}{2a}$. By solving equation (A.18), we can obtain an explicit expression for $A(t)$, as given by Eq. (3.4). Hence, we have completed the proof of Theorem 3.1. □

Proof of Theorem 3.2

Proof. We suppose that the value function $J(t, x.m)$ follows an exponential affine structure. The value function is assumed to satisfy the following:

$$J(t,x,m) = -\frac{1}{\beta}\mathrm{e}^{-\beta a(t)x + C(t)m + D(t)}, \tag{A.19}$$

where $a(t)$, $C(t)$, and $D(t)$ are determined functions with respect to variable t. In addition, we know the terminal boundary condition $J(T,x,m) = U(x)$. Hence, we can get $a(T) = 1$, $C(T) = 0$, and $D(T) = 0$.

We follow the structure in Eq. (A.19) and omit the parameters to reduce notations. Then, the partial derivatives become

$$\begin{cases} J_t = J(t,x,m)(-\beta x a_t + C_t m + D_t), \\ J_x = -J(t,x,m)a(t)\beta, \\ J_{xx} = J(t,x,m)a^2(t)\beta^2, \\ J_m = J(t,x,m)C(t), \\ J_{mm} = J(t,x,m)C^2(t), \\ J_{mx} = -J(t,x,m)a(t)\beta C(t), \end{cases} \tag{A.20}$$

where a_t, C_t, and D_t are the first-order partial derivatives of $a(t)$, $C(t)$, and $D(t)$ with respect to t, respectively.

Now we can substitute Eq. (A.20) into Eq. (2.10), and then the HJB Eq. (2.10) can be rewritten as

$$\begin{aligned} \max_{(\pi_0,\pi_1)\in\Pi}\Big\{ & -\beta x a_t + (C_t m + D_t) - \beta a(t)\Big[\pi_1(t)(1-\tau)\lambda m + \pi_0(t)(r + \lambda m - r_1) + r_1\Big]x \\ & + \tfrac{1}{2}\beta^2 a^2(t)\Big[(1-\tau)\pi_1(t) + \pi_0(t)\Big]^2 x^2 m + C(t)k(\theta - m) + \tfrac{1}{2}C^2(t)\sigma^2 m \\ & + \lambda_1\left(e^{\alpha\beta a(t)\pi_0(t)x} - 1\right) - \beta C(t)a(t)\Big[\Big((1-\tau)\pi_1(t) + \pi_0(t)\Big)\sigma\rho x m\Big]\Big\} = 0. \end{aligned} \tag{A.21}$$

By using the first-order optimality conditions, we obtain the following equations:

$$\beta a(t)\left[(1-\tau)\pi_1(t)+\pi_0(t)\right]x-\lambda-C(t)\sigma\rho=0 \tag{A.22}$$

and

$$-(r-r_1)+\Big\{\beta a(t)\left[(1-\tau)\pi_1(t)+\pi_0(t)\right]x-\lambda-C(t)\rho\sigma\Big\}m+\lambda_1 e^{\alpha\beta a(t)\pi_0(t)x}\alpha=0. \tag{A.23}$$

Equations (A.22), (A.23), and $r_1-r=r\tau$ imply that $\pi_0^*(t)$ satisfies the following:

$$\pi_0^*(t) \;=\frac{1}{\alpha\beta a(t)x}\ln\frac{r\tau}{\lambda_1\alpha}. \tag{A.24}$$

Furthermore, Eq. (A.22) implies

$$\pi_1^*(t)=\frac{1}{\beta a(t)x(1-\tau)}\left[\lambda+C(t)\sigma\rho\right]-\frac{\pi_0^*(t)}{1-\tau}. \tag{A.25}$$

Now by Eqs. (A.24) and (A.25), we can obtain Eq. (3.10). Finally, the formula (3.11) follows by combing Eqs. (A.24) and (A.25), and using the equation $\omega^*(t)=1-\pi_0^*(t)-\pi_1^*(t)$.

Substituting Eqs. (A.24) and (A.25) into Eq. (A.21), we get the following equation:

$$\begin{aligned}&-\beta x a_t-\beta a(t)r_1x+D_t+C(t)k\theta-\lambda_1+\tfrac{r\tau}{\alpha}\left(1-\ln\tfrac{r\tau}{\lambda_1\alpha}\right)\\&\quad+m\left\{C_t+\tfrac{1}{2}C^2(t)\sigma^2\left(1-\rho^2\right)+C(t)[-k-\lambda\sigma\rho]-\tfrac{1}{2}\lambda^2\right\}=0.\end{aligned} \tag{A.26}$$

Furthermore, we split Eq. (A.26) into three equations by matching the coefficient of m and x:

$$-\beta a_t-\beta a(t)r_1=0, \tag{A.27}$$

$$C_t+\frac{1}{2}C^2(t)\sigma^2\left(1-\rho^2\right)+C(t)[-k-\lambda\sigma\rho]-\frac{1}{2}\lambda^2=0, \tag{A.28}$$

and

$$D_t+C(t)k\theta-\lambda_1+\frac{r\tau}{\alpha}\left(1-\ln\frac{r\tau}{\lambda_1\alpha}\right)=0, \tag{A.29}$$

with the boundary conditions $a(T)=1$, $C(T)=0$, and $D(T)=0$.

It is easy to obtain the solution of Eq. (A.27) as $a(t)=e^{r_1(T-t)}$. Furthermore, by Eq. (A.29), we have that

$$D(t)=k\theta\int_t^T C(s)ds+\left[\frac{r\tau}{\alpha}\left(1-\ln\frac{r\tau}{\lambda_1\alpha}\right)-\lambda_1\right](T-t). \tag{A.30}$$

Let $\hat{a}=-\frac{1}{2}\sigma^2\left(1-\rho^2\right)$, $\hat{b}=k+\lambda\sigma\rho$, $\hat{c}=\frac{1}{2}\lambda^2$. Then Eq. (A.28) can be rewritten as

$$C^2(t)\hat{a}+C(t)\hat{b}+\hat{c}=C_t. \tag{A.31}$$

Denote by $\Delta_2=\hat{b}^2-4\hat{a}\hat{c}$ the discriminant of Eq. (A.31). Since $\hat{a}<0$ and $\hat{c}>0$, it implies that $\Delta_2=\hat{b}^2-4\hat{a}\hat{c}>0$. Then the equation $\hat{a}x^2+\hat{b}x+\hat{c}=0$ has two solutions, which can be denoted as $n_1=\frac{-\hat{b}+\sqrt{\Delta_2}}{2\hat{a}}$ and $n_2=\frac{-\hat{b}-\sqrt{\Delta_2}}{2\hat{a}}$. Therefore, we can obtain the explicit expression of $C(t)$, as given in Eq. (3.12). □

Before proving Proposition 3.1, we need the following Lemma A.1.

Lemma A.1. *Let* $(m(t))_{t\in[0,T]}$ *satisfy Heston's stochastic volatility model:*

$$dm(t) = k(\theta - m(t))dt + \sigma\rho\sqrt{m(t)}\,dW_1(t) + \sigma\sqrt{(1-\rho^2)m(t)}\,dW_2(t), \qquad m(0) = m_0 > 0, \quad \text{(A.32)}$$

with constants $k > 0, \theta > 0, \sigma > 0$ *and the Feller condition* $2k\theta > \sigma^2$ *is satisfied. Then*

(i) $\sup_{t\in[0,T]} \mathrm{E}[m(t)] < \infty$ *and*

$$\mathrm{E}\left[\int_0^T m(s)\,ds\right] = \int_0^T \mathrm{E}[m(s)]\,ds < \infty.$$

(ii) Moreover, for any $p > 1$, $m(t)$ *satisfies the following:*

$$\mathrm{E}\left[\int_0^T m^p(s)\,ds\right] < \infty.$$

Proof. (i) Define the stopping times

$$\tau_n = \inf\{t \geq 0 : m(t) \geq n\} \wedge T.$$

It follows that $m(t) \leq n$ when $t \in [0, \tau_n]$ and

$$\mathrm{E}\left[\int_0^{t\wedge\tau_n} \sigma^2 m(u)du\right] \leq \sigma^2 nT < \infty.$$

This implies that

$$\int_0^{t\wedge\tau_n} \sigma\rho\sqrt{m(s)}dW_1(s) + \int_0^{t\wedge\tau_n} \sigma\sqrt{(1-\rho^2)m(s)}dW_2(s)$$

is a square integrable martingale. From Eq. (A.32), we get

$$\begin{aligned} m(t\wedge\tau_n) = m_0 &+ k\int_0^{t\wedge\tau_n}(\theta - m(s))\,ds + \int_0^{t\wedge\tau_n}\sigma\rho\sqrt{m(s)}dW_1(s) \\ &+ \int_0^{t\wedge\tau_n}\sigma\sqrt{(1-\rho^2)m(s)}dW_2(s). \end{aligned} \quad \text{(A.33)}$$

Consequently, taking expectations on both sides of (A.33), we obtain

$$\mathrm{E}\left[m(t\wedge\tau_n)\right] = m_0 + k\mathrm{E}\left[\int_0^{t\wedge\tau_n}(\theta - m(s))\,ds\right]. \quad \text{(A.34)}$$

Noting that $0 \leq m(t\wedge\tau_n) \uparrow m(t)$, the monotone convergence theorem implies that $\mathrm{E}\left[m(t\wedge\tau_n)\right] \uparrow \mathrm{E}\left[m(t)\right]$. Therefore, the mean of $m(t)$ is given by

$$\mathrm{E}\left[m(t)\right] = m_0 + k\int_0^t (\theta - \mathrm{E}\left[m(s)\right])\,ds. \quad \text{(A.35)}$$

By direct calculation, we have

$$\mathrm{E}[m(t)] = \theta + (m_0 - \theta)e^{-kt}. \quad \text{(A.36)}$$

This implies that the mean of $m(t)$ is bounded when $t \in [0,T]$. Therefore, we have $\sup_{t\in[0,T]} \mathrm{E}[m(t)] < \infty$ and $\mathrm{E}\int_0^T m(s)ds = \int_0^T \mathrm{E}[m(s)]ds < \infty$.

(ii) For $p > 1$, applying Itô's formula to $m^p(t)$, we have

$$\begin{aligned} d\left(m(t)^p\right) &= \left(pk\theta m(t)^{p-1} - pkm(t)^p + \frac{1}{2}p(p-1)\sigma^2 m(t)^{p-1}\right)dt \\ &+ p\sigma m(t)^{p-0.5}\left(\rho\sqrt{m(t)}\,dW_1(t) + \sqrt{(1-\rho^2)m(t)}\,dW_2(t)\right). \end{aligned} \tag{A.37}$$

For convenience, let $C_p = pk\theta + \frac{1}{2}p(p-1)\sigma^2$ and we have

$$d\left(m(t)^p\right) = \left(C_p m(t)^{p-1} - pkm(t)^p\right)dt + p\sigma m(t)^{p-0.5}\left(\rho\sqrt{m(t)}\,dW_1(t) + \sqrt{(1-\rho^2)m(t)}\,dW_2(t)\right). \tag{A.38}$$

Following the method of the proof used in Lemma A.1(i), and applying the stopping times technique and the monotone convergence theorem, we obtain

$$\mathrm{E}[m(t)^p] = m_0^p + \int_0^t C_p \mathrm{E}[m(s)^{p-1}]ds - \int_0^t pk\mathrm{E}[m(s)^p]ds. \tag{A.39}$$

From Eq. (A.39), we derive the following result:

$$\frac{d}{dt}\mathrm{E}[m(t)^p] = C_p\mathrm{E}[m(t)^{p-1}] - pk\mathrm{E}[m(t)^p]. \tag{A.40}$$

Using the inequality $x^{p-1} \le 1 + x^p$ for $x \ge 0$ and $p \ge 1$, we have $\mathrm{E}\left[m^{p-1}(t)\right] \le 1 + \mathrm{E}\left[m^p(t)\right]$. Therefore

$$\frac{d}{dt}\mathrm{E}[m(t)^p] \le C_p + (C_p - pk)\mathrm{E}[m(t)^p]. \tag{A.41}$$

By Grönwall's inequality, we obtain

$$\mathrm{E}[m(t)^p] \le m_0^p e^{(C_p-pk)t} + \frac{C_p(e^{(C_p-pk)t}-1)}{C_p-pk} = e^{(C_p-pk)t}\left(m_0^p + \frac{C_p(1-e^{-(C_p-pk)t})}{C_p-pk}\right). \tag{A.42}$$

Note that $C_p - pk$ and m_0^p are constants, and we have $\mathrm{E}[m(t)^p] < \infty$. Integrating over $[0,T]$ yields $\mathrm{E}\left[\int_0^T m^p(s)ds\right] < \infty$. □

Proof of Proposition 3.1

Proof. Observe that the control $\pi_0^*(t)$ is a constant, as shown in Eq. (3.1), and the function $A(t)$ solving the Riccati equation is continuous and bounded on $[0,T]$, which implies that $\pi_1^*(t)$ is bounded. By Lemma A.1, we have

$$\mathrm{E}\Big[\int_0^T ((\pi_0^*(t))^2 + (\pi_1^*(t))^2)m(t)\,dt\Big] < \sup_{t\in[0,T]} ((\pi_0^*(t))^2 + (\pi_1^*(t))^2)\mathrm{E}\Big[\int_0^T m(t)\,dt\Big] < \infty.$$

From Eq. (3.1), it can be seen that $1 - \alpha\pi_0^*(t) > 0$. By applying Itô's formula, the solution of Eq. (2.5) is obtained as follows:

$$X^{\pi^*}(T) = x_0 \exp\Big\{\int_0^T \left[(1-\tau)\pi_1^*(t) + \pi_0^*(t)\right]\sqrt{m(t)}dW_1(t) - \int_0^T \ln(1-\alpha\pi_0^*(t))dN(t)$$

$$
\begin{aligned}
&+ \int_0^T \Big[\pi_1^*(t)(1-\tau)\lambda m(t) + \pi_0^*(t)\left[r + \lambda m(t) - r_1\right] \\
&\qquad + r_1 - \tfrac{1}{2}\Big[(1-\tau)\pi_1^*(t) + \pi_0^*(t)\Big]^2 m(t)\Big]dt\Big\}.
\end{aligned} \tag{A.43}
$$

Observe that $\pi_0^*(t)$ and $\pi_1^*(t)$ are bounded, and it follows from Lemma 4.3 of [30] that

$$
\begin{aligned}
&\exp\Big\{(-\tfrac{1}{2}\int_0^T (1-\gamma)^2 \Big[(1-\tau)\pi_1^*(t) + \pi_0^*(t)\Big]^2 m(t)dt \\
&\qquad + \int_0^T (1-\gamma)\Big[(1-\tau)\pi_1^*(t) + \pi_0^*(t)\Big]\sqrt{m(t)}dW_1(t)\Big\}.
\end{aligned} \tag{A.44}
$$

is a martingale. In particular, since $\pi_0^*(t)$ is in fact constant, we simply denote it by π_0^*. Thus

$$
\begin{aligned}
\mathrm{E}\Big[\Big(X^{\pi^*}(T)\Big)^{1-\gamma}\Big] &= x_0^{1-\gamma}\exp\Big\{\Big[(1-\gamma)(r-r_1)\pi_0^* + r_1 + (1-\alpha\pi_0^*)^{\gamma-1}\lambda_1\Big]T\Big\} \\
&\times\ \mathrm{E}\left[\exp\Big\{\int_0^T \Big(\lambda(1-\gamma)\left[\pi_1^*(t)(1-\tau) + \pi_0^*\right] - \frac{\gamma(1-\gamma)}{2}\left[\pi_1^*(t)(1-\tau) + \pi_0^*\right]^2\Big)m(t)dt\Big\}\right].
\end{aligned} \tag{A.45}
$$

From Eq. (3.6), we obtain the following inequality:

$$
\mathrm{E}\Big[\Big(X^{\pi^*}(T)\Big)^{1-\gamma}\Big] \le x_0^{1-\gamma}\exp\Big\{\Big[(1-\gamma)(r-r_1)\pi_0^* + r_1 + (1-\alpha\pi_0^*)^{\gamma-1}\lambda_1\Big]T\Big\}\mathrm{E}\left[\exp\Big(\zeta\int_0^T m(t)dt\Big)\right]. \tag{A.46}
$$

By a direct application of Theorem 5.1 in [30], we have

$$
\mathrm{E}\Big[\Big(X^{\pi^*}(T)\Big)^{1-\gamma}\Big] < \infty. \tag{A.47}
$$

This implies $\mathrm{E}\Big[\,|U(X^{\pi}(T))|\,\Big] < \infty$. Therefore, optimal investment strategies $(\pi_0^*(t), \pi_1^*(t), \omega^*(t))$ presented in Theorem 3.1 are shown to be admissible. □

Lemma A.2. *Applying Itô's formula to $J(t, X^{\pi}(t), m(t))$, we obtain the following Itô–Lévy decomposition:*

$$
J(t, X^{\pi}(t), m(t)) = J(0, X^{\pi}(0), m(0)) + \int_0^t \mathcal{L}^{\pi} J(s, X^{\pi}(s), m(s))ds + \mathcal{M}^J(t),
$$

where $J(t, X^{\pi}(t), m(t))$ is defined in Theorem 3.1 and $\mathcal{M}^J(t)$ is a local martingale term associated with $J(t, X^{\pi}(t), m(t))$. Then the local martingale term $\mathcal{M}^J(t)$ in the Itô–Lévy decomposition of $J(t, X^{\pi}(t), m(t))$ is a martingale.

Proof. We first define a sequence of stopping times:

$$
\tau_n = \inf\{t \ge 0 :\ X^{\pi}(t) \notin (1/n, n) \text{ or } m(t) \ge n\} \wedge T.
$$

Since $A(t)$ and $B(t)$ in Theorem 3.1 are continuous functions on $[0, T]$, then they are bounded. By Eqs. (A.4) and (A.13), we find that the first-order and second-order partial derivatives of J with respect to variables x and m are uniformly bounded on $[0, \tau_n]$. Hence there exists a constant $Q_n < \infty$ such that for all $t \le \tau_n$,

$$
|J_x(t,x,m)| \le Q_n, \quad |J_m(t,x,m)| \le Q_n, \quad |J_{xm}(t,x,m)| \le Q_n, \quad |J_{mm}(t,x,m)| \le Q_n, \quad |J_{xx}(t,x,m)| \le Q_n.
$$

Let $\mathcal{M}^J(t) = \mathcal{M}^{Jc}(t) + \mathcal{M}^{Jj}(t)$, where $\mathcal{M}^{Jc}(t)$ and $\mathcal{M}^{Jj}(t)$ denote the continuous and jump components of $\mathcal{M}^J$, respectively. Using Itô's formula, we derive

$$\begin{aligned} \mathcal{M}^{Jc}(t) &= \int_0^t J_m \sigma\rho\sqrt{m(u)}dW_1(u) + \int_0^t J_m\sigma\sqrt{(1-\rho^2)m(u)}dW_2(u) \\ &\quad + \int_0^t J_x\left[(1-\tau)\pi_1(u) + \pi_0(u)\right]\sqrt{m(u)}dW_1(u) \end{aligned} \tag{A.48}$$

and

$$\mathcal{M}^{Jj}(t) = \int_0^t \Big(J(s, X^\pi(s-)(1-\alpha\pi_0(s)), m(s)) - J(s, X^\pi(s-), m(s))\Big)d\tilde{N}(s), \tag{A.49}$$

where $\tilde{N}(t) = N(t) - \lambda_1 t$ is the compensated Poisson process.

The Brownian integrand in Itô's formula consists of terms involving J_x, J_m multiplied by the diffusion coefficients of X^π and m. Since these coefficients are bounded on $[0, \tau_n]$, it follows from Lemma A.1 that

$$\mathrm{E}\left[\int_0^{T\wedge\tau_n} \left|\text{Brownian integrand}\right|^2 dt\right] < \infty. \tag{A.50}$$

Hence, $\mathcal{M}^{Jc}(t\wedge\tau_n)$ is a square-integrable martingale.

Furthermore, $X^\pi(s-)$ and $m(s)$ are bounded on $[0, \tau_n]$, and the condition $1 - \alpha\pi_0(s) > 0$ ensures positivity of post-jump wealth. Consequently, the jump intensity λ_1 is finite, and we derive the quadratic variation of $\mathcal{M}^{Jj}(t\wedge\tau_n)$ as follows:

$$\mathrm{E}[\mathcal{M}^{Jj}\langle(\cdot\wedge\tau_n)\rangle_T] = \int_0^{T\wedge\tau_n}\Big(J(s, X^\pi(s-)(1-\alpha\pi_0(s)), m(s)) - J(s, X^\pi(s-), m(s))\Big)^2\lambda_1 ds < \infty. \tag{A.51}$$

Thus, $\mathcal{M}^{Jj}(t\wedge\tau_n)$ is also a square-integrable martingale. Combining Eqs. (A.50) and (A.51), it follows that

$$\mathrm{E}[\mathcal{M}^J\langle(\cdot\wedge\tau_n)\rangle_T] < \infty. \tag{A.52}$$

Hence, $\mathcal{M}^J(t\wedge\tau_n)$ is a square martingale and $\mathrm{E}[\mathcal{M}^J(t\wedge\tau_n)] = 0$.

By the Burkholder–Davis–Gundy inequality, there exists a constant $C > 0$ such that

$$\mathrm{E}\left[\sup_{0\le s\le t}|\mathcal{M}^J(s\wedge\tau_n)|\right] \le C\,\mathrm{E}\left[\langle\mathcal{M}^J(\cdot\wedge\tau_n)\rangle_t^{1/2}\right] < \infty.$$

Hence, $\{\mathcal{M}^J(t\wedge\tau_n)\}_{n\ge1}$ is uniformly integrable. Letting $n\to\infty$ and using $\tau_n \uparrow T$ almost surely, we obtain

$$\mathrm{E}[\mathcal{M}^J(t)] = \lim_{n\to\infty}\mathrm{E}[\mathcal{M}^J(t\wedge\tau_n)] = 0, \qquad \forall\, t\in[0,T].$$

Therefore, $\mathcal{M}^J(t)$ is a martingale.

$\square$

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